Proceedings of the Edinburgh Mathematical Society (2006) **49**, 399–425 ©
DOI:10.1017/S0013091504001555 Printed in the United Kingdom

THE ROLE OF FUNNELS AND PUNCTURES IN THE GROMOV HYPERBOLICITY OF RIEMANN SURFACES

ANA PORTILLA, JOSÉ M. RODRÍGUEZ AND EVA TOURÍS

Departamento de Matemáticas, Universidad Carlos III de Madrid,
Avenida de la Universidad, 30, 28911 Leganés, Madrid, Spain
(apferrei@math.uc3m.es; jomaro@math.uc3m.es; etourismath.uc3m.es)

(Received 22 December 2004)

Abstract We prove results on geodesic metric spaces which guarantee that some spaces are not hyperbolic in the Gromov sense. We use these theorems in order to study the hyperbolicity of Riemann surfaces. We obtain a criterion on the genus of a surface which implies non-hyperbolicity. We also include a characterization of the hyperbolicity of a Riemann surface S^* obtained by deleting a closed set from one original surface S. In the particular case when the closed set is a union of continua and isolated points, the results clarify the role of punctures and funnels (and other more general ends) in the hyperbolicity of Riemann surfaces.

Keywords: hyperbolicity; Riemann surface; funnel; puncture

2000 *Mathematics subject classification:* Primary 30F20; 30F45

1. Introduction

A good way to understand the important connections between graphs and 'potential theory' on Riemannian manifolds (see, for example, [**3**, **9**, **13**, **17**–**20**, **25**, **26**, **30**]) is to study Gromov hyperbolic spaces. This approach allows one to establish a general setting to work simultaneously with graphs and manifolds, in the context of metric spaces. Besides, the idea of Gromov hyperbolicity grasps the essence of negatively curved spaces, and has been successfully used in the theory of groups (see, for example, [**14**–**16**] and the references therein).

Although there exist some interesting examples of hyperbolic spaces (see the examples after Definition 2.1), the literature gives no good guide about how to determine whether or not a space is hyperbolic.

Recently, some interesting results of Balogh and Buckley [**4**] about the hyperbolicity of Euclidean bounded domains with their quasihyperbolic metric have represented significant progress in this direction (see also [**7**] and the references therein).

Originally, we were interested in studying when non-exceptional Riemann surfaces equipped with their Poincaré metric were Gromov hyperbolic. However, we have proved theorems on hyperbolicity for general metric spaces, which are interesting in themselves (see § 3) and have important consequences for Riemann surfaces (see § 5). Although one should expect Gromov hyperbolicity in non-exceptional Riemann surfaces due to its constant curvature -1, this turns out to be untrue in general, since topological obstacles can impede it: for instance, the two-dimensional jungle gym (a $\mathbb{Z}^2$-covering of a torus with genus two) is not hyperbolic. Let us recall that in the case of modulated plane domains, quasihyperbolic and Poincaré metrics are equivalent.

In [**29**], Rodríguez and Tourís prove that there is no inclusion relationship between hyperbolic Riemann surfaces and the usual classes of Riemann surfaces, such as O_{G}, O_{HP}, O_{HB}, O_{HD}, surfaces with hyperbolic isoperimetric inequality, or the complements of these classes (even in the case of plane domains). This fact is surprising and important, since it shows that the study of hyperbolic Riemann surfaces is more complicated and interesting than one might think at first sight. One can find results on hyperbolicity of Riemann surfaces in [**27**], [**28**], [**29**] and [**23**].

Here we present the outline of the main results. We refer to later sections for the definitions and the precise statements of the theorems.

In § 3 we obtain some lower bounds on the hyperbolicity constants of metric spaces, which will be useful in § 5. In these theorems we study the role of punctures and funnels (and more general ends) in the hyperbolicity of Riemann surfaces.

The main aim of this paper is to obtain global results on hyperbolicity from local information. That was the idea that led us to identify some of the ends of a surface S^* with closed sets $\{E_n\}_n$ removed from an original surface S, in such a way that $S^* = S \setminus \bigcup_n E_n$.

Theorem 5.14 allows us, in many cases, to study the hyperbolicity of a Riemann surface in terms of the local hyperbolicity of its ends; this fact is a significant simplification in the study of the hyperbolicity. This theorem provides, in fact, a necessary and sufficient condition. Besides, we have determined which are the relevant parameters in the hyperbolicity constant of S^*. Thanks to the theorems on Gromov spaces appearing in § 3, we have obtained this significant improvement of the results in [**23**], since now the topological context is much more general.

Theorem 5.15 allows one, in many cases, to forget punctures and funnels in order to study the hyperbolicity of a Riemann surface; this fact is a significant simplification in the topology of the surface, and therefore makes the problem easier. This theorem also gives a necessary and sufficient condition for hyperbolicity.

Theorem 5.6 is an important tool in the proof of Theorems 5.14 and 5.15. It guarantees the hyperbolicity of surfaces of finite type, with hyperbolicity constants which only depend on the topology of the surface and some metric restrictions. It is important by itself, since it can be also viewed as a result on uniform hyperbolicity and stability of the hyperbolicity of Riemann surfaces.

We also prove two general criteria which guarantee that many surfaces are not hyperbolic (see Theorems 5.2 and 5.3).

Notation

We denote by X or X_n geodesic metric spaces. By d_X and L_X we shall denote, respectively, the distance and the length in the metric of X.

We denote by S or S_i non-exceptional Riemann surfaces. We assume that the metric defined on these surfaces is the Poincaré metric, unless we specify to the contrary.

By $\#A$ we mean the cardinality of the set A. Finally, we denote by c_i, k_i positive constants which can assume different values in different theorems.

2. Background in Gromov spaces

In our study of hyperbolic Gromov spaces we use the notation of [**14**]. We give the basic facts about these spaces below. We refer to [**14**] for more background and further results.

Definition 2.1. Let us fix a point w in a metric space (X, d). We define the *Gromov product* of $x, y \in X$ with respect to the point w as

$$(x \mid y)_w := \tfrac{1}{2}(d(x, w) + d(y, w) - d(x, y)) \geqslant 0.$$

We say that the metric space (X, d) is *δ-hyperbolic* ($\delta \geqslant 0$) if

$$(x \mid z)_w \geqslant \min\{(x \mid y)_w, (y \mid z)_w\} - \delta,$$

for every $x, y, z, w \in X$. We say that X is *hyperbolic* (in the Gromov sense) if the value of δ is not important.

It is convenient to remark that this definition of hyperbolicity is not universally accepted, since sometimes the word hyperbolic refers to negative curvature or to the existence of Green's function. However, in this paper we only use the word *hyperbolic* in the sense of Definition 2.1.

Examples.

(1) Every bounded metric space X is $(\operatorname{diam} X)$-hyperbolic.

(2) Every complete simply connected Riemannian manifold with sectional curvature which is bounded from above by $-k$, with $k > 0$, is hyperbolic.

(3) Every tree with edges of arbitrary length is 0-hyperbolic.

We refer the reader to [**7, 14**] and [**11**] for further examples.

Definition 2.2. If $\gamma : [a, b] \to X$ is a continuous curve in a metric space (X, d), we can define the length of γ as

$$L(\gamma) := \sup\biggl\{\sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) : a = t_0 < t_1 < \cdots < t_n = b\biggr\}.$$

We say that γ is a *geodesic* if it is an isometry, i.e. $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$. We say that X is a *geodesic metric space* if for every $x, y \in X$ there exists a geodesic joining x and y; we denote by $[x, y]$ any of these geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but it is convenient).

https://doi.org/10.1017/S0013091504001555 Published online by Cambridge University Press

Definition 2.3. If X is a geodesic metric space and J is a polygon whose sides are $J_1, J_2, \dots, J_n$, we say that J is δ-*thin* if for every $x \in J_i$ we have that $d(x, \bigcup_{j \neq i} J_j) \leqslant \delta$. If $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of three geodesics $[x_1, x_2]$, $[x_2, x_3]$ and $[x_3, x_1]$. The space X is δ-*thin* (or satisfies the *Rips condition* with constant δ) if every geodesic triangle in X is δ-thin.

A basic result is that hyperbolicity is equivalent to the Rips condition.

Theorem A (see p. 41 in [14]). *Let us consider a geodesic metric space X.*

(1) *If X is δ-hyperbolic, then it is 4δ-thin.*

(2) *If X is δ-thin, then it is 4δ-hyperbolic.*

Now we present the class of maps which plays the main role in the theory.

Definition 2.4. A function between two metric spaces $f : X \to Y$ is a *quasi-isometry* if there are constants $a \geqslant 1$, $b \geqslant 0$ with

$$(1/a) d_X(x_1, x_2) - b \leqslant d_Y(f(x_1), f(x_2)) \leqslant a d_X(x_1, x_2) + b, \quad \text{for every } x_1, x_2 \in X.$$

Such a function is called an (a, b)-*quasi-isometry*. An (a, b)-*quasi-geodesic* in X is an (a, b)-quasi-isometry between an interval of $\mathbb{R}$ and X.

Notice that a quasi-isometry can be discontinuous.

Quasi-isometries are important since they are maps which preserve hyperbolicity.

Theorem B (see p. 88 in [14]). *Let us consider an (a, b)-quasi-isometry between two geodesic metric spaces $f : X \to Y$. If Y is δ-hyperbolic, then X is δ'-hyperbolic, where δ' is a constant which depends only on δ, a and b.*

Definition 2.5. Let us consider $H > 0$, a metric space X, and subsets $Y, Z \subseteq X$. The set $V_H(Y) := \{x \in X : d(x, Y) \leqslant H\}$ is called the *H-neighbourhood* of Y in X. The *Hausdorff distance* of Y to Z is defined by $\mathcal{H}(Y, Z) := \inf\{H > 0 : Y \subseteq V_H(Z),\ Z \subseteq V_H(Y)\}$.

The following is a beautiful and useful result.

Theorem C (see p. 87 in [14]). *For each $\delta, b \geqslant 0$ and $a \geqslant 1$, there exists a constant $H = H(\delta, a, b)$ with the following property.*

Let us consider a δ-hyperbolic geodesic metric space X and an (a, b)-quasi-geodesic g joining x and y. If γ is a geodesic joining x and y, then $\mathcal{H}(g, \gamma) \leqslant H$.

This property is known as geodesic stability. Bonk [**6**] has proved that, in fact, geodesic stability is equivalent to hyperbolicity.

Throughout this paper we will work with topological subspaces of a geodesic metric space X. There is a natural way to define a distance in these spaces.

Definition 2.6. If X_0 is a path-connected subset of a metric space (X, d), then we associate with it the *intrinsic distance*

$$\begin{aligned} d_{X_0}(x, y) &:= d_X|_{X_0}(x, y) \\ &:= \inf\{L(\gamma) : \gamma \subset X_0 \text{ is a continuous curve joining } x \text{ and } y\} \geqslant d_X(x, y). \end{aligned}$$

https://doi.org/10.1017/S0013091504001555 Published online by Cambridge University Press

If X_0 is not path connected, we also use this definition if x and y belong to the same path-connected component of X_0; if x and y belong to distinct path-connected components of X_0, we define $d_{X_0}(x, y) := \infty$.

Definition 2.7. A polygon whose sides are (a, b)-quasi-geodesics is said to be (a, b)-*quasi-geodesic*.

3. Results in metric spaces

We remark that almost every constant appearing in the results of this paper depends on only a small number of parameters.

The following result will be useful in order to check that a geodesic metric space is not hyperbolic (see Theorems 5.2 and 5.3).

Theorem 3.1. *Let us consider a geodesic metric space X, and $X_1, X_2 \subset X$ path-connected closed subspaces such that $X_1 \cup X_2 = X$, $X_1 \cap X_2 = \bigcup_{i\in A} \eta_i$, with $\#A \geqslant 2$, η_i closed sets and $d_{X_2}(\eta_i, \eta_j) \geqslant c$ for every $i, j \in A$, $i \neq j$. Let us also assume that each curve with finite length in X intersects at most finitely many η_i. Then, for each $\varepsilon > 0$ there exists a $(1, \varepsilon)$-quasi-geodesic triangle $T = \{A_0, B_0, C_0\}$ in X and $x \in A_0$ with $d_X(x, B_0 \cup C_0) \geqslant c/4 - \varepsilon$.*

Remarks.

(1) Notice that the condition $d_{X_2}(\eta_i, \eta_j) \geqslant c$ is much less restrictive than $d_X(\eta_i, \eta_j) \geqslant c$, since in the applications we usually know $d_{X_2}(\eta_i, \eta_j)$, but we do not have any lower bound of $d_X(\eta_i, \eta_j)$ at all (see Theorems 5.3, 5.14 and 5.15 and Proposition 5.12).

(2) X_1 and X_2 are required to be closed sets merely in order to guarantee that any curve joining X_1 and X_2 must pass through $X_1 \cap X_2$.

Proof. Let us consider a graph $G := (V, E)$ with vertices $V = \{v_1, v_2\} \cup \{v^i\}_{i\in A}$ and edges $E = \{[v_1, v^i], [v_2, v^i]\}_{i\in A}$, which will be used as a model of the connections between X_1 and X_2: X_1, X_2 are identified with the vertices v_1, v_2, respectively, and each set η_i is identified with v^i, for $i \in A$.

First of all, we define a map F, such that $F(\gamma)$ is a closed curve in G, for each closed curve γ with finite length in X. We define F in the following way.

(1) If γ is a non-closed curve starting and finishing in η_i, with $\gamma \cap (\bigcup_{j\in A\setminus\{i\}} \eta_j) = \varnothing$, then $F(\gamma) := v^i$.

(2) If γ is a non-closed curve starting in η_i and finishing in η_j $(i \neq j)$, γ only intersects $\eta_i \cup \eta_j$ at its endpoints, and $\gamma \cap (\bigcup_{k\in A\setminus\{i,j\}} \eta_k) = \varnothing$, it is clear that this curve is contained in some X_n $(n = 1, 2)$. We then define $F(\gamma) := [v_n, v^i] \cup [v_n, v^j]$.

(3) If γ is a closed curve in $X_n \setminus \bigcup_{i\in A} \eta_i$ $(n = 1, 2)$, we define $F(\gamma) := v_n$. If γ intersects $\bigcup_{i\in A} \eta_i$, then it can be decomposed in a unique way as a finite union of subcurves in (1) and/or (2); then we define $F(\gamma)$ as the union of the image by F of these subcurves (with the appropriate orientation so that $F(\gamma)$ is a continuous closed curve).

https://doi.org/10.1017/S0013091504001555 Published online by Cambridge University Press

Now, we are going to define a class of curves Γ in X: we say that a closed curve $\gamma \in \Gamma$ if and only if $F(\gamma)$ is non-simply connected in the graph G.

Notice that any curve $\gamma \in \Gamma$ satisfies $L(\gamma) \geqslant c$, since γ contains a subcurve joining some η_i and η_j $(i \neq j)$ in X_2: if γ does not contain such a subcurve, then $F(\gamma)$ is contained in $\bigcup_{i \in a}[v_1, v^i]$, which is a simply connected subset of G.

For each $\varepsilon > 0$, let us choose a curve $\gamma_\varepsilon \in \Gamma$ with $L(\gamma_\varepsilon) < \inf_{\gamma \in \Gamma} L(\gamma) + \varepsilon$. We want to prove that any subcurve γ_0 of γ_ε with $L(\gamma_0) \leqslant L(\gamma_\varepsilon)/2$ is a $(1, \varepsilon)$-quasi-geodesic.

In order to do this, we consider two points $p, q \in \gamma_\varepsilon$ and a geodesic g in X joining them. Since γ_ε is a closed curve, we can split it into two different curves γ', γ'' joining p and q, with $\gamma' \cup \gamma'' = \gamma_\varepsilon$. We now prove that $L(g) > \min\{L(\gamma'), L(\gamma'')\} - \varepsilon$. Seeking for a contradiction, suppose that $L(g) \leqslant \min\{L(\gamma'), L(\gamma'')\} - \varepsilon$. Then $L(g \cup \gamma'), L(g \cup \gamma'') \leqslant L(\gamma_\varepsilon) - \varepsilon < \inf_{\gamma \in \Gamma} L(\gamma)$.

Claim. We claim now that at least one of the closed curves $g \cup \gamma'$, $g \cup \gamma''$ belongs to Γ.

If we assume this claim to be true for the moment, then we obtain the required contradiction, since we have a curve of Γ with length less than $\inf_{\gamma \in \Gamma} L(\gamma)$.

Let us consider the arc-length parametrization $\gamma_0 : [0, l] \to X$ of a subcurve of γ_ε with $l = L(\gamma_0) \leqslant L(\gamma_\varepsilon)/2$. By definition of arc-length parametrization we have that $d_X(\gamma_0(t), \gamma_0(s)) \leqslant L(\gamma_0([s, t])) = |t - s|$. Since $l \leqslant L(\gamma_\varepsilon)/2$, we have proved that if g is a geodesic in X joining $\gamma_0(s)$ and $\gamma_0(t)$, then $d_X(\gamma_0(t), \gamma_0(s)) = L(g) > L(\gamma_0([s, t])) - \varepsilon = |t - s| - \varepsilon$. These inequalities guarantee that γ_0 is a $(1, \varepsilon)$-quasi-geodesic.

Let us now choose two points $p_0, q_0 \in \gamma_\varepsilon$ such that we can split γ_ε into two different curves γ', γ'' joining p_0 and q_0, with $\gamma' \cup \gamma'' = \gamma_\varepsilon$ and $L(\gamma') = L(\gamma'') = L(\gamma_\varepsilon)/2$. Consequently, γ' and γ'' are $(1, \varepsilon)$-quasi-geodesics in X, and $\{\gamma', \gamma''\}$ is a $(1, \varepsilon)$-quasi-geodesic triangle in X (it is a triangle since the definition of a triangle allows two vertices to be equal).

We consider the point $x \in \gamma'$ which splits γ' into two curves of equal length $L(\gamma_\varepsilon)/4$. We have that $d_X(x, \gamma'') \geqslant L(\gamma_\varepsilon)/4 - \varepsilon \geqslant c/4 - \varepsilon$.

Let us now prove the claim. Seeking for a contradiction, if both curves are not in Γ, then $F(g \cup \gamma')$, $F(g \cup \gamma'')$ are trivial in the graph G; therefore $F(g \cup \gamma') \cup F(g \cup \gamma'')$ is also trivial. We can construct a homotopy in X, which shows that $[g \cup \gamma'] * [g \cup \gamma''] = [\gamma' \cup \gamma''] = [\gamma_\varepsilon]$ (such homotopy can be a deformation to a single point of the two curves whose graph is g). In a similar way, we can construct a homotopy in G, which shows that $[F(g \cup \gamma')] * [F(g \cup \gamma'')] = [F(\gamma' \cup \gamma'')] = [F(\gamma_\varepsilon)]$ (although the image by F of the homotopy in X is not the homotopy in G). This is a contradiction because $F(\gamma_\varepsilon)$ is trivial in G but $\gamma_\varepsilon \in \Gamma$. □

In order to apply Theorem 3.1, we need the following elementary result (see, for example, [**22**, Lemma 2.16] for a proof).

Lemma A. *For each $\delta, b \geqslant 0$ and $a \geqslant 1$, there exists a constant $K = K(\delta, a, b)$ with the following property.*

If X is a δ-hyperbolic geodesic metric space and $T \subseteq X$ is an (a, b)-quasi-geodesic triangle, then T is K-thin. Furthermore, $K = 4\delta + 2H(\delta, a, b)$, where H is the constant in Theorem C.

Theorem 3.1 and Lemma A give the following result directly.

Theorem 3.2. *Let us consider a geodesic metric space X, and $X_1^n, X_2^n \subset X$ path-connected closed subspaces such that $X_1^n \cup X_2^n = X$, $X_1^n \cap X_2^n = \bigcup_{i \in A^n} \eta_i^n$, with $\# A^n \geqslant 2$, η_i^n closed sets and $d_{X_2^n}(\eta_i^n, \eta_j^n) \geqslant c_n$ for every $i, j \in A^n$, $i \neq j$. Let us also assume that, for each fixed n, each curve with finite length in X intersects at most finitely many η_i^n. If $\limsup_{n\to\infty} c_n = \infty$, then X is not hyperbolic.*

The following elementary result is a direct consequence of Theorem 3.2.

Corollary 3.3. *Let us consider a graph G which is a geodesic metric space, with a sequence of edges $\{e_n\}_n$ such that the graph $G \setminus e_n$ is connected for every n, and $\lim_{n\to\infty} L(e_n) = \infty$. Then G is not hyperbolic.*

In order to prove Proposition 5.12 and Theorems 5.14 and 5.15, we need a result similar to Theorem 3.1, but decomposing the space X in more than two subspaces and replacing condition '$d_{X_2}(\eta_i, \eta_j) \geqslant c$ for every $i, j \in A$, $i \neq j$', by '$d_{X_2}(\eta_i, \eta_j) \geqslant c$ for some $i, j \in A$'; however, some additional requirements are necessary by way of compensation. Next, let us start with an elementary fact.

Lemma 3.4. *Let X be a geodesic metric space, let γ be a geodesic in X and let S be a subset of X. Let us assume that there exists a geodesic η joining γ with S, such that $L(\eta) = d(\gamma, S)$, whose endpoints are $x_1 \in \gamma$ and $x_2 \in S$. Let us choose two arbitrary points, $x_3 \in S$, $x_4 \in \gamma$, and denote by $A := d(x_1, x_4)$, $B := d(x_2, x_3)$ and $C := d(x_3, x_4)$. Then, $A \leqslant B + 2C$.*

Proof. We define $D := d(x_1, x_2) = d(\gamma, S)$. Notice that the condition $D = d(\gamma, S)$ implies $D \leqslant C$. By the triangle inequality it is obvious that $A \leqslant B + C + D \leqslant B + 2C$. $\square$

We are now going to introduce the main result of this section. It will be essential in the proofs of Proposition 5.12 and Theorems 5.14 and 5.15.

Theorem 3.5. *Let X be a geodesic metric space, and let X_1, X_2, X_3 be closed subsets of X, with X_1, X_2 path connected, $X_1 \cup X_2 \cup X_3 = X$, $X_1 \cap X_2 = \bigcup_{i=1}^r \eta_i$ $(r \geqslant 2)$, $X_2 \cap X_3 = \bigcup_{i=r+1}^k \eta_i$, and $X_1 \cap X_3 = \varnothing$, where the η_i are closed sets. Let us assume that there exist two positive constants, c_1, c_2 such that $\operatorname{diam}_{X_1}(\eta_i) \leqslant c_1$ for every $1 \leqslant i \leqslant r$, $\operatorname{diam}_{X_2}(\eta_i) \leqslant c_1$ for every $1 \leqslant i \leqslant k$, and $d_{X_2}(\eta_i, \eta_j) \geqslant c_2$ for some $1 \leqslant i, j \leqslant r$. If X is δ-thin, then $c_2 \leqslant 8(k-1)(c_1/2 + (2k + 2r - 6)\delta + 2H(4\delta, 2, c_1/2))$, where H is the constant in Theorem C.*

Remarks.

(1) The case $X_3 = \varnothing$ is allowed.

(2) The hypothesis $X_1 \cap X_3 = \varnothing$ is not restrictive at all, since if some connected components of X_3 intersect X_1, we can consider these components as if they belonged to X_1.

(3) Since we do not require X_3 to be connected, the conclusion of Theorem 3.5 also holds if we consider $X = X_1 \cup X_2 \cup \cdots \cup X_n$, with $X_2 \cap (X_3 \cup \cdots \cup X_n) = \bigcup_{i=r+1}^k \eta_i$.

https://doi.org/10.1017/S0013091504001555 Published online by Cambridge University Press

Proof. Without loss of generality, we can assume that X_1 and X_2 are geodesic spaces, since, if this is not so, whenever we need a geodesic joining $x, y \in X_i$, for any $\varepsilon > 0$ we can take a curve γ_ε joining them with $L(\gamma_\varepsilon) < d_{X_i}(x, y) + \varepsilon$ (in a similar way as in the proof of Theorem 3.1). As only a finite number of geodesics are employed in the proof, it is still valid, taking ε into account when necessary; afterwards, it is sufficient to make $\varepsilon \to 0$, since the dependence on ε of the constants involved is continuous. Analogously, a geodesic of minimum length in X_2 can be assumed to exist between η_i and η_j, for i, j with $d_{X_2}(\eta_i, \eta_j) \geqslant c_2$.

Let us assume that there exists a $(2, c_1/2)$-quasi-geodesic polygon, with at most $2k + 2r - 4$ sides, that is δ_0-thin with δ_0 the sharpest constant and

$$\delta_0 \geqslant \frac{c_2}{8(k-1)} - \frac{c_1}{2}.$$

Since the space X is δ-thin, it is 4δ-hyperbolic by Theorem A; it can easily be deduced that a $(2, c_1/2)$-quasi-geodesic polygon with at most $2k + 2r - 4$ sides is δ_1-thin, with $\delta_1 = (2k + 2r - 6)\delta + 2H(4\delta, 2, c_1/2)$, where H is the constant in Theorem C. Therefore,

$$\delta_1 \geqslant \delta_0 \geqslant \frac{c_2}{8(k-1)} - \frac{c_1}{2}.$$

Consequently,

$$c_2 \leqslant 8(k-1)(c_1/2 + (2k + 2r - 6)\delta + 2H(4\delta, 2, c_1/2)).$$

To continue, let us construct such a quasi-geodesic polygon: without loss of generality, we can assume that η_1, η_r are the sets such that $d_{X_2}(\eta_1, \eta_r) \geqslant d_{X_2}(\eta_i, \eta_j)$, for every $1 \leqslant i, j \leqslant r$.

We denote by γ_2 a geodesic in X_2 joining η_1 with η_r, such that $L(\gamma_2) = d_{X_2}(\eta_1, \eta_r) \geqslant c_2$; let us assume that γ_2 starts in $a \in \eta_1$ and finishes in $b \in \eta_r$. We denote by γ_1 a geodesic in X_1 joining a and b. Therefore, $\gamma := \gamma_1 \cup \gamma_2$ is a closed curve in X.

Our goal is to construct a quasi-geodesic polygon contained in γ, where a and b are two of its vertices. We will choose the other vertices in two consecutive steps.

First step. We denote by σ_i^1 a geodesic of minimum length in X_1 between η_i and γ_1, with $2 \leqslant i \leqslant r-1$ (such a geodesic exists there since η_i is closed and γ_1 is compact, and X_1 is a geodesic space), and by σ_i^2 a geodesic of minimum length in X_2 between η_i and γ_2, with $2 \leqslant i \leqslant k$ and $i \neq r$. We define $x_i^j := \sigma_i^j \cap \eta_i$ and $y_i^j := \sigma_i^j \cap \gamma_j$ for every $2 \leqslant i \leqslant r-1$ if $j = 1$, and for every $2 \leqslant i \leqslant k$, $i \neq r$ if $j = 2$.

We take as a vertex the point y_i^j for every $2 \leqslant i \leqslant r-1$ if $j = 1$, and for every $2 \leqslant i \leqslant k$, $i \neq r$ if $j = 2$ (we can define $y_1^j := \sigma_1^j := a$ and $y_r^j := \sigma_r^j := b$).

Second step. Between every two consecutive vertices described in the previous step, we consider as a new vertex its middle point in γ.

Now, we are going to prove that this polygon, with at most $2k + 2r - 4$ sides, is $(2, c_1/2)$-quasi-geodesic: let α, β be points in the same side L_1 of the polygon. Without loss of generality, we can assume that $L_1 \subset X_2$, since the other case is similar. Notice

https://doi.org/10.1017/S0013091504001555 Published online by Cambridge University Press

that, by the construction of the polygon, there is an adjacent side, $L_2 \subset X_2$, such that $L(L_2) = L(L_1) =: l$ and $L_1 \cap L_2$ is one of the vertices chosen on the second step. Let g be a geodesic in X, joining α and β such that $g(0) = \alpha$, $g(T) = \beta$.

It is clear that $T := d_X(\alpha, \beta) \leqslant d_{X_2}(\alpha, \beta)$.

Let us now prove the other inequality. Let us suppose that g intersects $\eta_{i_1}, \eta_{i_2}, \dots, \eta_{i_s}$, in this order. Then we can define $t_0 := \max\{t \in [0,T] : g(t) \in \eta_{i_s}\}$, since η_{i_s} is a closed set.

Without loss of generality, we can assume that

$$d_{X_2}(\beta, L_2) = \min\{d_{X_2}(\alpha, L_2), d_{X_2}(\beta, L_2)\};$$

we define $\beta' := g(t_0) \in \eta_{i_s}$. It is clear that $d_{X_2}(\alpha, \beta) \leqslant d_{X_2}(\beta, y^2_{i_s})$, since it is not possible to have $y^2_{i_s} = L_1 \cap L_2$ (recall that $L_1 \cap L_2$ is one of the vertices chosen in the second step).

We construct the quadrilateral in X_2 with vertices β, $y^2_{i_s}$, $x^2_{i_s}$ and β', and sides $g|_{[\beta,\beta']}$, $\gamma_2|_{[\beta,y^2_{i_s}]}$, $\sigma^2_{i_s}$ and $[\beta', x^2_{i_s}]$ (a geodesic in X_2). Applying Lemma 3.4, where

$$A := d_{X_2}(\beta, y^2_{i_s}), \quad B := d_{X_2}(\beta', x^2_{i_s}) \quad \text{and} \quad C := d_{X_2}(\beta, \beta') = d_X(\beta, \beta') \leqslant d_X(\alpha, \beta),$$

we have

$$d_{X_2}(\alpha, \beta) \leqslant d_{X_2}(\beta, y^2_{i_s}) = A \leqslant B + 2C \leqslant 2d_X(\alpha, \beta) + c_1.$$

Consequently, we obtain

$$\tfrac{1}{2} d_{X_2}(\alpha, \beta) - \tfrac{1}{2} c_1 \leqslant d_X(\alpha, \beta) \leqslant d_{X_2}(\alpha, \beta),$$

and we have proved that our polygon is, actually, $(2, c_1/2)$-quasi-geodesic. Let us now show that it is δ_0-thin, with δ_0 the sharpest constant and

$$\delta_0 \geqslant \frac{c_2}{8(k-1)} - \frac{c_1}{2}.$$

As there are at most $2k-2$ sides of the polygon in X_2, there exist at least two adjacent sides in X_2 whose length is greater than or equal to $c_2/(2k-2)$. Let us choose one of them, and name its vertices v_1 and v_2. Let p be the middle point between them in γ_2, and let S be the union of the rest of the sides of the polygon. Our current aim is to estimate $d_X(p, S)$.

Let g be a geodesic in X such that $L(g) = d_X(p, S)$. There are two possibilities.

(1) If g is contained in X_2, then

$$d_X(p, S) = d_{X_2}(p, S) = d_{X_2}(p, \{v_1, v_2\}) \geqslant \frac{c_2}{4(k-1)}.$$

(2) If g is not contained in X_2, the first time g gets out of X_2 is through some η_i, $1 \leqslant i \leqslant k$, at a certain point q. Notice that it is not possible that $i \in \{1, r\}$, since γ_2 is a minimizing geodesic between η_1 and η_r. Let us define a quadrilateral with vertices p, q, x^2_i, y^2_i (where x^2_i and y^2_i are the endpoints of σ^2_i, defined at the

https://doi.org/10.1017/S0013091504001555 Published online by Cambridge University Press

beginning of the proof). The sides of this polygon are $\gamma_2|_{[p,y_i^2]}$, σ_i^2, $g|_{[p,q]}$ and $[x_i^2, q]$ (a geodesic in X_2). Applying Lemma 3.4, where $A := d_{X_2}(p, y_i^2)$, $B := d_{X_2}(x_i^2, q)$ and $C := d_{X_2}(p, q) = d_X(p, q) \leqslant d_X(p, S)$, it can be deduced that

$$\frac{c_2}{4(k-1)} \leqslant d_{X_2}(p, \{v_1, v_2\}) \leqslant d_{X_2}(p, y_i^2) = A \leqslant B + 2C \leqslant 2d_X(p, S) + c_1.$$

Consequently,

$$\delta_0 \geqslant d_X(p, S) \geqslant \frac{c_2}{8(k-1)} - \frac{c_1}{2}.$$

□

Theorem 3.5 and Lemma A imply the following result.

Theorem 3.6. *Let X be a geodesic metric space, and let X_1^n, X_2^n, X_3^n be closed subsets of X, with X_1^n, X_2^n path connected, $X_1^n \cup X_2^n \cup X_3^n = X$, $X_1^n \cap X_2^n = \bigcup_{i=1}^{r_n} \eta_i^n$ $(r_n \geqslant 2)$, $X_2^n \cap X_3^n = \bigcup_{i=r_n+1}^{k_n} \eta_i^n$ and $X_1^n \cap X_3^n = \varnothing$, where the η_i^n are closed sets. Let us assume that there exist positive constants c_1, c_2^n such that $\operatorname{diam}_{X_1^n}(\eta_i^n) \leqslant c_1$ for every $1 \leqslant i \leqslant r_n$, $\operatorname{diam}_{X_2^n}(\eta_i^n) \leqslant c_1$ for every $1 \leqslant i \leqslant k_n$, and $d_{X_2^n}(\eta_i^n, \eta_j^n) \geqslant c_2^n$ for some $1 \leqslant i, j \leqslant r_n$. If $k_n \leqslant k$ and $\limsup_{n\to\infty} c_2^n = \infty$, then X is not hyperbolic.*

We finish this section with one theorem that will be very useful in the proof of the main results of this paper. In order to state them, we need a definition.

Definition 3.7. We say that a geodesic metric space X has a *decomposition* if there exists a family of geodesic metric spaces $\{X_n\}_{n\in\Lambda}$ with $X = \bigcup_{n\in\Lambda} X_n$ and $X_n \cap X_m = \sigma_{nm}$, where, for each $n \in \Lambda$, $\{\sigma_{nm}\}_m$ are pairwise disjoint closed subsets of X_n ($\sigma_{nm} = \varnothing$ is allowed); furthermore, any geodesic in X with finite length meets at most a finite number of σ_{nm}.

We say that X_n, with $n \in \Lambda$, is a (k_1, k_2, k_3)-*tree-piece* if it satisfies the following properties.

(a) If $\sigma_{nm} \neq \varnothing$, then $X \setminus \sigma_{nm}$ is not connected and a, b are in different connected components of $X \setminus \sigma_{nm}$ for any $a \in X_n \setminus \sigma_{nm}$, $b \in X_m \setminus \sigma_{nm}$.

(b) $\operatorname{diam}_{X_n}(\sigma_{nm}) \leqslant k_1$ for every $m \neq n$, and there exists $A_n \subseteq \Lambda$ such that $\operatorname{diam}_{X_n}(\sigma_{nm}) \leqslant k_2 d_{X_n}(\sigma_{nm}, \sigma_{nk})$ if $m \neq k$ and $m, k \in A_n$, and

$$\sum_{m\notin A_n} \operatorname{diam}_{X_n}(\sigma_{nm}) \leqslant k_3.$$

We say that a geodesic metric space X has a (k_1, k_2, k_3)-*tree-decomposition* if it has a decomposition such that every X_n, with $n \in \Lambda$, is a (k_1, k_2, k_3)-tree-piece.

We wish to emphasize that condition $\operatorname{diam}_{X_n}(\sigma_{nm}) \leqslant k_1$ is not very restrictive: if the space is 'wide' at every point (in the sense of long injectivity radius, as in the case of simply connected spaces) or 'narrow' at every point (as in the case of trees), it is easier to study its hyperbolicity; if we can find narrow parts (such as σ_{nm}) and wide parts, the problem is more difficult and interesting.

Remarks.

(1) Obviously, condition (b) is required only for $\sigma_{nm}, \sigma_{nk} \neq \varnothing$.

(2) The sets Λ and A_n do not need to be countable.

(3) The hypothesis $\operatorname{diam}_{X_n}(\sigma_{nm}) \leqslant k_2 d_{X_n}(\sigma_{nm}, \sigma_{nk})$ holds if we have $d_{X_n}(\sigma_{nm}, \sigma_{nk}) \geqslant k_2'$, since $\operatorname{diam}_{X_n}(\sigma_{nm}) \leqslant k_1$.

(4) Condition (a) for every $n \in \Lambda$ guarantees that the graph $R = (V, E)$ constructed in the following way is a tree: $V = \bigcup_{n\in\Lambda}\{v_n\}$ and $[v_n, v_m] \in E$ if and only if $\sigma_{nm} \neq \varnothing$.

(5) If X is a Riemann surface, $\{X_n\}_{n\in\Lambda}$ are bordered Riemann surfaces and $\sigma_{nm} \subset \partial X_n \cap \partial X_m$, then the condition '$a$, b are in different components of $X \setminus \sigma_{nm}$ for any $a \in X_n \setminus \sigma_{nm}$, $b \in X_m \setminus \sigma_{nm}$' in (a) is a consequence of '$X \setminus \sigma_{nm}$ is not connected'.

The following result can be applied to the study of the hyperbolicity of Riemann surfaces (see the proofs of Theorems 5.6 and 5.14). Explicit expressions for the constants involved are supplied in [**22**].

Theorem D (see Theorem 2.9 in [22]). *Let us consider a (k_1, k_2, k_3)-tree-decomposition $\{X_n\}_{n\in\Lambda}$ of a geodesic metric space X. Then X is δ-hyperbolic if and only if there exists a constant k_4 such that X_n is k_4-hyperbolic for every $n \in \Lambda$. Furthermore, if X is δ-hyperbolic, then k_4 depends only on δ, k_1, k_2 and k_3; if k_4 exists, then δ depends only on k_1, k_2, k_3 and k_4.*

4. Background in Riemann surfaces

In both this section and the next one we always work with the Poincaré metric; consequently, curvature is always -1. In fact, many concepts appearing here (as punctures or funnels) only make sense with the Poincaré metric.

Below we collect some definitions concerning Riemann surfaces which will be referred to afterwards.

An *open non-exceptional* Riemann surface S (or a non-exceptional Riemann surface without boundary) is a Riemann surface whose universal covering space is the unit disc $\boldsymbol{D} = \{z \in \mathbb{C} : |z| < 1\}$, endowed with its Poincaré metric, i.e. the metric obtained by projecting the Poincaré metric of the unit disc $\mathrm{d}s = 2|\mathrm{d}z|/(1 - |z|^2)$ or, equivalently, the upper half plane $\boldsymbol{U} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, with the metric $\mathrm{d}s = |\mathrm{d}z|/\operatorname{Im} z$. Notice that, with this definition, every compact non-exceptional Riemann surface without boundary is open. With this metric, S is a geodesically complete Riemannian manifold with constant curvature -1, and therefore S is a geodesic metric space. The only Riemann surfaces which are left out are the sphere, the plane, the punctured plane and the tori. It is easy to study the hyperbolicity of these particular cases.

Let S be an open non-exceptional Riemann surface with a puncture q (if $S \subset \mathbb{C}$, every isolated point in ∂S is a puncture). A *collar* in S about q is a doubly connected domain in S 'bounded' by both q and a Jordan curve (called the boundary curve of the collar) orthogonal to the pencil of geodesics emanating from q.

We have used the word *geodesic* in the sense of Definition 2.2, that is to say, as a global geodesic or a minimizing geodesic; however, we now need to deal with a special type of local geodesics: simple closed geodesics, which obviously cannot be minimizing geodesics. We will continue using the word geodesic with the meaning of Definition 2.2, unless we are dealing with closed geodesics.

A *collar* in S about a simple closed geodesic γ is a doubly connected domain in S 'bounded' by two Jordan curves (called the boundary curves of the collar) orthogonal to the pencil of geodesics emanating from γ; such a collar is equal to $\{p \in S : d_S(p, \gamma) < d\}$, for some positive constant d. The constant d is called the *width* of the collar. The 'collar lemma' [**24**] says that there exists a collar of γ of width d, for every $0 < d \leqslant d_0$, where $\cosh d_0 = \coth(L_S(\gamma)/2)$ (see also [**8**, Chapter 4]).

We say that S is a *bordered non-exceptional Riemann surface* (or a non-exceptional Riemann surface with boundary) if it can be obtained by deleting an open set V from an open non-exceptional Riemann surface R, such that

(1) S is connected and $d_S := d_R|_S$ (recall Definition 2.6),

(2) any ball in R intersects at most a finite number of connected components of V,

(3) the boundary of S is locally Lipschitz.

Any such surface S is a bordered orientable Riemannian manifold of dimension 2 and its Riemannian metric has constant negative curvature -1. It is not difficult to see that S is a geodesic metric space.

A *funnel* is a bordered non-exceptional Riemann surface which is topologically a cylinder and whose boundary is a simple closed geodesic. Given a positive number a, there is a unique (up to conformal mapping) funnel such that its boundary curve has length a. Every funnel is conformally equivalent, for some $\beta > 1$, to the subset $\{z \in \mathbb{C} : 1 \leqslant |z| < \beta\}$ of the annulus $\{z \in \mathbb{C} : 1/\beta < |z| < \beta\}$. In fact, we can obtain any annulus by pasting two isometric funnels.

Every doubly connected end of an open non-exceptional Riemann surface is a puncture (if there are homotopically non-trivial curves with arbitrary small length) or a funnel (if this is not so).

A *Y-piece* is a bordered non-exceptional Riemann surface that is conformally equivalent to a sphere without three open discs and whose boundary curves are simple closed geodesics. Given three positive numbers a, b, c, there is a unique (up to conformal mapping) Y-piece such that their boundary curves have lengths a, b, c (see, for example, [**8**, p. 109]). They are a standard tool for constructing Riemann surfaces. A clear description of these Y-pieces and their use is given in [**10**, Chapter X.3] and [**8**, Chapter 3].

A *generalized Y-piece* is a non-exceptional Riemann surface (with or without boundary) which is conformally equivalent to a sphere without n open discs and m points, where n and m are non-negative integers such that $n + m = 3$. Besides, the n boundary curves are simple closed geodesics and the m deleted points are punctures. Note that a generalized Y-piece is topologically the union of a Y-piece and m cylinders, with $0 \leqslant m \leqslant 3$.

If we delete an open set U from a non-exceptional Riemann surface S, we consider $S \setminus U$ as a bordered non-exceptional Riemann surface, with $d_{S\setminus U} = d_S|_{S\setminus U}$.

If we delete a closed set E from an open non-exceptional Riemann surface S, we consider $S \setminus E$ as an open non-exceptional Riemann surface as well, with its (own) Poincaré metric; consequently, $d_{S\setminus E} \neq d_S|_{S\setminus E}$, since $(S \setminus E, d_{S\setminus E})$ is geodesically complete.

5. Results in Riemann surfaces

Intuition tells us that negative curvature in Riemann surfaces must imply hyperbolicity; in fact this is what happens when there are no topological 'obstacles' (as in the case of the Poincaré disc $\boldsymbol{D}$) or if there are a finite number of them (see Theorems E and 5.6 below). However, if there are infinitely many topological 'obstacles', hyperbolicity can fail, as in the case of the two-dimensional jungle gym (a $\mathbb{Z}^2$-covering of a torus with genus two).

The results in this section are useful since they not only provide many examples of hyperbolic Riemann surfaces, but also allow us to establish criteria for deciding whether a Riemann surface is hyperbolic or not.

Definition 5.1. If c is a positive constant, we say that an open non-exceptional Riemann surface S has *c-wide genus* if every simple closed geodesic $\gamma \subset S$ such that $S \setminus \gamma$ is connected verifies $L_S(\gamma) \geqslant c$. We say that S has *narrow genus* if there is not a $c > 0$ such that S has c-wide genus.

The two following general criteria guarantee that many surfaces are not hyperbolic.

Theorem 5.2. *Let us consider an open non-exceptional Riemann surface S, a closed set E in S with $S \setminus E$ path connected, and $X_1^n, X_2^n \subset S$ bordered surfaces such that $X_i^n \setminus E$ is path connected, $X_1^n \cap X_2^n = \partial X_1^n \cap \partial X_2^n = \bigcup_{i\in A^n} \eta_i^n$, $\#A^n \geqslant 2$, and $d_{X_2^n}(\eta_i^n, \eta_j^n) \geqslant c_n$ for every $i, j \in A^n$, $i \neq j$. If $\limsup_{n\to\infty} c_n = \infty$, then S and $S \setminus E$ are not hyperbolic.*

Proof. It is clear that S is not hyperbolic, as a consequence of Theorem 3.2 (recall that, for each fixed n, any ball intersects at most a finite number of η_i^n, by the definition of bordered non-exceptional Riemann surface, and then each η_i^n is a closed set). In order to apply Theorem 3.2 to $\tilde{X} = S \setminus E$, let us define $\tilde{X}_i^n = X_i^n \setminus E$ and $\tilde{\eta}_i^n = \eta_i^n \setminus E$ (which is a closed set in $S \setminus E$). It is well known (see, for example, Lemma B below) that if γ is a curve in $S \setminus E$, then $L_{S\setminus E}(\gamma) \geqslant L_S(\gamma)$; since every curve in $\tilde{X}_i^n$ is contained in X_i^n, it follows that

$$d_{\tilde{X}_2^n}(\tilde{\eta}_i^n, \tilde{\eta}_j^n) \geqslant d_{X_2^n}(\tilde{\eta}_i^n, \tilde{\eta}_j^n) \geqslant d_{X_2^n}(\eta_i^n, \eta_j^n) \geqslant c_n$$

for every $i, j \in A^n$, $i \neq j$. Then Theorem 3.2 implies that $S \setminus E$ is not hyperbolic. □

Theorem 5.3. *Let us consider an open non-exceptional Riemann surface S with narrow genus and a closed set E in S, with $S \setminus E$ path connected and $\Pi_1(S) \leqslant \Pi_1(S \setminus E)$. Then S and $S \setminus E$ are not hyperbolic.*

Proof. Since S has narrow genus, we can choose a sequence of simple closed geodesics $\{\gamma_n\}_n$ in S with $S \setminus \gamma_n$ connected and $\lim_{n\to\infty} L_S(\gamma_n) = 0$.

The 'collar lemma' [**24**] says that there exists a collar of γ_n of width d, for every $0 < d \leqslant d_n$, where $\cosh d_n = \coth(L_S(\gamma_n)/2)$. We define X_2^n to be the collar of γ_n of width $d_n/2$, X_1^n as the closure in S of $S \setminus X_2^n$, $A^n = \{1, 2\}$ and η_1^n, η_2^n the connected components of $X_1^n \cap X_2^n$.

Then $d_{X_2^n}(\eta_1^n, \eta_2^n) = d_n \to \infty$, and consequently S is not hyperbolic by Theorem 3.2 (recall that A^n has just two elements).

In order to study $\tilde{X} = S \setminus E$, let us consider for each n a simple closed curve g_n in S transversal to γ_n. Since $\varPi_1(S) \leqslant \varPi_1(S \setminus E)$, we can assume that $g_n \subset S \setminus E$, and even that g_n is a simple closed geodesic in $S \setminus E$. We denote by h_n a segment of g_n joining η_1^n and η_2^n in X_2^n. Let us define $\tilde{X}_2^n$ as the connected component of $X_2^n \setminus E$ containing h_n, $\tilde{X}_1^n$ as the closure in $\tilde{X}$ of $\tilde{X} \setminus \tilde{X}_2^n$, $\tilde{A}^n = \{1, 2\}$ and $\tilde{\eta}_1^n$, $\tilde{\eta}_2^n$, the connected components of $\tilde{X}_1^n \cap \tilde{X}_2^n$. Since

$$d_{\tilde{X}_2^n}(\tilde{\eta}_1^n, \tilde{\eta}_2^n) \geqslant d_{X_2^n}(\tilde{\eta}_1^n, \tilde{\eta}_2^n) \geqslant d_{X_2^n}(\eta_1^n, \eta_2^n) = d_n,$$

Theorem 3.2 allows us to conclude that $S \setminus E$ is not hyperbolic. □

We say that a Riemann surface is doubly connected if its fundamental group is isomorphic to $\mathbb{Z}$.

Definition 5.4. Let us consider a non-exceptional Riemann surface S of finite type (with or without boundary); if S is bordered, we also require that the components of ∂S with infinite length are local geodesics. An *outer loop* in S is a simple closed geodesic which is either the boundary curve of a funnel or freely homotopic to some component of ∂S. A *generalized funnel* in S is a doubly connected Riemann surface isometric to a subset of an annulus, whose boundary is a simple closed curve. A *generalized outer loop* in S is a simple closed geodesic in S which is either the boundary curve of a generalized funnel or freely homotopic to some component of ∂S. The *characteristic* of S is $a = 2g - 2 + n$, where g is the genus of S and n is the sum of the number of punctures of S and the number of generalized outer loops of S.

Remark. If γ is a closed curve not freely homotopic either to a point or to the boundary of a collar of a puncture, it is well known that there exists a unique simple closed geodesic in the free homotopy class of γ in S.

Notice that if S does not have a boundary, then every generalized outer loop in S is an outer loop.

Definition 5.5. We denote by $\mathcal{S}(a, l)$ the set of non-exceptional Riemann surfaces of finite type S verifying the following properties: if S is bordered, then the components of ∂S with infinite length are local geodesics, S has characteristic less than or equal to a and no genus, and every generalized outer loop has length less than or equal to l. We denote by $\mathcal{S}_{\mathrm{G}}(a, l)$ the set of Riemann surfaces $S \in \mathcal{S}(a, l)$ verifying the following additional property: if S is bordered, then ∂S is the union of local geodesics (closed or non-closed).

We need the following result.

Theorem E (see Theorem 3.4 in [29]). *For each $l \geqslant 0$ and each non-negative integer a, there exists a constant $\delta = \delta(a, l)$, which depends only on a and l, such that every surface in $\mathcal{S}_{\mathrm{G}}(a, l)$ is δ-hyperbolic.*

The hyperbolicity constants of Riemann surfaces in $\mathcal{S}(a, l)$ can be uniformly bounded by means of the following result. This theorem can also be viewed as a result on stability of the hyperbolicity of Riemann surfaces. Theorem 5.6 plays a fundamental role in the proofs of Theorems 5.14 and 5.15.

Theorem 5.6. *For each $l \geqslant 0$ and each non-negative integer a, there exists a constant $\delta = \delta(a, l)$, which depends only on a and l, such that every surface in $\mathcal{S}(a, l)$ is δ-hyperbolic.*

Proof. The idea of the proof is to see a surface in $\mathcal{S}(a, l)$ as a subset of a surface in $\mathcal{S}_{\mathrm{G}}(a, l)$, and then to check that we can apply Theorem D. Let us consider $S \in \mathcal{S}(a, l)$ and R_0 an open non-exceptional Riemann surface with $S \subseteq R_0$.

If there is no simple closed geodesic in R_0 that is freely homotopic to some closed curve in ∂S, then the fundamental group of R_0 is isomorphic to some subgroup of the fundamental group of S (every closed curve in ∂S is either trivial in R_0 or homotopic to a puncture in R_0). In this case we define $R := R_0$.

If this is not so, we denote by $\gamma_1, \dots, \gamma_k$ the simple closed geodesics in R_0 that are freely homotopic to some closed curve in ∂S. If we cut R_0 along $\gamma_1, \dots, \gamma_k$, we obtain bordered surfaces $R_0^1, \dots, R_0^m$. We see that the fundamental group $\Pi_1(S \cap R_0^j)$ may have at most one generator, for at most one value of j. Then we can assume that the fundamental group $\Pi_1(S \cap R_0^j)$ has at most one generator for $j = 2, \dots, m$, and that $\Pi_1(S \cap R_0^1)$ is not trivial. Then $\gamma_1, \dots, \gamma_k$ are the simple closed geodesics in ∂R_0^1; let us consider funnels $F^1, \dots, F^k$, with $L(\partial F^j) = L(\gamma_j)$ for $j = 1, \dots, k$. If we paste $F^1, \dots, F^k$ to R_0^1, we obtain an open non-exceptional Riemann surface R.

In any case, we can see S as a subset of R; we also have that the fundamental group of R is isomorphic to some subgroup of the fundamental group of S (some closed curves in ∂S can be trivial in R); therefore $R \in \mathcal{S}_{\mathrm{G}}(a, l)$, and the closure of $R \setminus S$ is the union of simply or doubly connected bordered surfaces $R^1, \dots, R^s$, with $s \leqslant a + 2$ (some R^i can be a neighbourhood of a puncture). We have that $\delta(R) \leqslant \delta_1(a, l)$ by Theorem E and, hence, Theorem D allows us to obtain that $\delta(S) \leqslant \delta(a, l)$, since $L(\partial S) \leqslant (a+2)l$ implies that $\{S, R^1, \dots, R^s\}$ is a $(l, 0, (a+2)l)$-tree-decomposition of R (taking $A_n = \varnothing$). □

We say that a Riemann surface is triply connected if it has characteristic 1 and genus 0, or, equivalently, if its fundamental group is generated by two disjoint simple closed curves.

We need the following results in order to prove our next theorem.

Theorem F (see Proposition 3.2 in [23]). *Let S be a triply connected bordered non-exceptional Riemann surface. Let us assume that ∂S is the union of two simple closed curves verifying $L_S(\partial S) \leqslant l$. Then S is δ-hyperbolic, where δ is a constant which depends only on l.*

The arguments in the proof of Theorem 5.6 (using Theorems E and F) allow us to deduce the following result.

https://doi.org/10.1017/S0013091504001555 Published online by Cambridge University Press

Lemma 5.7. *Let S be a triply connected non-exceptional Riemann surface (with or without boundary). Let us assume that there are two generalized outer loops in S with length less than or equal to l. Then S is δ-hyperbolic, where δ is a constant which depends only on l.*

Remark. In Theorem F and Lemma 5.7 we can see a puncture as an outer loop with zero length.

Lemma B (see Lemma 3.1 in [2]). *Let us consider an open non-exceptional Riemann surface S, a closed non-empty subset C of S, and a positive number ε. If $S^* := S \backslash C$, then we have that $1 < L_{S^*}(\gamma)/L_S(\gamma) < \coth(\varepsilon/2)$ for every curve $\gamma \subset S$ with finite length in S such that $d_S(\gamma, C) \geqslant \varepsilon$.*

Definition 5.8. Given a doubly connected domain D in a non-exceptional Riemann surface, there exists $0 \leqslant \mu < 1$ such that $\{z : \mu < |z| < 1\}$ is conformally equivalent to D. We define the *modulus* of D as $\operatorname{mod} D := (1/2\pi)\log(1/\mu)$.

Remark. The modulus of a doubly connected domain D can be defined in terms of extremal length (see [**1**, p. 224]). It is well known that the simple closed geodesic in D (with respect to the Poincaré metric in D) has length $\pi/\operatorname{mod} D$.

Definition 5.9. An *N-normal neighbourhood* of a subset F of a Riemann surface S is a bordered Riemann surface V such that $F \subset V \subset S$, verifying either

(i) V is compact and ∂V is the union of n closed curves ($1 \leqslant n \leqslant N$), which generate the fundamental group of V, or

(ii) V is homeomorphic to a funnel (then V is isometric to a non-compact subset of an annulus or of the punctured disc $\boldsymbol{D}^*$; recall that a collar of a puncture is homeomorphic to a funnel).

A set $E = \bigcup_n E_n$ in an open non-exceptional Riemann surface S, where $\{E_n\}_n$ are compact sets, is called (r, s, N)-*uniformly separated* in S if there exist N-normal neighbourhoods $\{V_n\}_n$ of E_n such that $V_n \setminus E_n$ is connected, $d_S(\partial V_n, E_n) \geqslant r$, $L_S(\partial V_n) \leqslant s$ for every n, and $d_S(V_n, V_m) \geqslant r$ for every $n \neq m$.

$E = \bigcup_n E_n$ is called (r, s, t, N)-*uniformly separated* in S if it is (r, s, N)-uniformly separated in S, E_n is simply connected for every n, and it verifies the following property: if V_n is isometric to a non-compact subset of an annulus or if ∂V_n contains at least three closed curves, then there exists a simply connected domain D_n in S, with $E_n \subset D_n$ and $\operatorname{mod}(D_n \setminus E_n) \geqslant t$.

Remarks 5.10.

(1) If E is (r, s, t, N)-uniformly separated, each E_n is simply connected and then it creates a puncture (if E_n is a single point) or a funnel (otherwise) in S^*. Although this is an important case for us, let us observe that we also deal with general compact sets E_n if E is (r, s, N)-uniformly separated.

https://doi.org/10.1017/S0013091504001555 Published online by Cambridge University Press

(2) Notice that an N-normal neighbourhood has genus 0, and, consequently, $V_n \in \mathcal{S}(\max\{N-2,1\},s) \subseteq \mathcal{S}(N,s)$ if E is $(0,s,N)$-uniformly separated in S.

(3) Note that we do not require $D_n \subset V_n$.

(4) If E_n is a single point, we have $\operatorname{mod}(D_n \setminus E_n) = \infty > t$, for any choice of D_n and t. If V_n is compact and ∂V_n is the union of one or two closed curves, or if V_n is isometric to a non-compact subset of the punctured disc $\boldsymbol{D}^*$, then there is no condition on E_n about modulus.

The uniformly separated sets play a central role in the study of hyperbolic isoperimetric inequalities in open Riemann surfaces (see [**2**, Theorem 1] and [**12**, Theorems 3 and 4]), and in other topics in complex analysis, such as harmonic measure (see [**21**]). Interesting relationships exist between the hyperbolic isoperimetric inequality and other conformal invariants of a Riemann surface (see, for example, [**2**], [**10**, p. 95], [**12**] and [**31**, p. 333]).

We need the following definition in order to state one of our main theorems.

Definition 5.11. Let S be an open non-exceptional Riemann surface and let $E = \bigcup_n E_n$ be an (r,s,N)-uniformly separated set in S. For each choice of $\{V_n\}_n$ we define

$$\begin{aligned} D_S &= D_S(\{V_n\}_n) \\ &:= \sup_{n,i,j}\{d_S|_{V_n}(\eta_i^n,\eta_j^n) : \eta_i^n,\ \eta_j^n \text{ are different connected components of } \partial V_n \text{ and} \\ &\qquad\qquad \eta_i^n,\ \eta_j^n \text{ are in the same connected component of } S \setminus \operatorname{int} V_n\}, \\ D_{S^*} &= D_{S^*}(\{V_n\}_n) \\ &:= \sup_{n,i,j}\{d_{S^*}|_{V_n\setminus E_n}(\eta_i^n,\eta_j^n) : \eta_i^n,\ \eta_j^n \text{ are different connected components of } \partial V_n \text{ and} \\ &\qquad\qquad \eta_i^n,\ \eta_j^n \text{ are in the same connected component of } S \setminus \operatorname{int} V_n\}. \end{aligned}$$

Proposition 5.12. *Let S be an open non-exceptional Riemann surface and let $E = \bigcup_n E_n$ be an (r,s,N)-uniformly separated set in S. Let us also assume that we can choose the sets $\{V_n\}_n$ such that $D_S(\{V_n\}_n) = \infty$ (respectively, $D_{S^*}(\{V_n\}_n) = \infty$). Then S (respectively, S^*) is not hyperbolic.*

Remark. The conclusion 'S is not hyperbolic' is also true if $E = \bigcup_n E_n$ is a $(0,s,N)$-uniformly separated set in S; in fact, in this part of the proof we do not use the set E at all.

Proof. Let us assume that $D_S = \infty$. For each V_n we consider the connected components $\{\eta_i^n\}_i$ of ∂V_n. By hypothesis, there exist n_k, i_k, j_k such that

$$\lim_{k\to\infty} d_{V_{n_k}}(\eta_{i_k}^{n_k},\eta_{j_k}^{n_k}) = \infty,$$

with $\eta_{i_k}^{n_k},\eta_{j_k}^{n_k}$ in the same connected component of $S \setminus \operatorname{int} V_{n_k}$.

Let us define $X_2^k := V_{n_k}$, X_1^k as the connected component of $S \setminus \operatorname{int} V_{n_k}$ containing $\eta_{i_k}^{n_k} \cup \eta_{j_k}^{n_k}$ and X_3^k as the union of the other components of $S \setminus \operatorname{int} V_{n_k}$ (if there are any).

https://doi.org/10.1017/S0013091504001555 Published online by Cambridge University Press

Since there are at most N terms in the union of i in $\{\eta_i^n\}_i$ and $\sum_i L_S(\eta_i^n) = L_S(\partial V_n) \leqslant s$, Theorem 3.6 guarantees that S is not hyperbolic.

If $D_{S^*} = \infty$, we obtain a similar result for S^*, since $d_S(\partial V_n, E_n) \geqslant r$ and Lemma B imply the inequality $L_{S^*}(\partial V_n) \leqslant s \coth(r/2)$. □

Since $D_S(\{V_n\}_n) \leqslant D_{S^*}(\{V_n\}_n)$, we deduce the following result.

Corollary 5.13. *Let S be an open non-exceptional Riemann surface and let $E = \bigcup_n E_n$ be an (r, s, N)-uniformly separated set in S. Let us also assume that we can choose the sets $\{V_n\}_n$ such that $D_S(\{V_n\}_n) = \infty$. Then S and S^* are not hyperbolic.*

Next we will state the main result of the paper. It allows one, in many cases, to study the hyperbolicity of a Riemann surface in terms of the local hyperbolicity of its ends; this fact is a significant simplification in the study of the hyperbolicity. Besides, we have determined which are the relevant parameters in the hyperbolicity constants.

Theorem 5.14. *Let S be an open non-exceptional Riemann surface and let $E = \bigcup_n E_n$ be an (r, s, N)-uniformly separated set in S. Then, $S^* := S \setminus E$ is δ^*-hyperbolic if and only if S is δ-hyperbolic, $D_{S^*}(\{V_n\}_n)$ is finite and $V_n \setminus E_n$ is k-hyperbolic for every n (with $d_{S^*}|_{V_n \setminus E_n}$).*

Furthermore, if $D_{S^*}(\{V_n\}_n)$ is finite and $V_n \setminus E_n$ is k-hyperbolic for every n, then δ^* (respectively, δ) is a universal constant which depends only on r, s, N, k, $D_{S^*}(\{V_n\}_n)$ and δ (respectively, r, s, N, $D_{S^*}(\{V_n\}_n)$ and δ^*).

Remark. Recall that $d_{S^*} \neq d_S|_{S^*}$, since (S^*, d_{S^*}) is a geodesically complete Riemannian manifold (the points of E are at infinite d_{S^*}-distance from the points of S^*; in fact, S^* is an open non-exceptional Riemann surface).

Proof. If $D_{S^*}(\{V_n\}_n) = \infty$, Proposition 5.12 gives that S^* is not hyperbolic. We now see that if $D_{S^*}(\{V_n\}_n) < \infty$, S^* is hyperbolic if and only if S is hyperbolic and $V_n \setminus E_n$ is k-hyperbolic for every n. This fact completes the proof.

The heart of the proof is to construct two tree-decompositions $\{X_n\}_{n\in\Lambda}$ of S and $\{X_n^*\}_{n\in\Lambda}$ of S^* which, thanks to Theorem D, will allow us to establish a relationship between the hyperbolicity of S and that of S^*.

In order to obtain the tree-decompositions, we need to construct open sets U_n with better properties than V_n. On the one hand, if every connected component η of ∂V_n disconnects S (in particular, if ∂V_n is connected), we define $U_n := \operatorname{int} V_n$. On the other hand, if ∂V_n has a connected component η, with $S \setminus \eta$ connected (and then we have another connected component with the same property), we obtain an open set U_n by modifying V_n in the following way: we consider every pair of different connected components η_i^n, η_j^n of ∂V_n which are in the same connected component of $S \setminus \operatorname{int} V_n$; if $d_S|_{V_n}(\eta_i^n, \eta_j^n) < r/2$, let us denote by s_{ij}^n a geodesic in V_n (with $d_S|_{V_n}$) joining η_i^n and η_j^n with $L_S(s_{ij}^n) = d_S|_{V_n}(\eta_i^n, \eta_j^n) < r/2$; then $d_S|_{V_n}(s_{ij}^n, E_n) \geqslant r/2$, and $d_S(s_{ij}^n, E) \geqslant r/2$ (since $d_S(\partial V_n, E_n) \geqslant r$ and $d_S(V_n, V_m) \geqslant r$), and hence Lemma B gives

$$L_{S^*}(s_{ij}^n) \leqslant \coth(r/4) L_S(s_{ij}^n) \leqslant (r/2)\coth(r/4);$$

https://doi.org/10.1017/S0013091504001555 Published online by Cambridge University Press

if $d_S|_{V_n}(\eta_i^n, \eta_j^n) \geqslant r/2$, let us denote by s_{ij}^n a geodesic in $V_n \setminus E_n$ (with $d_{S^*}|_{V_n \setminus E_n}$) joining η_i^n and η_j^n with $L_{S^*}(s_{ij}^n) = d_{S^*}|_{V_n \setminus E_n}(\eta_i^n, \eta_j^n) \leqslant D_{S^*}(\{V_n\}_n)$. Let us define $D'_{S^*} := \max\{(r/2)\coth(r/4), D_{S^*}\}$; then $L_{S^*}(s_{ij}^n) \leqslant D'_{S^*}$.

It is clear that

$$U_n := \operatorname{int} V_n \setminus \bigcup_{i \neq j} s_{ij}^n$$

is an open set; if $\eta_{i_1}^n, \dots, \eta_{i_q}^n$ are in the same connected component of $S \setminus \operatorname{int} V_n$, then they are contained in the same connected component $\eta_{i_1,\dots,i_q}^n$ of ∂U_n (notice that $S \setminus \eta_{i_1,\dots,i_q}^n$ is not connected); if η_i^n is a connected component of ∂V_n and it is also a connected component of ∂U_n, then it disconnects S; hence every connected component of ∂U_n disconnects S.

It is clear that $E_n \subset U_n$ (since s_{ij}^n is a geodesic in V_n with $L_S(s_{ij}^n) < r/2$ or a geodesic in $V_n \setminus E_n$). We also have $\bar{U}_n = V_n$ and $d_S(U_n, U_m) = d_S(V_n, V_m) \geqslant r$.

Let us denote by K the set of indices of n (K is finite or countable). For each $n \in K$, let us define $X_n := \bar{U}_n = V_n$ and $X_n^* := \bar{U}_n \setminus E_n = V_n \setminus E_n$.

Let us consider the connected components $\{X_n\}_{n \in J}$ of $S \setminus \bigcup_{n \in K} U_n$. If we define $X_n^* := X_n$ for $n \in J$, and $\Lambda := K \cup J$, then

$$S = \bigcup_{n \in \Lambda} X_n \quad \text{and} \quad S^* = \bigcup_{n \in \Lambda} X_n^*.$$

Claim. We now claim that $\{X_n\}_{n \in \Lambda}$ and $\{X_n^*\}_{n \in \Lambda}$ are $(k_1, k_1/r, Nk_1)$-tree-decompositions of S and S^*, respectively, where $k_1 := s\coth(r/2) + D'_{S^*}$.

We continue the proof, assuming this claim to be true for the moment.

For any $n \in K$, we have that $X_n = \bar{U}_n = V_n$ belongs to $\mathcal{S}(N, s)$ (see Remark 5.10 (2)); consequently, Theorem 5.6 gives that X_n is k_5-hyperbolic, with a constant k_5 which depends only on N and s.

If $n \in J$, let us recall that $X_n = X_n^*$ is a union of bordered Riemann surfaces and geodesics. If s_{ij}^m is one of such curves, we consider two cases.

(i) If $d_S|_{V_m}(\eta_i^m, \eta_j^m) < r/2$, we have, by Lemma B, $1 < L_{S^*}(s_{ij}^m)/L_S(s_{ij}^m) < \coth(r/4)$.

(ii) If $d_S|_{V_m}(\eta_i^m, \eta_j^m) \geqslant r/2$, then $D_{S^*} \geqslant L_{S^*}(s_{ij}^m)$ and $L_S(s_{ij}^m) \geqslant r/2$, and we conclude that $1 < L_{S^*}(s_{ij}^m)/L_S(s_{ij}^m) \leqslant 2D_{S^*}/r$.

Since

$$\max\{\coth(r/4), 2D_{S^*}/r\} = (2/r)\max\{(r/2)\coth(r/4), D_{S^*}\} = 2D'_{S^*}/r,$$

in both cases we then have $1 < L_{S^*}(s_{ij}^m)/L_S(s_{ij}^m) \leqslant 2D'_{S^*}/r$. Consequently, we can define a map $i_n : X_n \to X_n^*$, which is the identity in each bordered Riemann surface and a dilatation in the geodesics joining the bordered surfaces. In the bordered surfaces the identity is a $(\coth(r/2), 0)$-quasi-isometry by Lemma B. Since $\coth(r/2) \leqslant 2D'_{S^*}/r$, then this map i_n (and i_n^{-1}) is a $(2D'_{S^*}/r, 0)$-quasi-isometry.

Consequently, we obtain from Theorem B that if X_n^* is k_4^*-hyperbolic for every $n \in J$, then X_n is k_4-hyperbolic for every $n \in J$, where k_4 depends only on r, D_{S^*} and k_4^*, and that, if X_n is k_4-hyperbolic for every $n \in J$, then X_n^* is k_4^*-hyperbolic for every $n \in J$, where k_4^* depends only on r, D_{S^*} and k_4.

Let us assume that S^* is δ^*-hyperbolic. Hence, Theorem D guarantees that X_n^* is k_4^*-hyperbolic for every $n \in \Lambda$ (where k_4^* depends only on r, s, N, D_{S^*} and δ^*); consequently, $V_n \setminus E_n$ is k_4^*-hyperbolic for every $n \in K$ and X_n is k_4-hyperbolic for every $n \in J$ (where k_4 depends only on r, s, N, D_{S^*} and δ^*). Since X_n is k_5-hyperbolic for every $n \in K$, if we apply Theorem D again, we obtain that S is δ-hyperbolic, where δ depends only on r, s, N, D_{S^*} and δ^*.

Let us now assume that S is δ-hyperbolic and that X_n^* is k-hyperbolic for every $n \in K$. Hence, Theorem D guarantees that X_n is k_4-hyperbolic for every $n \in J$ (where k_4 depends only on r, s, N, D_{S^*} and δ), and consequently X_n^* is k_4^*-hyperbolic for every $n \in J$ (where k_4^* depends only on r, s, N, D_{S^*} and δ). If we apply Theorem D again, we obtain that S^* is δ^*-hyperbolic, where δ^* depends only on r, s, k, N, D_{S^*} and δ.

Let us now prove the claim.

If $n \in K$, we have that each X_n (with $d_S|_{X_n}$) and X_n^* (with $d_{S^*}|_{X_n^*}$) are bordered non-exceptional Riemann surfaces; hence they are geodesic metric spaces.

Notice that, for each $n \in J$, X_n is a bordered surface or a union of bordered surfaces $\bigcup_m M_m$, with M_{m_1} and M_{m_2} joined by geodesics in S and/or in S^*. These geodesics are contained in $\bigcup_k V_k$, and there are at most a finite number of them in each V_k. The condition $d_S(V_{n_1}, V_{n_2}) \geqslant r$ for every $n_1 \neq n_2$ guarantees that any ball in S (or in S^*) intersects at most a finite number of V_k. Hence, X_n and X_n^* are geodesic metric spaces.

(a) We have $X_n \cap X_m = X_n^* \cap X_m^* =: \sigma_{nm}$, where σ_{nm} is connected: it is clear if $n \in K$, since then every connected component σ of ∂U_n disconnects S; if $n \in J$ and $X_n \cap X_m \neq \varnothing$, then $m \in K$, and we can apply the last argument with m instead of n (notice that $X_n \cap X_m = \varnothing$ if $n, m \in J$ or $n, m \in K$). Notice that, if $n \in K$, σ_{nm} is a connected component of ∂U_n; we have already seen during the construction of X_n that $X_n \setminus \sigma_{nm}$ is not connected. It is obvious that $\{\sigma_{nm}\}_m$ are pairwise disjoint closed subsets of X_n.

Any geodesic in S with finite length meets at most a finite number of σ_{nm}, since $d_S(U_n, U_m) \geqslant r$ for any $n \neq m$, and $\{\sigma_{nm}\}_m$ is a set with at most N elements, for any $n \in K$. The same result is true in S^*.

(b) Lemma B guarantees that

$$\operatorname{diam}_{X_n}(\sigma_{nm}) \leqslant \operatorname{diam}_{X_n^*}(\sigma_{nm}) \leqslant L_{S^*}(\partial V_n) + D'_{S^*} \leqslant s \coth(r/2) + D'_{S^*} = k_1,$$

if $n \in K$; if $n \notin K$, then $m \in K$ and we obtain the same result.

If $n \in K$, we choose $A_n = \varnothing$; then we have

$$\sum_m \operatorname{diam}_{X_n}(\sigma_{nm}) \leqslant \sum_m \operatorname{diam}_{X_n^*}(\sigma_{nm}) \leqslant N k_1.$$

If $n \in J$, we choose $A_n = \Lambda$; then

$$\begin{aligned} d_{X_n^*}(\sigma_{nm}, \sigma_{nk}) \geqslant d_{X_n}(\sigma_{nm}, \sigma_{nk}) \geqslant d_S(V_m, V_k) \geqslant r \\ \geqslant (r/k_1) \operatorname{diam}_{X_n^*}(\sigma_{nm}) \geqslant (r/k_1) \operatorname{diam}_{X_n}(\sigma_{nm}). \end{aligned}$$

These facts prove the claim. □

The next result is a consequence of Theorem 5.14; it allows one, in many cases, to forget punctures and funnels (and more general ends) in order to study the hyperbolicity of a Riemann surface; this fact can be a significant simplification in the topology of the surface, and therefore makes the study of its hyperbolicity easier. Recall that to delete an isolated point from S gives a puncture in S^*, and that to delete a closed simply connected set from S gives a funnel in S^*.

The statement of Theorem 5.15 has one more hypothesis about E than Theorem 5.14, and therefore this allows us to obtain a simpler conclusion.

Theorem 5.15. *Let S be an open non-exceptional Riemann surface and let $E = \bigcup_n E_n$ be an (r, s, t, N)-uniformly separated set in S. Then, $S^* := S \setminus E$ is δ^*-hyperbolic if and only if S is δ-hyperbolic and $D_{S^*}(\{V_n\}_n)$ is finite.*

Furthermore, if $D_{S^*}(\{V_n\}_n)$ is finite, then δ^* (respectively, δ) is a universal constant which depends only on r, s, t, N, $D_{S^*}(\{V_n\}_n)$ and δ (respectively, r, s, N, $D_{S^*}(\{V_n\}_n)$ and δ^*).

Proof. In order to apply Theorem 5.14, we need to prove only that $X_n^* := V_n \setminus E_n$ is k-hyperbolic for every $n \in K$, where k is a constant which depends only on r, s, t and N.

Recall that for any $n \in K$, $X_n = V_n$ is compact and belongs to $\mathcal{S}(N, s)$, or V_n is homeomorphic to a funnel.

Let us denote by γ_n (respectively, γ_n^*) the simple closed geodesic in X_n^* (respectively, in S^*) which 'surrounds' E_n (if E_n is a single point, we see γ_n as a puncture and $L_{S^*}(\gamma_n) = 0$).

If ∂V_n contains at least three closed curves, we denote by γ_n' the simple closed geodesic in $D_n \setminus E_n$; since $D_n \setminus E_n \subseteq S^*$, we have

$$L_{S^*}(\gamma_n^*) \leqslant L_{S^*}(\gamma_n') \leqslant L_{D_n \setminus E_n}(\gamma_n') = \pi/\operatorname{mod}(D_n \setminus E_n) \leqslant \pi/t.$$

Since Lemma B implies $L_{S^*}(\partial V_n) < L_S(\partial V_n)\coth(r/2) \leqslant s\coth(r/2)$, we deduce that

$$L_{S^*}(\gamma_n) \leqslant L_{S^*}(\gamma_n^*) + L_{S^*}(\partial V_n) < s\coth(r/2) + \pi/t.$$

Notice that any generalized outer loop γ, which different from γ_n in $X_n^* = V_n \setminus E_n$, is freely homotopic to some closed curve in ∂V_n; then Lemma B guarantees that $L_{S^*}(\gamma) \leqslant L_{S^*}(\partial V_n) < s\coth(r/2)$. Hence, $X_n^* \in \mathcal{S}(N+1, s\coth(r/2)+\pi/t)$. Theorem 5.6 guarantees that X_n^* is k_5^*-hyperbolic, with a constant k_5^* which depends only on r, s, t and N.

If V_n is isometric to a non-compact subset of an annulus or of the punctured disc $\boldsymbol{D}^*$, a similar argument to the last one (now using Lemma 5.7 instead of Theorem 5.6) gives the same conclusion; in the case of $\boldsymbol{D}^*$, we do not need the condition about modulus, since there are two closed curves of bounded length in X_n^*: ∂V_n and the puncture.

Let us now consider any $n \in K$ such that V_n is compact and ∂V_n is the union of one or two closed curves. A similar argument (now using Theorem 5.6 or Theorem F, respectively) gives the same conclusion. □

Theorem 5.15 has the following direct consequence.

https://doi.org/10.1017/S0013091504001555 Published online by Cambridge University Press

Corollary 5.16. *Let S be an open non-exceptional Riemann surface and let $E = \bigcup_n E_n$ be an (r, s, t, N)-uniformly separated set in S. Let us also assume that we can choose the sets $\{V_n\}_n$ such that every connected component of each ∂V_n disconnects S. Then $S^* := S \setminus E$ is δ^*-hyperbolic if and only if S is δ-hyperbolic. Furthermore, δ^* (respectively, δ) is a universal constant which depends only on r, s, t, N and δ (respectively, r, s, N and δ^*).*

We can also obtain the following improvements of Theorem 5.15.

Corollary 5.17. *The conclusion of Theorem 5.15 also holds if we weaken the definition of an (r, s, t, N)-uniformly separated set in the following way: for an arbitrary subset of values of n with V_n isometric to a non-compact subset of an annulus, we can substitute the hypothesis 'there exists a simply connected domain D_n in S, with $E_n \subset D_n$ and $\operatorname{mod}(D_n \setminus E_n) \geqslant t$' with '$\operatorname{diam}_S(E_n) \leqslant 1/t$ and $d_S(\partial V_n, E_n) \leqslant 1/t$'.*

Proof. Let us denote by K_0 the set of indices n with V_n isometric to a non-compact subset of an annulus, $\operatorname{diam}_S(E_n) \leqslant 1/t$ and $d_S(\partial V_n, E_n) \leqslant 1/t$. In order to follow the proof of Theorem 5.15, we only need to check that X_n^* is k_5^*-hyperbolic for every $n \in K_0$, with a constant k_5^* which depends only on r, s and t. Lemma 5.7 implies this fact if we can find two generalized outer loops in X_n^* with length less than or equal to $c = c(r, s, t)$.

Fix $n \in K_0$. Let us denote by g_n the simple closed geodesic in S freely homotopic to ∂V_n and by F_n the funnel in S with boundary g_n; g_n does exist since V_n is isometric to a non-compact subset of an annulus and thus V_n cannot be a neighbourhood of a puncture. We obviously have $L_S(g_n) \leqslant L_S(\partial V_n) \leqslant s$.

If ∂V_n intersects F_n, we define $l := d_S(\partial V_n, g_n)$; otherwise, we define $l := 0$. Let us consider the boundary curve g_n' of the collar of g_n of width l which is contained in F_n. It is well known that $L_S(g_n') = L_S(g_n) \cosh l$; this computation can be easily checked using Fermi coordinates (see, for example, [**10**, p. 247]). We also have $L_S(g_n') \leqslant L_S(\partial V_n) \leqslant s$ (see, for example, [**5**, Lemma 4]). Consequently, $L_S(g_n)\mathrm{e}^l \leqslant 2s$.

Let us denote by g_n'' the boundary curve of the collar of g_n of width $x := l + 2/t + s/2 + r$ that is contained in F_n. Since $L_S(g_n'') = L_S(g_n) \cosh x$, we deduce that $L_S(g_n'') \leqslant L_S(g_n)\mathrm{e}^l \mathrm{e}^{2/t+s/2+r} \leqslant 2s\mathrm{e}^{2/t+s/2+r}$.

Notice that

$$\begin{aligned} d_S(g_n'', E_n) &\geqslant d_S(g_n'', g_n) - d_S(g_n, g_n') - \operatorname{diam}_S(\partial V_n) - d_S(\partial V_n, E_n) - \operatorname{diam}_S(E_n) \\ &\geqslant x - l - s/2 - 1/t - 1/t = r. \end{aligned}$$

Hence, Lemma B implies that

$$L_{S^*}(g_n'') \leqslant 2s\mathrm{e}^{2/t+s/2+r} \coth(r/2) \quad \text{and} \quad L_{S^*}(\partial V_n) \leqslant s \coth(r/2).$$

If ∂V_n intersects F_n, the curve g_n'' is contained in X_n^*, since ∂V_n is contained in the collar of g_n of width $l+s/2$. If ∂V_n does not intersect F_n, we also have that the curve g_n'' is contained in X_n^*, since $g_n'' \subset F_n \subset V_n = X_n$. If g_n^* is the generalized outer loop in X_n^* that is freely homotopic to g_n'' in X_n^*, it is clear that $L_{S^*}(g_n^*) \leqslant L_{S^*}(g_n'') \leqslant 2s\mathrm{e}^{2/t+s/2+r} \coth(r/2)$, and Lemma 5.7 completes the proof. □

With similar arguments we can prove the following result.

Corollary 5.18. *The conclusion of Theorem 5.15 also holds if we weaken the definition of an (r, s, t, N)-uniformly separated set in the following way: for an arbitrary subset of values of n with V_n isometric to a non-compact subset of an annulus, we can substitute the hypothesis 'there exists a simply connected domain D_n in S, with $E_n \subset D_n$ and $\mathrm{mod}(D_n \setminus E_n) \geqslant t$' with '$\min\{L_{S^*}(\gamma_2^n), L_{S^*}(\gamma_3^n)\} \leqslant 1/t$', where γ_2^n, γ_3^n are the outer loops in S^* corresponding to $V_n \setminus E_n$.*

The condition '$\min\{L_{S^*}(\gamma_2^n), L_{S^*}(\gamma_3^n)\} \leqslant 1/t$' for an arbitrary subset of values of n with V_n isometric to a non-compact subset of an annulus is sharp; in fact, it is equivalent to '$V_n \setminus E_n$ is k-hyperbolic' for every n in that subset of values of n. This equivalence is a direct consequence of Corollary 5.20 below. In order to prove Corollary 5.20 we need the following result, which is interesting in itself.

Theorem 5.19. *Let us consider $L_1, L_2 > 0$ and the generalized Y-piece Y_0 with simple closed geodesics γ_1, γ_2, γ_3 of lengths $l_1 \leqslant l_2 \leqslant l_3$ verifying $0 \leqslant l_1 \leqslant L_1$ and $L_2 \leqslant l_2, l_3$. Let S be any non-exceptional Riemann surface (with or without boundary) containing Y_0 such that γ_2, γ_3 are outer loops in S. The sharpest hyperbolicity constant δ of S satisfies $\delta \geqslant D(L_1, L_2)$, where $\lim_{L_2\to\infty} D(L_1, L_2) = \infty$, for any fixed L_1.*

Remark. As we will see in the proof, the hypothesis 'γ_2, γ_3 are outer loops in S' can be substituted by 'a geodesic in S joining two points of γ_2 can only get out of Y_0 through γ_1'. We have examples which show that the conclusion of Corollary 5.18 does not hold if a geodesic in S joining two points of γ_2 can get out of Y_0 through γ_2 or γ_3.

In order to prove Theorem 5.19, we need the following elementary lemma (see [**28**, Lemma 3.1] for a proof).

Lemma C. *Let us consider a geodesic metric space X and let $\varepsilon > 0$. If γ is a continuous curve joining $x, y \in X$ with $L_X(\gamma) \leqslant d_X(x, y) + \varepsilon$, then γ is a $(1, \varepsilon)$-quasi-geodesic with its arc-length parametrization.*

Proof of Theorem 5.19. The idea that lies behind the proof is that, given two points in γ_2, the distance between them is approximately the length of a subcurve of γ_2 joining them. Let us denote by $p_2 \in \gamma_2$, $p_3 \in \gamma_3$ the points with $d_{Y_0}(p_2, p_3) = d_{Y_0}(\gamma_2, \gamma_3) =: s$. We choose the points $q_2 \in \gamma_2$, $q_3 \in \gamma_3$ with $d_{Y_0}(p_2, q_2) = l_2/2$ and $d_{Y_0}(p_3, q_3) = l_3/2$. If we split Y_0 along the geodesics that start orthogonally to γ_2 in p_2 and q_2 and to γ_3 in q_3, then we obtain two isometric right-angled hexagons H_1, H_2. Each H_i has three alternate sides with lengths $l_1/2$, $l_2/2$, $l_3/2$.

We consider the locally geodesic bigon γ_2 in S with vertices $\{p_2, q_2\}$. We now prove that this bigon is $(1, L_1)$-quasi-geodesic in S.

Since H_1 and H_2 are isometric and γ_2, γ_3 are outer loops in S, if a geodesic in S joining p_2 with q_2 is not contained in γ_2, then it must join p_2 with γ_1. By Lemma C we only need to prove that $l_2/2 \leqslant d_S(p_2, q_2) + L_1$. We denote by B the opposite side to $[p_3, q_3]$ and by z' the point $z' := B \cap \gamma_1$. Choose the point $z_0 \in \gamma_1$ with $d_{H_1}(p_2, z_0) = d_{H_1}(p_2, \gamma_1)$. Let us consider the right-angled quadrilateral $\{p_2, q_2, z', z_0\}$ in H_1. If $L(B) \geqslant l_1/2$, then

https://doi.org/10.1017/S0013091504001555 Published online by Cambridge University Press

$d_{H_1}(z', z_0) \leqslant L(B)$, and hyperbolic trigonometry gives $l_2/2 \leqslant d_{H_1}(p_2, z_0) = d_{H_1}(p_2, \gamma_1)$; hence, $\gamma_2 \cap H_1$ is a geodesic in S and $l_2/2 = d_S(p_2, q_2)$. If $d_S(p_2, q_2) < l_2/2$, then $d_S(p_2, q_2) > d_{H_1}(p_2, \gamma_1)$, and consequently $L(B) < l_1/2$. Hence the triangle inequality implies that

$$l_2/2 \leqslant d_{H_1}(p_2, z_0) + d_{H_1}(z_0, z') + L(B) < d_{H_1}(p_2, \gamma_1) + L_1/2 + L_1/2 < d_S(p_2, q_2) + L_1.$$

Now we study the thin constant of the $(1, L_1)$-quasi-geodesic bigon.

Let us choose the point $z \in \gamma_2 \cap H_1$ such that $d_{H_1}(z, p_2) = d_{H_1}(z, q_2) = l_2/4$. Let us consider the point $z'' \in \gamma_1$ with $d_{H_1}(z, z'') = d_{H_1}(z, \gamma_1)$.

Considering the right-angled triangle $\{z, q_2, z'\}$ in H_1 we obtain

$$d_{H_1}(z, z') \geqslant d_{H_1}(z, q_2) = l_2/4.$$

Hence,

$$d_{H_1}(z, \gamma_1) = d_{H_1}(z, z'') \geqslant d_{H_1}(z, z') - d_{H_1}(z', z'') \geqslant l_2/4 - l_1/2 \geqslant L_2/4 - L_1/2.$$

Now we deal with a bound of $d_{H_1}(z, A)$, where A is the opposite side to $[p_2, q_2]$ in H_1. Standard hyperbolic trigonometry (see, for example, [**5**, p. 161]) gives

$$\cosh s = \frac{\cosh(l_1/2) + \cosh(l_2/2)\cosh(l_3/2)}{\sinh(l_2/2)\sinh(l_3/2)} \geqslant \operatorname{cotanh}(l_2/2)\operatorname{cotanh}(l_3/2).$$

Let us consider the geodesic γ_0 which gives the distance in H_1 between $[p_2, q_2]$ and A; we define $x := \gamma_0 \cap [p_2, q_2]$ and $y := \gamma_0 \cap A$. The geodesic γ_0 splits H_1 into two right-angled pentagons. Hyperbolic trigonometry for pentagons (see, for example, [**5**, p. 159]) gives $\cosh L(\gamma_0) = \sinh s \sinh(l_3/2)$. Then

$$\begin{aligned}
\cosh^2 L(\gamma_0) &= \sinh^2 s \sinh^2(l_3/2) \geqslant (\operatorname{cotanh}^2(l_2/2)\operatorname{cotanh}^2(l_3/2) - 1)\sinh^2(l_3/2) \\
&\geqslant ((1 + 2\mathrm{e}^{-l_2})^2(1 + 2\mathrm{e}^{-l_3})^2 - 1)\sinh^2(l_3/2) \\
&\geqslant (4\mathrm{e}^{-l_2} + 4\mathrm{e}^{-l_3})\frac{\mathrm{e}^{l_3} - 2}{4} = (\mathrm{e}^{-l_2} + \mathrm{e}^{-l_3})(\mathrm{e}^{l_3} - 2) \\
&= 1 + \mathrm{e}^{l_3 - l_2} - 2(\mathrm{e}^{-l_2} + \mathrm{e}^{-l_3}).
\end{aligned}$$

Hence, $\sinh^2 L(\gamma_0) \geqslant 1 - 4\mathrm{e}^{-L_2}$. Without loss of generality we can assume that $L_2 > 2\log 2$ (and then $1 - 4\mathrm{e}^{-L_2} > 0$), since the conclusion of the theorem deals with the limit as L_2 tends to infinity). Hyperbolic trigonometry for pentagons gives $\sinh d_{H_1}(x, q_2) \sinh L(\gamma_0) = \cosh(l_1/2)$, and consequently

$$\sinh d_{H_1}(x, q_2) = \frac{\cosh(l_1/2)}{\sinh L(\gamma_0)} \leqslant \frac{\cosh(L_1/2)}{\sqrt{1 - 4\mathrm{e}^{-L_2}}}.$$

Let σ be the geodesic in H_1 that joins z with A, and such that $L(\sigma) = d_{H_1}(z, A)$. If $u := \sigma \cap A$, the hyperbolic trigonometry for the right-angled quadrilateral $\{z, u, y, x\}$

gives $\sinh d_{H_1}(z, A) = \sinh L(\sigma) = \sinh L(\gamma_0) \cosh d_{H_1}(x, z)$. We have

$$\tfrac{1}{4}\mathrm{e}^{2d_{H_1}(x,q_2)} \leqslant \cosh^2 d_{H_1}(x, q_2) \leqslant \frac{\cosh^2(L_1/2)}{1-4\mathrm{e}^{-L_2}} - 1 \leqslant \frac{\sinh^2(L_1/2) + 4\mathrm{e}^{-L_2}}{1-4\mathrm{e}^{-L_2}},$$

$$\mathrm{e}^{-d_{H_1}(x,q_2)} \geqslant \frac{1}{2}\sqrt{\frac{1-4\mathrm{e}^{-L_2}}{\sinh^2(L_1/2) + 4\mathrm{e}^{-L_2}}} \geqslant \frac{1}{2}\frac{\sqrt{1-4\mathrm{e}^{-L_2}}}{\sinh(L_1/2) + 2\mathrm{e}^{-L_2/2}}.$$

Consequently, we obtain

$$\begin{aligned}\sinh d_{H_1}(z, A) &= \sinh L(\gamma_0) \cosh d_{H_1}(x, z) = \sinh L(\gamma_0) \cosh(l_2/4 - d_{H_1}(x, q_2)) \\ &\geqslant \sqrt{1-4\mathrm{e}^{-L_2}}\,\tfrac{1}{2}\mathrm{e}^{l_2/4 - d_{H_1}(x,q_2)} \geqslant \frac{\mathrm{e}^{L_2/4}}{4}\frac{1-4\mathrm{e}^{-L_2}}{\sinh(L_1/2) + 2\mathrm{e}^{-L_2/2}}.\end{aligned}$$

Since $\operatorname{arcsinh} t \geqslant \log(2t)$, we deduce that

$$d_{H_1}(z, A) \geqslant \log\biggl(2\frac{\mathrm{e}^{L_2/4}}{4}\frac{1-4\mathrm{e}^{-L_2}}{\sinh(L_1/2) + 2\mathrm{e}^{-L_2/2}}\biggr) = \frac{L_2}{4} - \log\biggl(\frac{2\sinh(L_1/2) + 4\mathrm{e}^{-L_2/2}}{1-4\mathrm{e}^{-L_2}}\biggr).$$

Let us consider a geodesic η in S joining z with $\gamma_2 \cap H_2$ such that $L(\eta) = d_S(z, \gamma_2 \cap H_2)$. Since γ_2, γ_3 are outer loops in S, if η is not contained in γ_2, then it must intersect A or γ_1. Consequently, using Lemma A,

$$\begin{aligned}K(\delta, 1, L_1) &\geqslant d_S(z, \gamma_2 \cap H_2) \geqslant \min\biggl\{\frac{L_2}{4} - \frac{L_1}{2}, \frac{L_2}{4} - \log\biggl(\frac{2\sinh(L_1/2) + 4\mathrm{e}^{-L_2/2}}{1-4\mathrm{e}^{-L_2}}\biggr)\biggr\} \\ &= \frac{L_2}{4} - \max\biggl\{\frac{L_1}{2}, \log\biggl(\frac{2\sinh(L_1/2) + 4\mathrm{e}^{-L_2/2}}{1-4\mathrm{e}^{-L_2}}\biggr)\biggr\} =: E(L_1, L_2).\end{aligned}$$

Let us fix L_1. Since $K(\delta, 1, L_1) = 4\delta + 2H(\delta, 1, L_1)$ is an increasing function of δ, we can consider its inverse function $F(t, L_1)$. Then $\delta \geqslant D(L_1, L_2) := F(E(L_1, L_2), L_1)$.

We have that $\lim_{\delta\to\infty} K(\delta, 1, L_1) = \infty$, since $K(\delta, 1, L_1) = 4\delta + 2H(\delta, 1, L_1)$. It is clear that $\lim_{L_2\to\infty} E(L_1, L_2) = \infty$ and $\lim_{t\to\infty} F(t, L_1) = \infty$ for any fixed L_1. Hence, $\lim_{L_2\to\infty} D(L_1, L_2) = \infty$ for any fixed L_1. □

Corollary 5.20. *Let us consider $L_1, L_2 > 0$ and S a triply connected bordered non-exceptional Riemann surface such that ∂S contains a simple closed curve g_1 with $L(g_1) \leqslant L_1$. Let us also assume that S contains two simple closed geodesics γ_2, γ_3 not freely homotopic to g_1 with $L(\gamma_2), L(\gamma_3) \geqslant L_2$. Then the sharpest hyperbolicity constant δ of S satisfies $\delta \geqslant \Lambda(L_1, L_2)$, where $\lim_{L_2\to\infty} \Lambda(L_1, L_2) = \infty$ for any fixed L_1.*

Proof. Let us denote by $M(L_1, L_2)$ the set of Riemann surfaces verifying the hypotheses of Corollary 5.20. We define $\Lambda(L_1, L_2)$ as the infimum of the hyperbolicity constants of the surfaces $S \in M(L_1, L_2)$.

Let us assume that the conclusion of Corollary 5.20 does not hold. Since $\Lambda(L_1, L_2)$ is a non-decreasing function in L_2, there exists some constant c_1 such that $\Lambda(L_1, L_2) < c_1$ for some fixed L_1 and for every L_2. For this L_1 and for each L_2 we can take $S \in M(L_1, L_2)$

with $\delta(S) \leqslant c_1$. Such a surface S is contained in some triply connected non-exceptional Riemann surface R with simple closed geodesics $\gamma_1, \gamma_2, \gamma_3$ such that γ_1 is freely homotopic to g_1 (γ_1 can be a puncture) and $\partial R = \partial S \setminus \{g_1\}$; therefore $L(\gamma_1) \leqslant L(g_1) \leqslant L_1$. It is clear that γ_2, γ_3 are outer loops in R.

Consider the doubly connected bordered non-exceptional Riemann surface S_0 defined as the closure of $R \setminus S$; we have $S \cup S_0 = R$ and $S \cap S_0 = \partial S_0 = g_1$. We obtain $\delta(S_0) \leqslant c_2$ from Theorem 5.6. Since the pair S, S_0 is an $(L_1, 0, 0)$-tree-decomposition of R, Theorem D gives $\delta(R) \leqslant c_3$. This fact is a contradiction of Theorem 5.19, since $\delta(R) \geqslant D(L_1, L_2)$. □

Acknowledgements. The research of all authors was partly supported by grant BFM 2003-04870 from DGI, Spain, while the research of E.T. was also partly supported by grant BFM 2000-0022, also from DGI, Spain.

We thank Professor José Luis Fernández for some useful discussions. Also, we thank the referees for their careful reading of the manuscript and for some helpful suggestions.

References

1. L. Ahlfors and L. Sario, *Riemann surfaces* (Princeton University Press, 1960).
2. V. Alvarez, D. Pestana and J. M. Rodríguez, Isoperimetric inequalities in Riemann surfaces of infinite type, *Rev. Mat. Iber.* **15** (1999), 353–427.
3. V. Alvarez, J. M. Rodríguez and V. A. Yakubovich, Subadditivity of p-harmonic 'measure' on graphs, *Michigan Math. J.* **49** (2001), 47–64.
4. Z. M. Balogh and S. M. Buckley, Geometric characterizations of Gromov hyperbolicity, *Invent. Math.* **153** (2003), 261–301.
5. L. Bers, An inequality for Riemann surfaces, in *Differential geometry and complex analysis* (ed. C. Isaac and M. F. Hershel), pp. 87–93 (Springer, 1985).
6. M. Bonk, Quasi-geodesic segments and Gromov hyperbolic spaces, *Geom. Dedicata* **62** (1996), 281–298.
7. M. Bonk, J. Heinonen and P. Koskela, *Uniformizing Gromov hyperbolic spaces*, Astérisque, vol. 270 (Société Mathématique de France, Paris, 2001).
8. P. Buser, *Geometry and spectra of compact Riemann surfaces* (Birkhäuser, Boston, MA, 1992).
9. A. Cantón, J. L. Fernández, D. Pestana and J. M. Rodríguez, On harmonic functions on trees, *Potent. Analysis* **15** (2001), 199–244.
10. I. Chavel, *Eigenvalues in Riemannian geometry* (Academic, 1984).
11. M. Coornaert, T. Delzant and A. Papadopoulos, *Notes sur les groups hyperboliques de Gromov* (IRMA, Strasbourg, 1989).
12. J. L. Fernández and J. M. Rodríguez, The exponent of convergence of Riemann surfaces, Bass Riemann surfaces, *Ann. Acad. Sci. Fenn. Ser. AI* **15** (1990), 165–183.
13. J. L. Fernández and J. M. Rodríguez, Area growth and Green's function of Riemann surfaces, *Arch. Math.* **30** (1992), 83–92.
14. E. Ghys and P. de la Harpe, *Sur les groupes hyperboliques d'après Mikhael Gromov*, Progress in Mathematics, vol. 83 (Birkhäuser, Boston, MA, 1990).
15. M. Gromov, Hyperbolic groups, in *Essays in group theory* (ed. S. M. Gersten), MSRI Publications, vol. 8, pp. 75–263 (Springer, 1987).
16. M. Gromov, *Metric structures for Riemannian and non-Riemannnian spaces* (with appendices by M. Katz, P. Pansu and S. Semmes), Progress in Mathematics, vol. 152 (Birkhäuser, Boston, MA, 1999).

17. I. Holopainen and P. M. Soardi, p-harmonic functions on graphs and manifolds, *Manuscr. Math.* **94** (1997), 95–110.
18. M. Kanai, Rough isometries and combinatorial approximations of geometries of non-compact Riemannian manifolds, *J. Math. Soc. Jpn* **37** (1985), 391–413.
19. M. Kanai, Rough isometries and the parabolicity of Riemannian manifolds, *J. Math. Soc. Jpn* **38** (1986), 227–238.
20. M. Kanai, Analytic inequalities and rough isometries between non-compact Riemannian manifolds, in *Curvature and Topology of Riemannian Manifolds, Katata, 1985*, Lecture Notes in Mathematics, vol. 1201, pp. 122–137 (Springer, 1986).
21. J. Ortega and K. Seip, Harmonic measure and uniform densities, *Indiana Univ. Math. J.* **53** (2004), 905–924.
22. A. Portilla, J. M. Rodríguez and E. Tourís, Gromov hyperbolicity through decomposition of metric spaces, II, *J. Geom. Analysis* **14** (2004), 123–149.
23. A. Portilla, J. M. Rodríguez and E. Tourís, The topology of balls and Gromov hyperbolicity of Riemann surfaces, *Diff. Geom. Applic.* **21** (2004), 317–335.
24. B. Randol, Cylinders in Riemann surfaces, *Comment. Math. Helv.* **54** (1979), 1–5.
25. J. M. Rodríguez, Isoperimetric inequalities and Dirichlet functions of Riemann surfaces, *Publ. Mat.* **38** (1994), 243–253.
26. J. M. Rodríguez, Two remarks on Riemann surfaces, *Publ. Mat.* **38** (1994), 463–477.
27. J. M. Rodríguez and E. Tourís, Gromov hyperbolicity through decomposition of metric spaces, *Acta Math. Hungar.* **103** (2004), 53–84.
28. J. M. Rodríguez and E. Tourís, A new characterization of Gromov hyperbolicity for Riemann surfaces, preprint.
29. J. M. Rodríguez and E. Tourís, Gromov hyperbolicity of Riemann surfaces, *Acta Math. Sinica*, in press.
30. P. M. Soardi, Rough isometries and Dirichlet finite harmonic functions on graphs, *Proc. Am. Math. Soc.* **119** (1993), 1239–1248.
31. D. Sullivan, Related aspects of positivity in Riemannian geometry, *J. Diff. Geom.* **25** (1987), 327–351.