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A Microlocal Riemann–Hilbert Correspondence*

O. NETO
Departamento de Matemática, Faculdade de Ciências da Universidade de Lisboa, Lisbon, Portugal. e-mail: orlando@lmc.fc.ul.pt and CMAF, Universidade de Lisboa, Av. Gama Pinto, 2, 1649-003 Lisboa, Portugal

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Abstract. We present a microlocal version of the Riemann–Hilbert correspondence for regular holonomic $\mathcal{D}$-modules. We show that a regular holonomic system of microdifferential equations is associated to a perverse sheaf concentrated in degree 0. Moreover, we show that this perverse sheaf can be recovered from the local system it determines on the complementary of its singular locus. We characterize the classes of perverse sheaves and local systems associated to regular holonomic systems of microdifferential equations.

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Irreducible holonomic $\mathcal{D}$-modules characterize multivalued holomorphic functions modulo the action of the fundamental group in the same way that irreducible polynomials characterize algebraic numbers modulo the action of the Galois group. To microlocalize a $\mathcal{D}$-module is equivalent to focalize our attention on the singularities of the multivalued holomorphic solutions of the system. Riemann showed with is study of the hypergeometric differential equation that this point of view can be quite efficient.

The main purpose of this paper is to study the perverse sheaves associated to $\mathcal{D}$-modules that come from systems of microdifferential equations. We show that the structure of these perverse sheaves is much simpler than the structure of an arbitrary perverse sheaf. These sheaves are concentrated in degree zero and they can be recovered from the local system they determine on the complementary of the ramification locus of the solutions of the system.

Several papers have been dedicated to the presentation of combinatorial descriptions of perverse sheaves (cf. [MV], [Ma], [Na], [GMV].). We show that in this case there is a very simple description in terms of certain linear representations of the fundamental group of the complementary of the ramification locus. We call these representations hypergeometric because the monodromy of a Gauss

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hypergeometric function is an example of an hypergeometric representation. The hypergeometric representation of a holonomic system of microdifferential equations of multiplicity one along the conormal of the cusp $y^2 = x^{2n+1}$ equals the monodromy of some Gauss hypergeometric function (cf. [NS]).

We can find in [A] another approach to the microlocalization of the de Rham functor. Our approach produces a statement that does not need to use the language of derived categories. Moreover, it provides a combinatorial description of the objects. It opens the door to a systematic study of the systems of PDE's with characteristic variety equal to a given Lagrangian variety.

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1. Introduction

We will assume that the reader has some knowledge of the theory of sheaves on manifolds (cf. [KS]) and of the theory of regular holonomic $\mathcal{D}$-modules (cf. [KK] or [Bj]).

Let X denote a complex manifold. Let π: $T^*X \to X$ be the cotangent bundle of X. Let Λ be a germ of a conic Lagrangian variety at $p \in T^*X \setminus T^*_X X$. We say that Λ is in *generic position* at p if $\pi^{-1}(\pi(p)) \cap \Lambda = \mathbf{C}p$. We will denote by $RH_p(\mathcal{D})$ the full subcategory of the category of regular holonomic $\mathcal{D}_{X,\pi(p)}$-modules $\mathcal{M}$ that satisfy the following conditions.

(a) $Char(\mathcal{M}) \cap \pi^{-1}(\pi(p)) = \mathbf{C}p$.
(b) If u is a germ of an holomorphic vector field at $\pi(p)$ such that $u(\pi(p)) \neq 0$ then the $\mathbf{C}$-linear map $u\,\cdot$: $\mathcal{M} \to \mathcal{M}$ is invertible.

We will denote by $RH_p(\mathcal{E})$ the full subcategory of the category of regular holonomic $\mathcal{E}_{X,p}$-modules $\mathcal{M}$ that satisfy the condition

(a) $\mathrm{supp}\mathcal{M} \cap \pi^{-1}(\pi(p)) = \mathbf{C}^*p$.

THEOREM 1.1 [KK]. *The functor*

$$\mu_p\colon RH_p(\mathcal{D}) \to RH_p(\mathcal{E}), \qquad \mathcal{M} \mapsto \mathcal{E}_{X,p} \otimes_{\mathcal{D}_{X,\pi(p)}} \mathcal{M}$$

is an equivalence of categories. Its inverse is the base change functor v_p that associates to an $\mathcal{E}_{X,p}$-module $\mathcal{N}$ the $\mathcal{D}_{X,\pi(p)}$-module $\mathcal{N}$.

Let us recall the Riemann–Hilbert correspondence for $\mathcal{D}$-modules.

DEFINITION 1.2. A $\mathbf{C}_X$-module is called *constructible* if there is a decreasing sequence

$$(X_j)_{j\in\mathbf{N}}\colon X = X_0 \supset X_1 \supset X_2 \cdots$$

of closed analytic subsets of X such that $\cap_{j \geqslant 0} X_j = \emptyset$ and, for each $j \geqslant 0$, the sheaf $F|_{X_j \setminus X_{j+1}}$ is a locally constant $\mathbf{C}_{X_j \setminus X_{j+1}}$-module.

Let $\mathbf{D}^b_c(X)$ be the full subcategory of the derived category $\mathbf{D}(X)$ whose objects are complexes

$$F = \cdots \to F_k \to F_{k+1} \to \cdots$$

such that $H^j(F) = 0$ for almost all j and the cohomology groups $H^j(F)$ are constructible.

DEFINITION 1.3. We say that a complex $F \in \mathbf{D}^b_c(X)$ satisfies the *conditions of support* if

$$\operatorname{codim}(\operatorname{supp} H^j(F)) \geqslant j, \quad \text{for all } j.$$

We say that a complex $F \in \mathbf{D}^b_c(X)$ is a *perverse sheaf* if F and its Verdier dual satisfy the conditions of support. We will denote by $\mathbf{Perv}(X)$ the full subcategory of $\mathbf{D}^b_c(X)$ whose objects are the perverse sheaves.

THEOREM 1.4 (cf. [K], [Mb1,Mb2]). *The functor*

$$\mathcal{DR}_X \colon \mathbf{RH}(\mathcal{D}_X) \longrightarrow \mathbf{Perv}(X),$$

$$\mathcal{M} \longmapsto \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M})$$

is an equivalence of categories.

It is possible to recover the characteristic variety of a coherent $\mathcal{D}_X$-module from $\mathcal{DR}_X(\mathcal{M})$.

DEFINITION 1.5. Let X be a C^1 manifold. Let $p \in T^*X$. Let $F \in D^b(X)$. We say that $p \notin SS(F)$ if there is an open neighbourhood U of p such that if $a \in X$, f is a C^1 real function defined on a neighbourhood of a and $df(a) \in U$, we have

$$\left(\mathbf{R}\Gamma_{\{x: f(x) \geqslant 0\}} F\right)_a = 0.$$

We call $SS(F)$ the *micro-support* of F.

THEOREM 1.6 [KS]. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Then*

$$\mathcal{C}har(\mathcal{M}) = SS(\mathcal{DR}_X \mathcal{M}).$$

The following characterization of the micro-support will be very useful.

THEOREM 1.7 [KS]. *Assume that X is an open subset of a finite-dimensional real vector space E. Let $p = (x_0, \xi_0) \in T^*X$ and let $F \in \mathrm{Ob}(D^b(X))$. Then $p \notin SS(F)$ if* *and only if there are an open neighbourhood U of* 0*, an* $\varepsilon > 0$ *and a proper closed convex cone* γ *verifying the following conditions.*

(a) $0 \in \gamma$ *and* $\gamma \setminus \{0\} \subset \{v \in E : \langle v, \xi_0 \rangle < 0\}$.
(b) *If* $H = \{x : \langle x - x_o, \xi_0 \rangle \geqslant -\varepsilon\}$ *and* $L = \{x : \langle x - x_o, \xi_0 \rangle = -\varepsilon\}$, *then* $H \cap (U + \gamma) \subset X$ *and we have a natural isomorphism*

$$\mathrm{R}\Gamma(H \cap (x + \gamma), F) \xrightarrow{\sim} \Gamma(L \cap (x + \gamma), F),$$

for all $x \in U$.

2. An Equivalence of Categories

Remark 2.1. Let Λ be a germ of a conic Lagrangian variety at $p \in T^*X \setminus T^*_X X$. If Λ is in generic position we can assume that X is a copy of $\mathbf{C}^{n+1}$ with coordinates (x, t), where $x = (x_1, \ldots, x_n)$, $p = (0, dt)$, and there is a triple (Ω, Δ, Y), where Δ is an open polydisk of $\mathbf{C}^{n+1}$, Ω is a conic open neighbourhood of p and Y is a closed hypersurface of Δ, defined by a Weierstrass polynomial $f(x, t) = t^m + \sum_{i=0}^{m-1} a_i(x) t^i$ such that $T^*_Y X \cap \pi^{-1}(\Omega)$ is a representative of Λ.

There are $C > 0$ and a neighbourhood U of 0 such that

$$|t| \leqslant C|x| \text{ if } (x, t) \in Y \cap \Delta. \tag{1}$$

LEMMA 2.2. *Let F be a perverse sheaf with micro-support in generic position. Then*

$$H^j(F) = 0, \qquad \text{for all } j \geqslant 2.$$

Proof. Let us fix $q \in X$. If $SS(F) \cap \pi^{-1}(q) = \{q\}$ then F is a local system in a neighbourhood of q and $H^j(F) = 0$ in a neighbourhood of q, for all $j \geqslant 1$. If $SS(F) \cap \pi^{-1}(q) \neq \{q\}$ then there is $p \in \pi^{-1}(q)$, $p \neq q$, such that

$$SS(F) \cap \pi^{-1}(q) = \mathbf{C}p.$$

Following Remark 2.1, we assume that $p = (0, \mathrm{d}t)$. Moreover, choosing Δ small enough, we can assume that the line

$$L_a = \{(x, t) \in \Delta : x = a\}$$

is non characteristic for all a. By the Cauchy–Kowalevsky–Kashiwara theorem (cf. chapter 5 of [K2]), $F|_{L_a}$ is a perverse sheaf for all a. Hence

$$H^j(F)_{(a,t)} = H^j(F|_{L_a})_t = 0,$$

for all $(a, t) \in \Delta$ and all $j \geqslant 2$. □

DEFINITION 2.3. Let $p \in T^*X \setminus T^*_X X$. We will denote by $\mathbf{Perv}_p(X)$ the category of germs at $\pi(p)$ of perverse sheaves F that satisfy the following conditions.

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(a) $SS(F) \cap \pi^{-1}(\pi(p)) = \mathbf{C}p$.
(b) $F_{\pi(p)} = 0$.
(c) F is concentrated in degree 0.

Let Y be the germ at $\pi(p)$ of an hypersurface Y such that $T^*_Y X \cap \pi^{-1}(\pi(p)) = \mathbf{C}p$. We will denote by $\mathbf{Perv}_p(X, Y)$ the category of germs at $\pi(p)$ of sheaves F such that $F \in \mathbf{Perv}_p(X)$ and $SS(F) \subset T^*_Y X \cup T^*_X X$.

Let $\mathcal{M}$ be a regular holonomic $\mathcal{E}_{X,p}$-module in generic position. Let $\mathcal{DR}_p(\mathcal{M})$ be the germ at $\pi(p)$ of the perverse sheaf $\mathcal{DR}(v_p\mathcal{M})$.

THEOREM 2.4. *We have an equivalence of categories*

$$\mathcal{DR}_p : \mathbf{RH}_p(\mathcal{E}_X) \to \mathbf{Perv}_p(X).$$

DEFINITION 2.5. We call the functor $\mathcal{DR}_p$ the *Microlocal Riemann–Hilbert Correspondence.*

Proof. We will follow the notations of Remark 2.1. Set $F = \mathcal{DR}(\mathcal{M})$. Locally, we can identify the germ at 0 of de Rham complex of $\mathcal{M}$ with the Koszul complex

$$K^{\cdot}\big(\mathcal{M}_0, \partial_{x_1}, \ldots, \partial_{x_n}, \partial_t\big). \tag{2}$$

Since $\partial_t : \mathcal{M} \to \mathcal{M}$ is invertible, (2) is exact. Hence

$$H^j(F)_0 = 0, \qquad \text{for all } j. \tag{3}$$

By Lemma 2.2

$$H^j(F) = 0, \qquad \text{for all } j \geqslant 2.$$

Let $\tau : \Delta \to \mathbf{C}^n$ be the projection on the first n coordinates. By Remark 2.1 and Proposition 5.4.17 of [KS] there is a neighbourhood of the origin of $\mathbf{C}^n$ where $SS(R\tau_* H^1(F))$ is contained in the zero section. Hence $R\tau_* H^1(F)$ is a local system in a neighbourhood of the origin of $\mathbf{C}^n$. Therefore $\Gamma(L_a, H^1(F))$ does not depend on a, for small a. By Remark 2.1

$$\Gamma(L_a, H^1(F)) = \oplus_{b \in Y \cap L_a} H^1(F)_b,$$

$$\Gamma(L_0, H^1(F)) = H^1(F)_0 = 0.$$

Hence, $H^1(F) = 0$.

We will now complete the proof by showing that, if $\mathcal{M} \in \mathrm{RH}(\mathcal{D}_X)$, if $\mathcal{C}har(\mathcal{M})$ is in generic position at $\pi(p)$ and if $\mathcal{DR}(\mathcal{M})_{\pi(p)} \in \mathrm{Perv}_p(X)$, then

$$\mathcal{M}_{\pi(p)} \in \mathrm{RH}_p(\mathcal{D}_X).$$

The $\mathcal{D}_{X,\pi(p)}$-module

$$M = \mathcal{E}_{X,p} \otimes_{\mathcal{D}_{X,\pi(p)}} \mathcal{M}_{\pi(p)}$$

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is in $\mathrm{RH}_p(\mathcal{D}_X)$. We have a $\mathcal{D}_{X,\pi(p)}$-linear morphism $\varphi : \mathcal{M}_{\pi(p)} \to M$, defined by $\varphi(u) = 1 \otimes u$. Let $\widetilde{M}$ be a representative of M in a small polydisk Δ. If Δ is small enough, the morphism φ extends into a morphism of $\mathcal{D}_\Delta$-modules. We have an exact sequence

$$0 \to N_1 \to \mathcal{M}|_\Delta \to \widetilde{M} \to N_2 \to 0.$$

If Δ is small enough, the $\mathcal{E}_\Delta$-modules

$$\mathcal{E}_\Delta \otimes_{\mathcal{D}_\Delta} \mathcal{M}|_\Delta \quad \text{and} \quad \mathcal{E}_\Delta \otimes_{\mathcal{D}_\Delta} \widetilde{M}$$

are isomorphic outside the zero section. Therefore N_1 and N_2 are integrable connections. Since $\mathcal{DR}(N_1)_{\pi(p)} = \mathcal{DR}(N_2)_{\pi(p)} = 0$, N_1 and N_2 vanish in a neighbourhood of $\pi(p)$. Therefore the $\mathcal{D}_{X,\pi(p)}$-modules $\mathcal{M}_{\pi(p)}$ and M are isomorphic. □

3. An Extension Result

DEFINITION 3.1. Let G be a group and let Φ a linear representation of G into a finite dimensional vector space E. Let $(g_i)_{i\in I}$ be a set of generators of G. We say that the family $(g_i)_{i\in I}$ is Φ-*hypergeometric* if there is a family $(E_i)_{i\in I}$ of linear subspaces of E such that $E = \oplus_{i\in I} E_i$, $E_i \neq 0$, for all i, and

$$\{x \in E : \Phi(g_i)(x) = x\} = \oplus_{j\neq i} E_j, \quad \text{for all } i.$$

We say that a linear representation Φ is *hypergeometric* if there is a family of generators $(g_i)_{i\in I}$ of G such that $(g_i)_{i\in I}$ is Φ-hypergeometric.

Let X_0 be a simply connected open subset of the complex plane. Let I be a finite subset of X_0. Set $X = X_0 \setminus I$. Choose $a \in X$ and $b \in I$. We say that a loop $\gamma \in \pi_1(X, a)$ is b-simple if γ is homotopic to the trivial loop in $X \cup \{b\}$. We say that a family of free generators $(\gamma_b)_{b\in I}$ of $\pi_1(X, a)$ is *simple* if γ_b is b-simple for all $b \in I$.

Let H be a local system on X. Let Φ_H be the associated linear representation of $\pi_1(X, a)$. We say that H is *hypergeometric* if there is a simple family of generators $(\gamma_b)_{b\in I}$ of $\pi_1(X, a)$ wich is Φ_H-hypergeometric.

Remark 3.2. Let Φ be a linear representation of a group G. Let $(g_i)_{i\in I}$ and $(h_i)_{i\in I}$ be two families of generators of G. If $(g_i)_{i\in I}$ is a Φ-hypergeometric family and h_i is conjugated to g_i for each i then $(h_i)_{i\in I}$ is a Φ-hypergeometric family.

EXAMPLE 3.3. Set $X_0 = \mathbf{C}$, $I = \{0, 1\}$. The sheaf of solutions of a Gauss hypergeometric differential equation on $\mathbf{C} \setminus \{0, 1\}$ is an hypergeometric local system.

DEFINITION 3.4. Let X be a complex manifold. Let Y be the germ at $q \in X$ of an hypersurface of X. We say that a system of local coordinates $(x_1, \ldots, x_n, t)$ defined on an open neighbourhood of q is adapted to Y if there is a Weierstrass polynomial

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$f(x,t) = t^m + \sum_{i=0}^{m-1} a_i(x)t^i$ defined on an open polydisk Δ such that $\widetilde{Y} \cap \Delta = f^{-1}(0)$, for some representative $\widetilde{Y}$ of Y.

Let H be the germ at q of a local system on $X \setminus Y$. We say that H is an *hypergeometric local system at q, relatively to a system of local coordinates* $(x_1, \ldots, x_n, t)$ *adapted to* Y if $\widetilde{H}|_{\tau^{-1}(a)\setminus Y}$ is an hypergeometric local system for all $a \in \tau(\Delta)$ such that $\tau^{-1}(a) \cap Y \subset Y_{reg}$. Here $\widetilde{H}$ denotes a representative of H and τ denotes the projection $(x_1, \ldots, x_n, t) \mapsto (x_1, \ldots, x_n)$.

We say that H is an *hypergeometric local system at q* if H is an hypergeometric local system at q relatively to some system of local coordinates $(x_1, \ldots, x_n, t)$ adapted to Y.

Remark 3.5. If H is an hypergeometric local system at q then $\Phi_{\widetilde{H}}$ is an hypergeometric representation of the fundamental group of the complementary of Y.

LEMMA 3.6. *Let X_0 be a simply connected open subset of the complex plane. Let I be a finite subset of X_0. Let F be a constructible sheaf on X_0 that is locally constant on $X_0 \setminus I$. Choose $a \in X \setminus I$, and paths $\delta_b : [0,1] \to X$, $b \in I$ such that $\delta_b([0,1]) \cap \delta_c([0,1]) = \{a\}$, if $b \neq c$, and $\delta_b^{-1}(I) = \{1\}$. Let $\varphi_b : F_b \to F_a$ denote the analytic continuation along δ_b, for all $b \in I$. We have an isomorphism of complexes*

$$\mathrm{R}\Gamma(X_0, F) \xrightarrow{\sim} F_a \oplus \oplus_{b\in I} F_b \xrightarrow{\varphi} F_a^I, \tag{4}$$

where $\varphi(s, (s_b)_{b\in I}) = (s + \varphi_b(s_b))_{b\in I}$.

Proof. Set $L = \cup_{b\in I}\delta_b([0,1])$. There is a family $(W_t)_{t\in\mathbf{R}}$ of limited open subsets of $\mathbf{C}$ such that

$$W_t = \cup_{s<t} W_s, \forall t \in \mathbf{R}, \cup_t W_t = X_0, \cap_t W_t = L.$$

By the non characteristic deformation lemma (Proposition 2.7.2 of [KS]), $H^k(W_t, F)$ does not depend on t. Therefore

$$H^k(X_0, F) = H^k(L, F), \quad \text{for all } k.$$

Since L is a locally compact topological space of dimension 1, $H^k(X_0, F) = 0$, for all $k \geqslant 2$. Set $U = L \setminus I$, $U_b = \delta_b([0,1])$, for all $b \in I$. We have

$$H^k(U \cap U_b, F) = H^k(U, F) = \begin{cases} F_a, & \text{if } k = 0, \\ 0, & \text{if } k \neq 0. \end{cases}$$

$$H^k(U_b, F) = \begin{cases} F_b, & \text{if } k = 0, \\ 0, & \text{if } k \neq 0. \end{cases}$$

□

LEMMA 3.7. *Let I be a finite set. Let E be a vector space and for each $j \in I$ let $\varphi_j : V_j \to E$ be a linear map. The following statements are equivalent.*

1. *The linear map from* $E \oplus \oplus_{i\in I} V_i$ *into* E^I, *defined by*

$$(x, (x_j)_{j\in I}) \mapsto (x + \varphi_j(x_j))_{j\in I},$$

is an isomorphism.
2. *The maps* φ_j *are injective and* E *is isomorphic to* $\oplus_{j\in I} E/V_j$.
3. *There is a family* $(E_j)_{j\in J}$ *of subspaces of* E *such that*

$$\varphi_k(V_k) = \oplus_{j\neq k} E_j, \quad \text{for all } k \in I. \qquad \square$$

In order to prove Theorem 3.10 we will need to prove a Hartogs type theorem for perverse sheaves with micro-support in generic position.

Let F be a constructible complex on a complex manifold X. Let p be a non singular point of $SS(F)$. We say that F is *perverse at* p if there is a submanifold Y of X and there is a finite dimensional vector space L such that F is isomorphic to a shift of L_Y in the localization $D^b(X; p)$ of $D^b(X)$.

By the results of section 10.3 of [KS], the following statements are equivalent.

(a) F is a perverse sheaf.
(b) For each non singular point p of $SS(F)$ such that π: $SS(F) \to X$ has constant rank in a neighbourhood of p, F is perverse at p.
(c) For each irreducible component Λ of $SS(F)$ there is a point $p \in \Lambda$ such that F is perverse at p.

THEOREM 3.8. *Let* F *be a constructible complex on a complex manifold* X. *If the micro-support of* F *is in generic position at* p, *for all* $p \in SS(F) \setminus T^*_X X$ *and if* F *is perverse outside an analytic subset* Z *of* X *of codimension two, then* F *is perverse.*

Proof. Let Λ be an irreducible component of $SS(F)$. Since $\Lambda \not\subset \pi^{-1}(Z)$, there is a point $p \in \Lambda$ in the conditions of statement (b). Hence all irreducible components of $SS(F)$ verify statement (c). $\square$

Let Y be a hypersurface of a complex manifold X. Consider the open inclusions $j : X \setminus Y \hookrightarrow X \setminus Y_{sing}$, $i : X \setminus Y_{sing} \hookrightarrow X$.

LEMMA 3.9. *Let* $p \in T^*X \setminus T^*_X X$. *Let* Y *be the germ at* $\pi(p)$ *of a hypersurface such that* $T^*_Y X$ *is in generic position. If* $F \in \mathrm{Perv}_p(X, Y)$ *then the following statements hold.*

(a) *There is an open neighbourhood* W *of* $\pi(p)$ *such that for all* $y \in Y_{reg} \cap W$ *there is an open neighbourhood* U *of* y, *there is a local system* S *on* U *and there is a perverse sheaf* G *on* U *such that* $F|_U = S \oplus G$ *and all the eigenvalues of the monodromy around* Y *of* $G|_{U\setminus Y}$ *are different from* 1.
(b) *The sheaves* $j_* j^{-1} i^{-1} F$ *and* $i^{-1} F$ *are isomorphic.*

Proof. We can assume that there is a system of local coordinates $(y_1, \dots, y_n, s)$ on U such that $y = 0$ and $Y \cap U = \{s = 0\}$. Let $\mathcal{M}_{k,\lambda}$ be the $\mathcal{D}_U$-module given by a

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generator u and relations

$$(s\partial_s - \lambda)u = 0, \quad \partial_{y_i}u = 0, \quad i = 1, \dots, n.$$

The monodromy along Y of the local system $\mathcal{DR}(\mathcal{M}_{k,\lambda})|_{U\setminus Y}$ equals ${}^tJ_{l,\lambda}^{-1}$.

Let M be an element of $RH_p(\mathcal{E}_X)$ associated to F by the microlocal Riemann–Hilbert correspondence. Let $\mathcal{M}$ be a representative of the $\mathcal{D}_{X,\pi(p)}$-module M on an open neighbourhood W of $\pi(p)$. We can assume that there is a section u of $\mathcal{M}$ on W such that the germ of u at z generates the fiber of $\mathcal{M}$ at z, for all $z \in W$. We can assume that the map π: $T^*_Y X \cap \pi^{-1}(W) \to W$ is injective. Given $y \in Y_{reg} \cap W$, let U be a simply connected open neighbourhood of y contained in U such that there is a system of local coordinates $(y_1, \dots, y_n, s)$ on U verifying the conditions refered above. Let q be a point of $T^*_Y X$ such that $\pi(q) = y$. We can assume that there is a $\mathcal{D}_U$-module $\mathcal{N}$ that is a representative of the $\mathcal{D}_{X,\pi(q)}$-module $\mathcal{M}_q$. Moreover, we can assume that there is a section v of $\mathcal{N}$ on U that is a representative of the element $1 \otimes u$ of the $\mathcal{D}_{X,\pi(q)}$-module $\mathcal{M}_q$. There is one and only one morphism of $\mathcal{D}_U$-modules $\psi : \mathcal{M}|_U \to \mathcal{N}$ such that $\psi(u|_U) = v$. The kernel and the cokernel of ψ are integrable connexions. Since the fiber at y of $\mathcal{DR}(\mathcal{N})$ vanishes, the cokernel of ψ vanishes. Setting $G = \mathcal{DR}(\mathcal{N})$, we have the exact sequence

$$0 \to S \to F|_U \to G \to 0, \tag{5}$$

where S is a local system on U. The $\mathcal{D}_U$-module $\mathcal{N}$ is a direct sum of $\mathcal{D}_U$-modules of type $\mathcal{M}_{k,\lambda}$. The exact sequence (5) splits since the following statements are equivalent.

(i) The complex $\mathcal{DR}(\mathcal{M}_{k,\lambda})_0$ vanishes.
(ii) The eigenvalues of the monodromy of $\mathcal{DR}(\mathcal{M}_{k,\lambda})|_{X\setminus Y}$ are all different from 1.
(iii) The complex number λ is not an integer.

Statement (b) is a straightforward consequence of statement (a). □

THEOREM 3.10. *Let $p \in T^*X \setminus T^*_X X$. Let Y be the germ at $\pi(p)$ of a hypersurface such that $T^*_Y X$ is in generic position. The following statements hold.*

1. *If $F \in \mathrm{Perv}_p(X, Y)$, then $F|_{X\setminus Y}$ is a hypergeometric local system at $\pi(p)$.*
2. *If S is a local system on $X \setminus Y$, $i_!j_*S \in \mathrm{Perv}_p(X, Y)$ if and only if S is a hypergeometric local system at $\pi(p)$.*
3. *If $F \in \mathrm{Perv}_p(X, Y)$, then $F = i_!j_*j^{-1}i^{-1}F$.*

Proof. Set $S = F|_{X\setminus Y}$. Let $(x_1, \dots, x_n, t)$ be a system of local coordinates adapted to Y. Choose $\alpha \in \tau(\Delta)$ such that $\tau^{-1}(\alpha) \cap Y \subset Y_{reg}$. Choose $a \in \tau^{-1}(\alpha) \setminus Y$. Set $I = \tau^{-1}(\alpha) \cap Y$, $E = S_a$. Let $(\gamma_b)_{b\in I}$ be a simple family of generators of $\tau^{-1}(\alpha) \setminus Y$. Set $V_b = \ker(\gamma_b - Id)$, for each $b \in I$. By Lemma 3.9 V_b equals the fiber of F at b. By Lemma 3.6 and Lemma 3.7 there is a family $(E_b)_{b\in I}$ of linear subspaces of E such that $V_b = \oplus_{c\neq b} E_c$. Therefore the family $(\gamma_b)_{b\in I}$ is Φ_S-hypergeometric.

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We will now prove statement (2). Let S be an hypergeometric local system on $X \setminus Y$ at $\pi(p)$. We can assume that we are in the conditions of Remark 2.1. Set $F = i_! j_* S$. It follows from (1) that we can assume that $\Delta = \tau^{-1}(\tau(\Delta))$.

By Lemma 3.9 and Theorem 3.8, in order to prove statement (2) it is enough to show that

$$SS(F) \subset T^*_Y X \cup T^*_X X. \tag{6}$$

By Theorem 8.5.5 of [KS], the micro-support of F is a closed Lagrangian subvariety of T^*X. Hence we can assume that there is a finite family $(Z_i)_{i\in I}$ of irreducible complex analytic subsets of X such that

$$SS(F) = T^*_Y X \cup T^*_X X \cup \cup_{i\in I} T^*_{Z_i} X, \tag{7}$$

codim $Z_i \geqslant 2$, for all i, and $0 \in Z_i$, for all i. We will assume I nonempty in order to reach a contradiction. Set $Z = \cup_{i\in I} Z_i$. It follows from the definition of F that $Z \subset Y_{sing}$. Let $\rho : X \to \mathbf{C}^l$ denote the restriction to X of the linear projection $(x_1, \ldots, x_n, t) \mapsto (x_1, \ldots, x_l)$, where $l = dim Z$. We can assume the following facts.

(α) The restriction of ρ to Z is a finite map. The set $\rho(Z)$ is an open neighbourhood of the origin of $\mathbf{C}^l$. There is a dense Zariski open set ω of $\rho(Z)$ such that

$$\rho^{-1}(\omega) \cap Z_i \neq \emptyset \Rightarrow dim Z_i = l,$$

$$\rho^{-1}(\omega) \cap Z \subset Z_{reg}.$$

The set $\mathbf{P}^*_Y X$ is the graph of a continous map from Y into the dual of $\mathbf{P}^n(\mathbf{C})$. Therefore we can assume that for each two-dimensional linear subspace V of the dual of $\mathbf{C}^{n+1}$

$$V \not\subset \cup_{a\in Y} (T^*_Y X)_a. \tag{8}$$

There is $y_0 \in Z \cap \rho^{-1}(\omega)$ such that Z is transversal to the fibers of ρ in a neighbourhood of y_0. There is one and only one i_0 such that $y_0 \in Z_{i_0}$. It follows from (8) that there is $\beta \in \left(T^*_{Z_{i_0}} X\right)_{y_0}$ such that

$$\beta \notin \cup_{a\in Y} \left(T^*_Y X\right)_a. \tag{9}$$

We will reach the desired contradiction showing that

$$\beta \notin SS(F)_{y_0}. \tag{10}$$

Let $\tilde{\rho}$ denote the restriction to X of the real linear projection $(x_1, \ldots, x_n, t) \mapsto (x_1, \ldots, x_l, \Im(x_n))$. Let σ denote the linear projection $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_l)$. We can assume that the following facts hold.

(β) There is an open neighbourhood $\tilde{\omega}$ of $\rho(y_0)$ such that the restriction of $\tilde{\rho}$ to $Z \cap \rho^{-1}(\tilde{\omega})$ is injective and $\beta = d(\Re x_n)$.

(γ) There is a dense Zariski open subset $\tilde{\omega}_0$ of $\sigma^{-1}(\tilde{\omega})$ such that $Y \cap \tau^{-1}(\tilde{\omega}_0) \subset Y_{reg}$.

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Let δ, ε be two positive real numbers. Set

$$H_\varepsilon = \{(x,t) : \Re(x_n) \geqslant -\varepsilon\},$$
$$L = \{(x,t) : \Re(x_n) = -\varepsilon\},$$
$$\gamma = \{(x,t) : x_1 = \cdots = x_{n-1} = 0, \quad \Im(x_n) = 0, \Re(x_n) \leqslant -\delta|t|\}.$$

Let E denote the real linear subspace of $\mathbf{C}^{n+1}$ spanned by γ. The following statements hold.

(δ) If ε is small enough $L \cap Y \cap (E+y) \subset Y_{reg}$, for all y close enough to y_0.

(ζ) If we choose δ small enough then $L \cap Y \cap (\gamma + y) = L \cap Y \cap (E+y)$, for all y close enough to y_0.

(η) If we choose δ small enough, the conormal of the regular part of the boundary of γ as a subset of E does not intersect $\cup_{a\in Y}(T^*_Y X)_a$.

(θ) If y is close enough to y_0, the set $Y \cap (E+y)$ contains at most a point of the singular locus of Y. If this point exists, it belongs to the set $\rho^{-1}(\tilde{\omega}) \cap Z$.

Since S is an hypergeometric local system and the intersection of Y with $L \cap (y+E)$ equals the intersection of Y with a fiber of τ, it follows from (δ), (ζ), Lemma 3.9, Lemma 3.7 and Lemma 3.6 that

$$\mathrm{R}\Gamma(L \cap (y+\gamma), F) = 0, \qquad \text{for all } y \text{ close enough to } y_0. \tag{11}$$

If

$$H_\varepsilon \cap (y+\gamma) \cap Y_{sing} = \emptyset \tag{12}$$

then

$$\mathrm{R}\Gamma(L \cap (y+\gamma), F) \xrightarrow{\sim} \mathrm{R}\Gamma(H_\varepsilon \cap (y+\gamma), F). \tag{13}$$

The proof of (13) is quite similar to the proof of the implication $(1)_\omega \Rightarrow (3)$ of the Proposition 5.1.1 of [KS]. The main ingredients of the proof of (13) are (η) and the non characteristic deformation lemma.

If the equality (12) does not hold, it follows from (θ) that there is $z_0 \in Z_{reg}$ such that

$$(y + H_\varepsilon) \cap (y+\gamma) \cap Y_{sing} = (y+E) \cap Y_{sing} = \{z_0\}.$$

We will assume that $y \neq z_0$. The proof in the case $y = z_0$ is quite similar. Notice that

$$z_0 + \theta_1(y - z_0) + \gamma \subset int_{y+E}(z_0 + \theta_2(y-z_0) + \gamma)$$

if and only if $\theta_1 < \theta_2$. Moreover,

$$\cap_{\theta>1}\,((z_0 + \theta(y-z_0) + \gamma) \cap (z_0 + H_{\theta\varepsilon})) = (y+\gamma) \cap (z_0 + H_\varepsilon), \tag{14}$$

$$\cap_{\theta>0}\,((z_0 + \theta(y-z_0) + \gamma) \cap (z_0 + H_{\theta\varepsilon})) = \{z_0\}. \tag{15}$$

It follows from (δ), (η), (8) and the non-characteristic deformation lemma that

$$\mathrm{R}\Gamma\big(int_{y+E}(z_0+\theta(y-z_0)+\gamma)\cap(z_0+H_{\theta\varepsilon})),F|_{y+E}\big),$$

does not depend on θ, for $\theta\in]0,\eta[$ and for some $\eta>1$. Since the sets $\{z_0\}$ and $H_\varepsilon\cap(y+\gamma)$ are compact,

$$0=\mathrm{R}\Gamma(\{z_0\},F)\xrightarrow{\sim}\mathrm{R}\Gamma((z_0+H_\varepsilon)\cap(y+\gamma),F).$$

We will now prove statement (3). It follows from Lemma 3.9 that we have an exact sequence

$$0\to i_!j_*j^{-1}i^{-1}F\to F\to G\to 0.$$

It follows from the Lemma 3.9 that $\pi(SS(G)\setminus T^*_XX)$ is an analytic subset of codimension greater or equal than two of X. Since $SS(G)$ is in generic position, $G=0$.

COROLLARY 3.11. *Let X be a complex manifold and q a point of X. Let Y be a germe at q of an hypersurface of X. Assume that there is $p\in\pi^{-1}(q)$ such that $T^*_YX\cap\pi^{-1}(q)=\mathbf{C}p$. Then a germ at q of a local system H on $X\setminus Y$ is hypergeometric if and only if H is hypergeometric relatively to all systems of local coordinates of X adapted to Y.*

Proof. By 3.10 (2) $i_!j_*H$ is a perverse sheaf at q. By the proof of 3.10 (1) H is a hypergeometric local system relatively to all systems of local coordinates of X adapted to Y. □

COROLLARY 3.12. *Let X be a complex manifold. Let Y be a germ of a point a of an hypersurface Y of X, such that T^*_YX is in generic position. Let p be a point of T^*_YX such that $\pi(p)=a$. Let Λ be the germ of a Lagrangian variety of T^*X that is isomorphic to T^*_YX. The category of germs at p of regular holonomic systems with support on Λ and the category of hypergeometric representations of the fundamental group of the complementary of Y are equivalent.*

References

[A] Andronikof, E.: A microlocal version of the Riemann–Hilbert correspondence, *Topol. Methods Nonlinear Anal.* **2** (1994), 417–425.

[Bj] Björk, J. E.: *Analytic $\mathcal{D}$-Modules and Applications*, Math. Appl. 247, Kluwer Acad. Publ., Dordrecht, 1993.

[GMV] Gelfand, S., MacPherson, R. and Vilonen, K.: Perverse sheaves and quivers, *Duke Math. J.* **83** (1996), 621–643.

[GR] Gunning, R. C. and Rossi, H.: *Analytic Functions of Several Complex Variables*, Prentice-Hall, Englewood Cliffs, 1965.

[K1] Kashiwara, M.: The Riemann–Hilbert problem for holonomic systems, *Publ. RIMS Univ. Kyoto* **20** (1984), 319–365.

[K2] Kashiwara, M.: *Systems of Microdifferential Equations*, notes by T. Monteiro-Fernandes, Progr. in Math. 34 Birkhäuser, Basel, 1983.

[KK] Kashiwara, M. and Kawai T.: On holonomic systems of microdifferential equations. III – systems with regular singularities, *Publ. RIMS, Kyoto Univ.* **17** (1981), 813–979.

[KS] Kashiwara, M. and Schapira, P.: *Sheaves on Manifolds*, Grundlehren Math. Wiss. 292, Springer-Verlag, Berlin, 1992.

[MV] MacPherson, R. and Vilonen, K.: Elementary construction of perverse sheaves, *Invent. Math.* **84** (1986), 403–435.

[Ma] Maisonobe, Ph.: $\mathcal{D}$-Modules et faisceaux pervers dont le suport singulier est une courbe plane, Thèse de doctorat d'Etat, Univ. de Nice (1985).

[Mb1] Mebkhout, Z.: Une équivalence de catégories, *Composito Math.* **51** (1984), 51–62.

[Mb2] Mebkhout, Z.: Une autre équivalence de catégories, *Composito Math.* **51** (1984), 63–88.

[NS] Neto, O. and Silva, P. C.: On holonomic systems of multiplicity one with support on the conormal of a generalized cusp, submitted.

[Na] Narvaez, L.: Cycles évanescents et faisceaux pervers: cas des courbes planes irreducibles, *Compositio Math.* **65** (1988), 321–347.

[S] Schapira P.: *Microdifferential Systems in the Complex Domain*, Springer-Verlag, New York, 1985.

[SKK] Sato, M., Kashiwara, M. and Kawai, T.: *Hyperfunctions and Pseudodifferential Equations*, Lecture Notes in Math. 287, Springer-Verlag, New York, 1973, pp. 265–529.