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COHOMOLOGY OF A REAL TORIC VARIETY AND SHELLABILITY OF POSETS ARISING FROM A GRAPH

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Abstract Given a graph G without loops, the pseudograph associahedron P_G is a smooth polytope, so there is a projective smooth toric variety X_G corresponding to P_G. Taking the real locus of X_G, we have the projective smooth real toric variety $X_G^{\mathbb{R}}$. The integral cohomology groups of $X_G^{\mathbb{R}}$ can be computed by studying the topology of certain posets of even subgraphs of G; such a poset is neither pure nor shellable in general. We completely characterize the graphs whose posets of even subgraphs are always shellable. It follows that we get a family of projective smooth real toric varieties whose integral cohomology groups are torsion-free or have only 2-torsion.

Keywords: shellable poset; chain-lexicographic-shellability; even subgraph; pseudograph associahedron; real smooth toric variety; Betti number

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1. Introduction

A convex polytope of dimension n is *simple* if there are exactly n facets intersecting at each vertex. A simple convex polytope P is *smooth* (or *Delzant*) if the normal fan of P is unimodular, that is, each cone in the fan is spanned by an integral basis. The importance of smooth polytopes stems from the fact that each smooth polytope P corresponds to a projective smooth toric variety X_P, see [11]. Taking the real locus of a projective smooth toric variety X_P, we obtain a smooth manifold $X_P^{\mathbb{R}}$ of dimension n, which is also known as a *projective smooth real toric variety*.

In the late 1970s, it was known that the cohomology of a smooth compact toric variety is torsion-free and the integral cohomology ring $H^*(X_P;\mathbb{Z})$ of a projective smooth toric variety X_P can be explicitly described by the corresponding smooth polytope P, see [13]. On the other hand, the topology of the real locus $X_P^{\mathbb{R}}$ is much more complicated than that of X_P in general. In 1985, Jurkiewicz [14] showed that $H^*(X_P^{\mathbb{R}};\mathbb{Z}_2)$ can be explicitly formulated via P just as $H^*(X_P;\mathbb{Z})$. Yet, $X_P^{\mathbb{R}}$ may have p-torsion for arbitrary $p > 1$ in its cohomology in general, and for that reason, there was no significant progress in this direction for a long time. Recently, there have been some attempts to computing the integral cohomology of a real toric variety [5, 10, 19].

Let P be a smooth polytope of dimension n and $\mathcal{F}(P) = \{F_1, \dots, F_m\}$ the set of all the facets of P. In the normal fan of P, each one-dimensional cone is generated by the primitive integral vector $\mathbf{n}_j$ normal to a facet $F_j, j = 1, \dots, m$. Define a map $\lambda\colon \mathcal{F}(P) \to \mathbb{Z}_2^n$ by taking $\lambda(F_j) \equiv \mathbf{n}_j \pmod 2$. Then λ can be represented by a $\mathbb{Z}_2$-matrix of size $n \times m$ as follows:

$$\Lambda_P = \begin{pmatrix} \lambda(F_1) & \cdots & \lambda(F_m) \end{pmatrix}.$$

For $\omega \in \mathbb{Z}_2^m$, let P_ω be the union of facets F_j such that the jth entry of ω is nonzero. For each $S \subset [n]$, by summing the ith rows of Λ_P for all $i \in S$, we obtain the vector $\omega_S \in \mathbb{Z}_2^m$. Let K_P denote the simplicial complex dual to P and $K_{P,S}$ denote the simplicial subcomplex of K_P dual to P_{ω_S}. That is,

> $K_{P,S}$ is the simplicial complex on $\{j \in [m] \mid \text{the } j\text{th entry of } \omega_S \text{ is nonzero}\}$ such that $\sigma = \{j_1, \dots, j_k\} \in K_{P,S}$ if and only if $F_{j_1} \cap \cdots \cap F_{j_k} \neq \emptyset$ in P_{ω_S}.

Recently, Cai and Choi [5, Theorem 1.1] showed that the integral cohomology of $X_P^{\mathbb{R}}$ is completely determined by the reduced integral cohomology of $K_{P,S}$ ($S \subset [n]$) and the h-vector of P. Their formulation says that the integral cohomology of $X_P^{\mathbb{R}}$ is torsion-free or has only 2-torsion if and only if the integral cohomology of $K_{P,S}$ is torsion-free for each $S \subset [n]$. Moreover, the reduced Betti numbers of $K_{P,S}$ ($S \subset [n]$) and the h-vector of P completely determine the integral cohomology of $X_P^{\mathbb{R}}$ if the integral reduced cohomology of $K_{P,S}$ is torsion-free for every $S \subset [n]$. In particular, the reduced Betti numbers of $K_{P,S}$ ($S \subset [n]$) determine the Betti numbers of $X_P^{\mathbb{R}}$. Naturally, the following question arises:

Question 1.1. Find a family of smooth polytopes P such that the integral cohomology of $K_{P,S}$ is torsion-free for every $S \subset [n]$, where $n = \dim P$. (This is equivalent to finding a family of projective smooth real toric varieties $X^{\mathbb{R}}$ such that $H^*(X^{\mathbb{R}}; \mathbb{Z})$ is torsion-free or has only 2-torsion.)

Given a simplicial complex K, it is not easy to determine whether $H^*(K; \mathbb{Z})$ is torsion-free or not. Furthermore, $H^*(K_{P,S}; \mathbb{Z})$ may have p-torsion for arbitrary $p > 1$, see [10]. Thus, the above question seems to cover a wide scope.

Shellability is a combinatorial property of simplicial complexes with strong topological consequences. A simplicial complex K is *shellable* if its maximal simplices can be arranged in a linear order $F_1, F_2, \dots, F_t$ such that the subcomplex $(\sum_{i=1}^{k-1} F_i) \cap F_k$ is pure and $(\dim F_k - 1)$-dimensional for all $k = 2, \dots, t$. It was shown in [3] that a shellable simplicial complex is homotopy equivalent to a wedge of spheres (in varying dimensions). It follows that the integral cohomology of a shellable simplicial complex is torsion-free. Thus, we can ask to find a family of smooth polytopes P such that $K_{P,S}$ is shellable for every subset S in the set $\{1, 2, \dots, \dim(P)\}$.

Now we restrict our attention to the real toric varieties arising from graphs. Throughout this paper, a graph permits multiple edges but no loops. A graph is *simple* if it does not have multiple edges. A *bundle* is a maximal set of multiple edges with the same pair of endpoints.

For a graph G, the pseudograph associahedron P_G is a smooth polytope, and hence we have the projective smooth toric variety X_{P_G} and the projective smooth real toric variety $X^{\mathbb{R}}_{P_G}$ as well. See §3.2 for the construction of P_G. For convenience, we use the notation $X_G = X_{P_G}$ and $X^{\mathbb{R}}_G = X^{\mathbb{R}}_{P_G}$. We also refer the readers to [8, §2] for a more detailed description of P_G.

For a simple graph G, it was shown in [9] that $\widetilde{H}^*(K_{P_G,S};\mathbb{Z})$ is torsion-free for every subset S in the set $\{1, 2, \ldots, \dim(P_G)\}$, which implies that the polytope P_G belongs to the family to be found in Question 1.1. The main contribution of their result is finding a pure shellable poset $\mathcal{P}^{\mathrm{even}}_H$ such that $\Delta(\overline{\mathcal{P}^{\mathrm{even}}_H})$, the order complex of the proper part of $\mathcal{P}^{\mathrm{even}}_H$, is homotopy equivalent to the complement $K_{P_G} \setminus K_{P_G,S}$. Here, H is the subgraph of G determined by S (see [9, §4]), and $\mathcal{P}^{\mathrm{even}}_H$ is a poset consisting of the induced subgraphs of H whose connected components are of even order, including $\emptyset$ and H, ordered by the subgraph containment. Note that the reduced cohomology groups of $K_{P_G,S}$ are determined by the reduced homology groups of $\Delta(\overline{\mathcal{P}^{\mathrm{even}}_H})$ by the Alexander duality since K_{P_G} is homeomorphic to a sphere.

The work of [9] on simple graphs was generalized to graphs (allowing multiple edges) in [8]. Namely, for each simplicial complex $K_{P_G,S}$, there is a poset $\mathcal{P}^{\mathrm{even}}_{H,A}$ such that $\Delta(\overline{\mathcal{P}^{\mathrm{even}}_{G,A}})$ is homotopy equivalent to the complement $K_{P_G} \setminus K_{P_G,S}$. Here, H is a graph determined by S, obtained from G by deleting some vertices and replacing some bundles with simple edges, and A is a set of vertices and multiple edges of H such that $|A \cap V(H')| \equiv 0 \pmod 2$ for each connected component H' of H with the following properties:

(1) each vertex that is incident to only simple edges of H is contained in A, and
(2) $B \cap A \neq \emptyset$ and $|B \cap A| \equiv 0 \pmod 2$ for each bundle B of H.

We call A an *admissible collection* of H. The poset $\mathcal{P}^{\mathrm{even}}_{H,A}$ is defined to be a poset consisting of all the subgraphs I of H such that I includes at least one edge between every pair of vertices in I if such edges exist in H, and each connected component of I has an even number of elements in A, including both $\emptyset$ and H, ordered by the subgraph containment. All definitions are elaborated in §3.

Now we let $\mathcal{A}^*(G)$ be the set of all pairs (H, A), where H is a graph obtained from G by deleting some vertices and replacing some bundles with simple edges, and A is an admissible collection of H. In order to show $K_{P_G,S}$ is torsion-free for every S, it is sufficient to check $\Delta(\overline{\mathcal{P}^{\mathrm{even}}_{H,A}})$ for every $(H, A) \in \mathcal{A}^*(G)$. Unlike simple graphs, for a non-simple graph H, the poset $\mathcal{P}^{\mathrm{even}}_{H,A}$ is neither pure nor shellable in general, see §4. Hence, it is natural to ask the following, which is a subproblem of Question 1.1.

Question 1.2. [8]. Find all graphs G such that $\mathcal{P}^{\mathrm{even}}_{H,A}$ is shellable for every $(H, A) \in \mathcal{A}^*(G)$.

Our main result is the following, which answers Question 1.2.

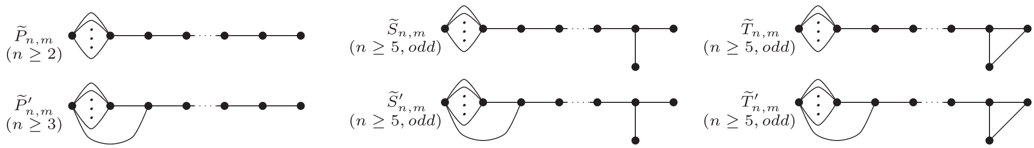

Figure 1.1. Non-simple connected graphs with n vertices and m multiple edges ($m \geq 2$).

Theorem 1.3. *Let G be a graph. Then $\mathcal{P}_{H,A}^{\text{even}}$ is shellable for every $(H, A) \in A^*(G)$ if and only if each connected component of G is a simple graph or one of the graphs in the following figure.*

To show the main theorem, we use the notion of chain-lexicographic shellability (CL-shellability for short) of posets; this tool is based on labelling the edges of the Hasse diagram of a poset in a certain way. Note that CL-shellability is stronger than shellability, and there is an example of a shellable poset with no CL-shelling, see [21]. We refer the readers to [20] and references therein regarding lexicographic shellability, but we lay out some basic facts in §2.

Björner and Wachs [2] proved that if a bounded poset $\mathcal{P}$ is CL-shellable, then the homotopy type of $\Delta(\overline{\mathcal{P}})$, the order complex of the proper part of $\mathcal{P}$, is determined by the information of the falling chains of $\mathcal{P}$ with a CL-labelling. Hence, our results give a way to compute the homotopy type of $K_{P_G,S}$, and hence we can compute the integral cohomology groups of $X_G^{\mathbb{R}}$ explicitly for a graph G in Theorem 1.3.

This paper is organized as follows. Section 2 collects some basic definitions and important facts about a poset and its shellability. In §3, we provide our motivation from the cohomology of real toric varieties associated with a graph. We also explain Theorem 1.3, which is the main theorem. Section 4 proves the necessary condition of Theorem 1.3, which gives a possible list of graphs G such that $\mathcal{P}_{H,A}^{\text{even}}$ is shellable for every $(H, A) \in \mathcal{A}^*(G)$. Section 5 proves the sufficient condition of Theorem 1.3, which shows the CL-shellability of each poset $\mathcal{P}_{H,A}^{\text{even}}$ for a graph G in the list and $(H, A) \in \mathcal{A}^*(G)$. In Section 6, we determine the homotopy type of $\Delta(\overline{\mathcal{P}_{G,A}^{\text{even}}})$ by considering the falling chains of $\mathcal{P}_{G,A}^{\text{even}}$ for a graph G in Figure 1.1. In §7, as an application of our result, we compute the Betti numbers of the projective smooth real toric variety associated with the graph $\widetilde{P}_{n,2}$. In Appendix 1, we add a sketch of the proof of Proposition 3.4, which is obtained by combining several results from [8].

2. Preliminaries: Shellability of a poset

In this section, we prepare some notions and basic facts about a poset and its shellability. See [20] for a more detailed explanation about this section.

We only consider a finite poset in this paper. Let $\mathcal{P}$ be a poset (partially ordered set). For two elements $x, y \in \mathcal{P}$, we say y *covers* x, denoted by $x \lessdot y$, if $x < y$, and there is no z such that $x < z < y$. We also call it a cover $x \lessdot y$. One represents $\mathcal{P}$ as a mathematical diagram, called a *Hasse diagram*, in a way that a point in the plane is drawn for each element of $\mathcal{P}$, and a line segment or curve is drawn upward from x to y whenever y covers x. A *chain* of $\mathcal{P}$ is a totally ordered subset σ of $\mathcal{P}$, and we say the *length* $\ell(\sigma)$ of σ is

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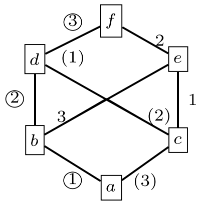
$|\sigma| - 1$. We say $\mathcal{P}$ is *pure* if all maximal chains have the same length. The *length* $\ell(\mathcal{P})$ of $\mathcal{P}$ is the length of a longest chain of $\mathcal{P}$. For $x \leq y$ in $\mathcal{P}$, let $[x, y]$ denote the (closed) interval $\{z \in \mathcal{P}\colon x \leq z \leq y\}$. We say $\mathcal{P}$ is *semimodular* if for all $x, y \in \mathcal{P}$ that cover $a \in \mathcal{P}$, there is an element $b \in \mathcal{P}$ that covers both x and y. If every closed interval of $\mathcal{P}$ is semimodular, then $\mathcal{P}$ is *totally semimodular*. If $\mathcal{P}$ has a unique minimum element, it is usually denoted by $\hat{0}$ and referred to as the bottom element. Similarly, the unique maximum element, if it exists, is denoted by $\hat{1}$ and referred to as the top element. An element of $\mathcal{P}$ that covers the bottom element is called an *atom*. We say $\mathcal{P}$ is *bounded* if it has the elements $\hat{0}$ and $\hat{1}$. The *order complex* of $\mathcal{P}$, denoted by $\Delta(\mathcal{P})$, is an abstract simplicial complex whose faces are the chains of $\mathcal{P}$. Note that if $\mathcal{P}$ has either $\hat{0}$ or $\hat{1}$, then $\Delta(\mathcal{P})$ is contractible; hence, we usually remove the top and bottom elements, and then study the topology of the remaining part. The *proper part* of a bounded poset $\mathcal{P}$ with length at least one is defined to be the poset $\overline{\mathcal{P}} := \mathcal{P} \setminus \{\hat{0}, \hat{1}\}$.

The notion of shellability first appeared in the middle of the nineteenth century in the computation of the Euler characteristic of a convex polytope [16]. A simplicial complex K is *shellable* if its maximal simplices can be arranged in a linear order $F_1, F_2, \ldots, F_t$ in such a way that the subcomplex $(\sum_{i=1}^{k-1} F_i) \cap F_k$ is pure and $(\dim F_k - 1)$-dimensional for all $k = 2, \ldots, t$. Such an ordering of the facets is called a *shelling*. A poset $\mathcal{P}$ is *shellable* if its order complex $\Delta(\mathcal{P})$ is shellable.

The idea of lexicographic shellability is based on a technique introduced by Stanley [17, 18] for showing that the Möbius function of rank-selected subposets of certain posets alternates in sign. This technique involved labelling the edges of the Hasse diagram of the poset in a certain way. Stanley conjectured that the posets that he was considering were topological and algebraic properties of simplicial complexes implied by shellability. This conjecture was proved by Björner [1] by finding a condition on edge labellings, which implies shellability of the poset, and then the theory of lexicographic shellability was further developed in a series of papers by Björner and Wachs [2–4]. In this paper, we consider CL-shellability. CL-shellability was introduced to establish the shellability of Bruhat order on a Coxeter group [2]. It is known that every CL-shellable poset is a shellable poset, but the converse is not true in general; see [21]. We refer to the readers [20, Lecture 4] and the references therein.

Let $\mathcal{P}$ be a bounded poset and $\mathcal{ME}(\mathcal{P})$ the set of pairs $(\sigma, x \lessdot y)$ consisting of a maximal chain σ and a cover $x \lessdot y$ along that chain. For $x, y \in \mathcal{P}$ and a maximal chain r of $[\hat{0}, x]$, the closed rooted interval $[x, y]_r$ of $\mathcal{P}$ is a subposet of $\mathcal{P}$ obtained from $[x, y]$ adding the chain r. A *chain-edge labelling* of $\mathcal{P}$ is a map $\rho\colon \mathcal{ME}(\mathcal{P}) \to L$ satisfying the following: if two maximal chains coincide along their bottom d covers, then their labels also coincide along these covers. Here, L is a poset. A *chain-lexicographic labelling* (*CL-labelling* for short) of a bounded poset $\mathcal{P}$ is a chain-edge labelling such that for each closed rooted interval $[x, y]_r$ of $\mathcal{P}$, there is a unique strictly increasing maximal chain, which lexicographically precedes all other maximal chains of $[x, y]_r$. A poset that admits a CL-labelling is *CL-shellable*. Figure 2.1 shows an example of a CL-shellable poset.

We recall well-known properties on shellability which we will use. The *product* $\mathcal{P} \times \mathcal{Q}$ of two posets $\mathcal{P}$ and $\mathcal{Q}$ is the new poset with partial order given by $(a, b) \leq (c, d)$ if and only if $a \leq c$ (in $\mathcal{P}$) and $b \leq d$ (in $\mathcal{Q}$).



Labeling of the covers in chain $a < b < d < f$ is 1, 2, 3 (marked as ①, ②, ③).
Labeling of the covers in chain $a < b < e < f$ is 1, 3, 2 (marked as ①, 3, 2).
Labeling of the covers in chain $a < c < d < f$ is 3, 2, 1 (marked as (3), (2), (1)).
Labeling of the covers in chain $a < c < e < f$ is 3, 1, 2 (marked as (3), 1, 2).

Figure 2.1. A CL-labelling of a poset with four maximal chains (same example in [20]).

Theorem 2.1. ([1, 3, 4]). *The following statements hold:*

(1) *Every (closed) interval of a shellable (respectively, CL-shellable) poset is shellable (respectively, CL-shellable).*
(2) *The product of bounded posets is shellable (respectively, CL-shellable) if and only if each of the posets is shellable (respectively, CL-shellable).*
(3) *If a bounded poset is pure and totally semimodular, then it is CL-shellable.*

For a bounded poset $\mathcal{P}$ with a CL-labelling $\rho \colon \mathcal{ME}(\mathcal{P}) \to \mathbb{Z}$, a chain $\sigma : x_0 \lessdot x_1 \lessdot \cdots \lessdot x_\ell$ of $\mathcal{P}$ is called a *falling chain* if it is a maximal chain such that $\rho(\sigma, x_{i-1} \lessdot x_i) \not<_L \rho(\sigma, x_i \lessdot x_{i+1})$ for every $1 \leq i < \ell$.

Theorem 2.2. ([3]). *If a bounded poset $\mathcal{P}$ is CL-shellable, then the order complex of the proper part of $\mathcal{P}, \Delta(\overline{\mathcal{P}})$, is homotopy equivalent to a wedge of spheres. Furthermore, for every fixed CL-labelling, the ith reduced Betti*[1] *number of $\Delta(\overline{\mathcal{P}})$ is equal to the number of falling chains of $\mathcal{P}$ of length $(i+2)$.*

The poset in Figure 2.1 has exactly one falling chain $a < c < d < f$. On the other hand, the order complex of the proper part of the poset is homotopy equivalent to S^1 whose first reduced Betti number is 1.

A recursive atom ordering is an alternative approach to lexicographic shellability, which is known to be an equivalent concept of CL-shellability.

Definition 2.3. *A bounded poset $\mathcal{P}$ is said to admit a recursive atom ordering if its length $\ell(\mathcal{P})$ is 1, or $\ell(\mathcal{P}) > 1$, and there is an ordering $\alpha_1, \ldots, \alpha_t$ of the atoms of $\mathcal{P}$ satisfying the following:*

(1) *For all $j = 1, \ldots, t$, the interval $[\alpha_j, \hat{1}]$ admits a recursive atom ordering in which the atoms of $[\alpha_j, \hat{1}]$ that belong to $[\alpha_i, \hat{1}]$ for some $i < j$ come first.*
(2) *For all i, j with $1 \leq i < j \leq t$, if $\alpha_i, \alpha_j < y$, then there exist an integer k and an atom z of $[\alpha_j, \hat{1}]$ such that $1 \leq k < j$ and $\alpha_k < z \leq y$.*

For example, for the poset in Figure 2.1, if we order the atoms of each interval by alphabetical order (for the atoms of $[a, f]$, the ordering is $b \prec c$; for the atoms of $[b, f]$,

[1] For a topological space X, the ith *reduced Betti number* of X, denoted by $\widetilde{\beta}^i(X)$, is the free rank of the reduced singular cohomology group $\widetilde{H}^i(X; \mathbb{Z})$, and the ith *reduced Betti number of X over a field F*, denoted by $\widetilde{\beta}^i_F(X)$, is the dimension of $\widetilde{H}^i(X; F)$ as a vector space over F.

the ordering is $d \prec e$; and for the atoms of $[c, f]$, the ordering is $d \prec e$), then it is a recursive atom ordering.

We note that any atom ordering of a pure totally semimodular bounded poset is a recursive atom ordering, which implies (3) of Theorem 2.1. We finish the section by introducing a sketch of the proof shown in [3] that the existence of a recursive atom ordering implies CL-shellability.

Theorem 2.4. ([3]). *A bounded poset admits a recursive atom ordering if and only if it is CL-shellable.*

Sketch of proof of the 'only if' part. Suppose that a bounded poset $\mathcal{P}$ admits a recursive atom ordering, and let the atoms of $\mathcal{P}$ be ordered as $\alpha_1, \ldots, \alpha_t$. Let us give an integer labelling ρ of the bottom covers of $\mathcal{P}$ such that $\rho(\hat{0}, \alpha_i) < \rho(\hat{0}, \alpha_j)$ for all $i < j$. For each j, let $F(\alpha_j)$ be the set of all atoms of $[\alpha_j, \hat{1}]$ that cover some α_i, where $i < j$. We label the bottom covers of $[\alpha_j, \hat{1}]$ consistently with the atom ordering of $[\alpha_j, \hat{1}]$ and satisfying

$$x \in F(\alpha_j) \Rightarrow \rho(\alpha_j, x) < \rho(\hat{0}, \alpha_j) \quad \text{and} \quad x \notin F(\alpha_j) \Rightarrow \rho(\alpha_j, x) > \rho(\hat{0}, \alpha_j),$$

where ρ denotes the labelling of the bottom covers of $[\alpha_j, \hat{1}]$ as well as the original labelling of the bottom covers of $\mathcal{P}$. This labelling inductively extends to an integer CL-labelling of $[\alpha_j, \hat{1}]$. Choosing such an extension at each α_j, we obtain a chain-edge labelling ρ of $\mathcal{P}$, which is a CL-labelling of $[\alpha_j, \hat{1}]$ for all $j = 1, \ldots, t$, and hence for every rooted interval whose bottom element is not $\hat{0}$, and which extends the original labelling of the bottom covers of $\mathcal{P}$. Then one can show that the unique lexicographically first maximal chain of each interval $[0, y]$ is the only increasing maximal chain of that interval. Hence, the labelling ρ is an integer CL-labellling on $\mathcal{P}$. $\square$

3. Real toric variety arising from a graph G and a poset $\mathcal{P}^{\text{even}}_{G,A}$

In this section, we first introduce the integral cohomology of a projective smooth real toric variety and then restrict our attention to projective smooth real toric varieties arising from graphs. For a graph G, we construct the pseudograph associahedron P_G, which defines the projective smooth real toric variety $X^{\mathbb{R}}_G$. We describe the cohomology of $X^{\mathbb{R}}_G$ in terms of posets of even subgraphs of G, and then introduce our main result.

3.1. Cohomology of a real toric variety

A *toric variety* of complex dimension n is a normal algebraic variety containing an algebraic torus $(\mathbb{C}^*)^n$ as a Zariski open dense subset such that the action of the torus on itself extends to the whole variety. A *real toric variety* is the real locus of a toric variety. The fundamental theorem of toric geometry says that there is a one-to-one correspondence between the class of toric varieties of complex dimension n and the class of fans in $\mathbb{R}^n$. In particular, for a complete smooth toric variety X, the corresponding fan Σ_X is complete and smooth. Furthermore, a complete smooth toric variety X is projective if and only if Σ_X can be realized as the normal fan of a smooth polytope in $\mathbb{R}^n$.

Although the integral cohomology ring of a complete smooth toric variety was studied by Danilov [12] and Jurkiewicz [13] in the late 1970s, only little is known about the

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cohomology of complete smooth real toric varieties. For a complete smooth toric variety X and its real locus $X^{\mathbb{R}}$, the cohomology ring $H^*(X^{\mathbb{R}};\mathbb{Z}_2)$ was computed by Jurkiewicz [14] in 1985, and it has a similar form to the integral cohomology ring $H^*(X;\mathbb{Z})$. Note that the dimension of $H^i(X^{\mathbb{R}};\mathbb{Z}_2)$ as a vector space over $\mathbb{Z}_2$ is equal to h_i, where $(h_0, h_1, \dots, h_n)$ is the h-vector of K_X, the underlying simplicial sphere of the fan Σ_X.

Recently, there were several efforts to compute the integral cohomology of a real toric variety. Let P be a smooth polytope of dimension n and let $\mathcal{F}(P) = \{F_1, \dots, F_m\}$ be the set of facets of P. Then the primitive outward normal vectors of P can be understood as a function ϕ from $\mathcal{F}(P)$ to $\mathbb{Z}^n$, and the composition map $\lambda\colon \mathcal{F}(P) \xrightarrow{\phi} \mathbb{Z}^n \xrightarrow{\text{mod } 2} \mathbb{Z}_2^n$ is called the (mod 2) *characteristic function* over P. Note that λ can be represented by a $\mathbb{Z}_2$-matrix Λ_P of size $n \times m$ as

$$\Lambda_P = \begin{pmatrix} \lambda(F_1) & \cdots & \lambda(F_m) \end{pmatrix},$$

where the ith column of Λ_P is $\lambda(F_i) \in \mathbb{Z}_2^n$. For $\omega \in \mathbb{Z}_2^m$, we define P_ω to be the union of facets F_j such that the jth entry of ω is nonzero. Then the following holds:

Theorem 3.1. ([10, 19]). *Let P be a smooth polytope of dimension n and $X_P^{\mathbb{R}}$ the projective smooth real toric variety associated with P. Then the Betti numbers of $X_P^{\mathbb{R}}$ is given as follows:*

$$\beta^i(X_P^{\mathbb{R}}) = \sum_{S \subset [n]} \widetilde{\beta}^{i-1}(P_{\omega_S}), \tag{3.1}$$

where ω_S is the sum of the kth rows of Λ_P for all $k \in S$.

For $S \subset [n]$, let $K_{P,S}$ be the simplicial subcomplex of K_P dual to P_{ω_S}. Note that $K_{P,S}$ and P_{ω_S} have the same homotopy type. Hence, we can rewrite Equation (3.1) by using $K_{P,S}$:

$$\beta^i(X_P^{\mathbb{R}}) = \sum_{S \subset [n]} \widetilde{\beta}^{i-1}(K_{P,S}). \tag{3.2}$$

In general, the integral cohomology of $K_{P,S}$ may have p-torsion for arbitrary $p > 1$, and hence it is not easy to compute. Furthermore, the torsion of $H^*(K_{P,S};\mathbb{Z})$ influences the torsion of $H^*(X_P^{\mathbb{R}};\mathbb{Z})$.

Theorem 3.2. ([5]). *Let P be a smooth polytope of dimension n and $X_P^{\mathbb{R}}$ the projective smooth real toric variety associated with P. Then the integral cohomology of $X_P^{\mathbb{R}}$ is completely determined by the reduced cohomology group of $K_{P,S}$ (for $S \subset [n]$) and the h-vector of P. Moreover, the following are equivalent:*

(1) *$H^*(X_P^{\mathbb{R}};\mathbb{Z})$ has no p-torsion for every $p > 2$.*
(2) *$\widetilde{H}^*(K_{P,S};\mathbb{Z})$ is torsion-free for every $S \subset [n]$.*

See Theorem 1.1 and Corollary 1.2 of [5] for more details of the above theorem.

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3.2. Real toric variety arising from a graph

Given a graph G, the pseudograph associahedron P_G is first introduced in [7] as a generalization of a graph associahedron in [6]. In this paper, we use the construction of a pseudograph associahedron in [8], slightly different from [7]. When G is connected, our pseudograph associahedron P_G is the same as the pseudograph associahedron $\mathcal{K}G$ in [7], but if G is disconnected, we get a different polytope. In our constructon, if G consists of connected components $G_1, \ldots, G_k$, then P_G is defined to be the product $P_{G_1} \times \cdots \times P_{G_k}$.[2] We introduce the construction of P_G briefly.

For a graph G, we label the vertices and the multiple edges of G. We write a subgraph H of G as the set of the vertices of H and the edges of H in a bundle of G. For simplicity, we omit the braces and commas to represent a subset of $\mathcal{C}_G$ and we always denote it in a way that the vertices precede the multiple edges. We say that a subgraph H of G is *induced* (respectively, *semi-induced*) if H is a subgraph that includes all edges (respectively, at least one edge) between every pair of vertices in H, if such edges exist in G. For instance, in Figure 3.1, the four subgraphs $12a, 12b, 123a, 123b$ are semi-induced but not induced subgraphs of G. It should be noted that our set expression makes sense for a semi-induced subgraph because semi-induced subgraphs of a given graph G can be distinguishable by the corresponding set.

We remark that when we consider a subgraph H of a graph G, the labels of H are inherited from the labels of G. Thus, if a graph H is considered as a subgraph of a graph G, then H may have a labelled simple edge, which is not in a bundle of H (actually, it is in a bundle of G). Note that for the graph G in Figure 3.1, $12a$ and $12b$ are different objects if they are considered as semi-induced subgraphs of G.

Let G be a connected graph on the vertex set $V = [n]$ and with exactly k bundles $B_1, \ldots, B_k$. Let Δ_V be the simplex $\Delta^{|V|-1}$ whose facets are labelled with the vertices of G. Then each face of Δ_V corresponds to a proper subset of vertices of G, defined by the intersection of the facets associated with those vertices.[3] For each $i = 1, \ldots, k$, let Δ_{B_i} be the simplex $\Delta^{|B_i|-1}$ whose vertices are labelled with the multiple edges in B_i. Then each face of Δ_{B_i} corresponds to a subset of B_i defined by the vertices spanning the face.[4] Now we define Δ_G as the product of simplices

$$\Delta_G := \Delta_V \times \Delta_{B_1} \times \cdots \times \Delta_{B_k}$$

endowed with the labels naturally induced from the labelling on Δ_V and Δ_{B_i} $(1 \leq i \leq k)$. Then the pseudograph associahedron P_G is obtained from Δ_G by truncating the faces corresponding to the proper connected semi-induced subgraphs of G in increasing order of dimension.[5] See Figure 3.1. Then the following hold:

[2] For a simple graph G, this modified construction is affinely equivalent to the graph associahedron in [15] based on the Minkowski sum.

[3] The simplex Δ^{n-1} is a smooth polytope; the (outward) normal vector of the facet labelled by j is the vector $-\mathbf{e}_j$ (respectively, $\sum_{j=1}^{n-1} \mathbf{e}_j$) if $1 \leq j < n$ (respectively, $j = n$).

[4] The simplex $\Delta^{|B_i|-1}$ is a smooth polytope; the (outward) normal vector of the facet opposite to the vertex labelled by $e_i^j, j = 1, \ldots, |B_i|$, is the vector $-\mathbf{e}_j$ (respectively, $\sum_{j=1}^{|B_i|-1} \mathbf{e}_j$) if $1 \leq j < |B_i|$ (respectively, $j = |B_i|$).

[5] Note that for a graph G, a proper connected semi-induced subgraph of some connected component of G is called a tube in [8].

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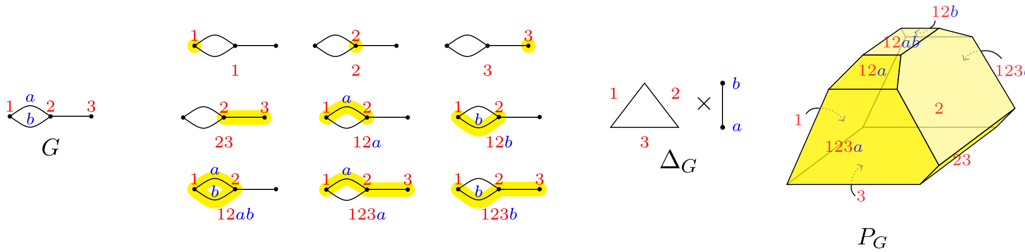

Figure 3.1. The proper connected semi-induced subgraphs of G and the facets of P_G

(1) There is a one-to-one correspondence between the facets F_I of P_G and the proper connected semi-induced subgraphs I of G.
(2) Two facets F_H and $F_{H'}$ of P_G intersect if and only if H and H' are disjoint and cannot be connected by an edge of G, or one contains the other.

If $G_1, \ldots, G_\ell$ are the connected components of G, then $P_G = P_{G_1} \times \cdots \times P_{G_\ell}$ and the dimension of P_G is

$$\dim P_G = |V(G)| - \ell + \sum_{i=1}^{k}(|B_i| - 1),$$

where B_i's are all the bundles of G. See [8, §2], where the readers may find examples, definitions and a much more detailed account of results for pseudograph associahedra.

Note that

- a product of smooth polytopes is a smooth polytope,
- any face of a smooth polytope is a smooth polytope, and
- for a smooth polytope P and a proper face F, there is a canonical truncation of P along F such that the result is a smooth polytope. (See [8, Lemma 2.5].)

For a connected graph G, since Δ_G is a smooth polytope, the pseudograph associahedron P_G can be realized as a smooth polytope canonically. In particular, the normal vector of the facet corresponding to a proper connected semi-induced subgraph H of G is determined by the label of H. Hence, under the canonical smooth realization, we get the projective smooth toric variety $X_G := X_{P_G}$ and the projective smooth real toric variety $X^{\mathbb{R}}_G := X^{\mathbb{R}}_{P_G}$, associated with a graph G. For example, it is known that if G is the simple path graph P_3, then the polytope P_G is a pentagon. Hence, X_G is $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$, obtained from $\mathbb{C}P^2$ by blowing up two fixed points; $X^{\mathbb{R}}_G$ is $\#3\mathbb{R}P^2$, the connected sum of three copies of the real projective plane $\mathbb{R}P^2$. If G is the graph with two vertices and two multiple edges, then X_G is $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $X^{\mathbb{R}}_G$ is $\mathbb{R}P^1 \times \mathbb{R}P^1$.[6]

[6] Note that only when every connected component of G has a few vertices and multiple edges, we can describe the variety X_G or $X^{\mathbb{R}}_G$ in terms of elementary smooth manifolds.

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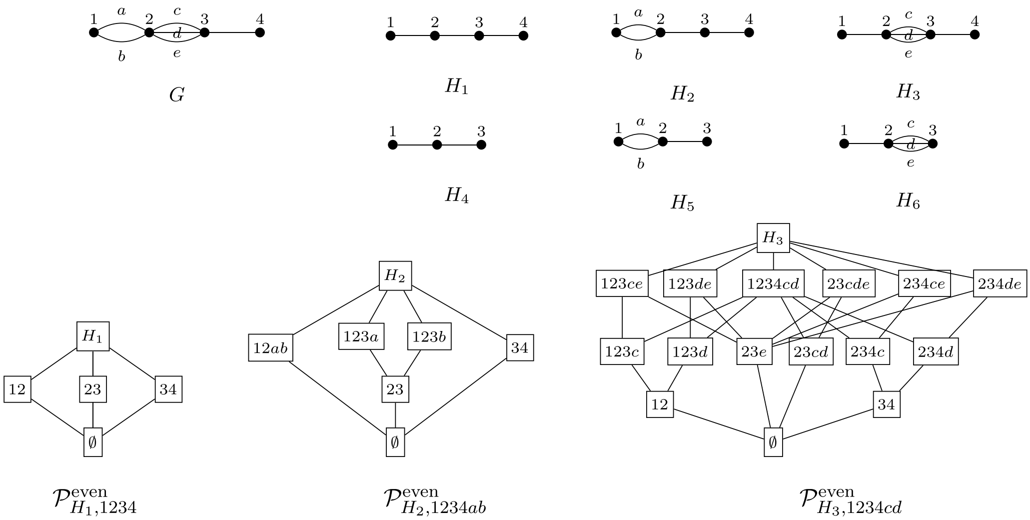

Figure 3.2. Examples for PI-graphs of G and the posets $\mathcal{P}^{\text{even}}_{H,A}$.

3.3. A poset of even subgraphs

For a graph G, a graph H is a *partial underlying graph* of G if H can be obtained from G by replacing some bundles with simple edges, that is, the set of all the bundles of H is a subset of G. A graph H is a *partial underlying induced* graph (*PI*-graph for short) of G if H is an induced subgraph of some partial underlying graph of G. Note that a graph is a PI-graph of itself. For instance, for the graph G with two bundles $\{a, b\}$ and $\{c, d, e\}$ in Figure 3.2,

- H_1, H_2 and H_3 are partial underlying graphs of G, and
- all H_i's are PI-graphs of G.

For a graph G, we let $\mathcal{C}_G$ be the set of all labels of G, i.e., $\mathcal{C}_G = V(G) \cup B_1 \cup \cdots \cup B_k$, where $B_1, \ldots, B_k$ are the bundles of G. For instance, $\mathcal{C}_G = \{1, 2, 3, 4, a, b, c, d, e\}$ and $\mathcal{C}_{H_3} = \{1, 2, 3, 4, c, d, e\}$ for the graphs G and H_3 in Figure 3.2.

Definition 3.3. *For a graph H, a subset A of $\mathcal{C}_H$ is called an admissible collection of H if $|A \cap V(H')| \equiv 0 \pmod 2$ for every connected component H' of H with the following properties:*

(1) *each vertex that is incident to only simple edges of H is contained in A, and*
(2) *$B \cap A \neq \emptyset$ and $|B \cap A| \equiv 0 \pmod 2$ for each bundle B of H.*

Let $\mathcal{A}(H)$ denote the set of all the admissible collections of H. The set of admissible collections each of the graphs H_i's in Figure 3.2 are as follows:

$$\begin{aligned}
&\mathcal{A}(H_1) = \{1234\}, \quad \mathcal{A}(H_2) = \{34ab, 1234ab\}, \quad \mathcal{A}(H_3) = \{14cd, 14ce, 14de, 1234cd, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1234ce, 1234de\}, \\
&\mathcal{A}(H_4) = \emptyset, \qquad\quad \mathcal{A}(H_5) = \{13ab, 23ab\}, \qquad \mathcal{A}(H_6) = \{12cd, 12ce, 12de, 13cd, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 13ce, 13de\},
\end{aligned}$$

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Let $A \subset \mathcal{C}_H$. We say that a semi-induced subgraph I of H is A-*even* (respectively, A-*odd*) if $|A \cap I'|$ is even (respectively, odd) for every connected component I' of I.[7] Then we define the poset $\mathcal{P}^{\text{even}}_{H,A}$ as follows. If $A \in \mathcal{A}(H)$, then the poset $\mathcal{P}^{\text{even}}_{H,A}$ is defined to be the poset consisting of all A-even semi-induced subgraphs of H ordered by the subgraph containment, including both $\emptyset$ and H. Hence, $\mathcal{P}^{\text{even}}_{H,A}$ is a bounded poset. If $A \notin \mathcal{A}(H)$, then we define $\mathcal{P}^{\text{even}}_{H,A}$ by the null poset. Figure 3.2 shows (the Hasse diagram of) the posets $\mathcal{P}^{\text{even}}_{H_1,1234}, \mathcal{P}^{\text{even}}_{H_2,1234ab}$ and $\mathcal{P}^{\text{even}}_{H_3,1234cd}$. Note that the first two posets are shellable but the last is not. For more examples of $\mathcal{P}^{\text{even}}_{H,A}$, see also Figure 5.3.

Combining Lemma 4.5 with Proposition 4.7 of [8], it holds the following.

Proposition 3.4. ([8]). *Let G be a graph.*

(1) *For every subset S in the set of integers $\{1, 2, \dots, \dim P_G\}$, the simplicial complex $K_{P_G,S}$ is contractible or there exist a PI-graph H of G and $A \in \mathcal{A}(H)$ such that $K_{P_G,S}$ is homotopy equivalent to the order complex of the proper part of $\mathcal{P}^{\text{odd}}_{H,A}$.*

(2) *The ith Betti number of $X^{\mathbb{R}}_G$ is*

$$\beta^i(X^{\mathbb{R}}_G) = \sum_{\substack{H:\,PI\text{-}graph\\ of\ G}} \sum_{A \in \mathcal{A}(H)} \tilde{\beta}^{i-1}\left(\Delta\left(\overline{\mathcal{P}^{\text{odd}}_{H,A}}\right)\right).$$

For the sake of convenience, we put a sketch of the proof of Proposition 3.4 in Appendix.

For a graph H, it was also noted in [8, §5] that $\Delta(\overline{\mathcal{P}^{\text{even}}_{H,A}})$ (respectively, $\Delta(\overline{\mathcal{P}^{\text{odd}}_{H,A}})$) is a geometric subdivision of the simplicial complex dual to the union of the facets F_I of the polytope P_H such that $|I \cap A|$ is even (respectively, odd). Hence, from the Alexander duality, we have

$$\widetilde{H}^i\left(\Delta\left(\overline{\mathcal{P}^{\text{odd}}_{H,A}}\right); \mathbb{Z}\right) \cong \widetilde{H}_{\dim(P_H)-i-2}\left(\Delta\left(\overline{\mathcal{P}^{\text{even}}_{H,A}}\right); \mathbb{Z}\right). \tag{3.3}$$

Therefore, if $K_{P_G,S}$ is homotopy equivalent to $\Delta(\overline{\mathcal{P}^{\text{odd}}_{H,A}})$ and $\mathcal{P}^{\text{even}}_{H,A}$ is shellable, then $\widetilde{H}^*(K_{P_G,S}; \mathbb{Z})$ is torsion-free.

Let H be a simple graph. Then $\mathcal{A}(H) = \{H\}$ if each connected component of H has an even number of vertices, and $\mathcal{A}(H) = \emptyset$ otherwise. Thus, we write $\mathcal{P}^{\text{even}}_H$ instead of $\mathcal{P}^{\text{even}}_{H,H}$.

Theorem 3.5. ([9, Proposition 4.9]). *For a simple graph H, $\mathcal{P}^{\text{even}}_H$ is pure and totally semimodular, so it is shellable.*

Recall that a pure and totally semimodular poset is CL-shellable by Theorem 2.1-(3). Hence, $\mathcal{P}^{\text{even}}_H$ is CL-shellable for every simple graph H.

In [8], there was an effort to extend results of [9] for a simple graph to a graph allowing multiple edges. Almost all results of [9] except for Theorem 3.5 were well-extended by using $\mathcal{P}^{\text{even}}_{H,A}$, where H is a PI-graph of G and $A \in \mathcal{A}(H)$. In fact, since the poset $\mathcal{P}^{\text{even}}_{H_3,1234cd}$

[7] Recall that each semi-induced subgraph of G is identified with a subset of $\mathcal{C}_G$ by the definition of semi-induced subgraphs.

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in Figure 3.2 is not shellable, Theorem 3.5 cannot be generalized to $\mathcal{P}^{\text{even}}_{H,A}$. Hence, it is natural to ask which $\mathcal{P}^{\text{even}}_{H,A}$ is shellable. Taking an interest in a projective smooth real toric variety associated with a graph, the following Question 1.2 was asked in [8]. For a graph G, let $\mathcal{A}^*(G) = \{(H, A) \mid H \text{is a PI-graph of } G \text{ and } A \in \mathcal{A}(H)\}$.

Question 1.2. ([8]) Find all graphs G such that $\mathcal{P}^{\text{even}}_{H,A}$ is shellable for every $(H, A) \in \mathcal{A}^*(G)$.

For simplicity, throughout the paper, let $\mathcal{G}^*$ be the family of all graphs G such that $\mathcal{P}^{\text{even}}_{H,A}$ is shellable for every $(H, A) \in \mathcal{A}^*(G)$. Clearly, the family $\mathcal{G}^*$ contains all simple graphs by Theorem 3.5. The answer to Question 1.2 is the following, which restates Theorem 1.3.

Theorem 1.3. *A graph G is in $\mathcal{G}^*$ if and only if each connected component of G is either a simple graph or one of the graphs in Figure 1.1.*

By Theorem 1.3, for every $G \in \mathcal{G}^*$, each poset $\mathcal{P}^{\text{even}}_{H,A}$ is shellable, and hence $\Delta(\overline{\mathcal{P}^{\text{even}}_{H,A}})$ is homotopy equivalent to a wedge of spheres. Thus, $\widetilde{H}^*(K_{P_G,S}; \mathbb{Z})$ is torsion-free for every subset S of the set of integers $\{1, 2, \ldots, \dim P_G\}$, and we get the following from [5, Corollary 1.2].

Corollary 3.6. *For a graph $G \in \mathcal{G}^*$, the integral cohomology of the projective smooth real toric variety $X^{\mathbb{R}}_G$ is torsion-free or has only 2-torsion elements:*

$$H^i\left(X^{\mathbb{R}}_G; \mathbb{Z}\right) = \mathbb{Z}^{\beta^i} \oplus \mathbb{Z}_2^{h_i - \beta^i},$$

where $\beta^i = \beta^i(X^{\mathbb{R}}_G)$ and $h_i = h_i(P_G)$.

As an immediate consequence of the proof in §5, we also get the following:

Theorem 3.7. *For every $G \in \mathcal{G}^*$, each $\mathcal{P}^{\text{even}}_{H,A}$ is CL-shellable for every $(H, A) \in \mathcal{A}^*(G)$.*

We finish the section by giving a remark that it is sufficient to consider a connected graph to prove Theorem 1.3 and Theorem 3.7. To see why, let $G_1, \ldots, G_k$ be the connected components of a graph G. Note that for a subgraph H of G and $A \in \mathcal{C}_H$, $(H, A) \in \mathcal{A}^*(G)$ if and only if $(H \cap G_i, A \cap \mathcal{C}_{G_i}) \in \mathcal{A}^*(G_i)$ for each i. Thus, for each $(H, A) \in \mathcal{A}^*(G)$, $\mathcal{P}^{\text{even}}_{H,A}$ is isomorphic to the product $\mathcal{P}^{\text{even}}_{H_1,A_1} \times \cdots \times \mathcal{P}^{\text{even}}_{H_k,A_k}$, where $H_i = H \cap G_i$ and $A_i = A \cap \mathcal{C}_{G_i}$ for each i. By (2) of Theorem 2.1, $\mathcal{P}^{\text{even}}_{H,A}$ is shellable if and only if $\mathcal{P}^{\text{even}}_{H_i,A_i}$ is shellable for each i. Thus, $G \in \mathcal{G}^*$ if and only if $G_i \in \mathcal{G}^*$ for each i.

4. Graphs that admit a non-shellable poset $\mathcal{P}^{\text{even}}_{H,A}$

In this section, we give the 'only if' part of Theorem 1.3. We will see that almost all graphs do not belong to the family $\mathcal{G}^*$. The result of this section is based on the following basic observation.

Lemma 4.1. *Let $\mathcal{P}_0$ be the poset in Figure 4.1 and let $\mathcal{Q}$ be any subposet that has at least two chains of length 3, with one containing a or b and another containing a' or b'. Then $\mathcal{Q}$ is not shellable.*

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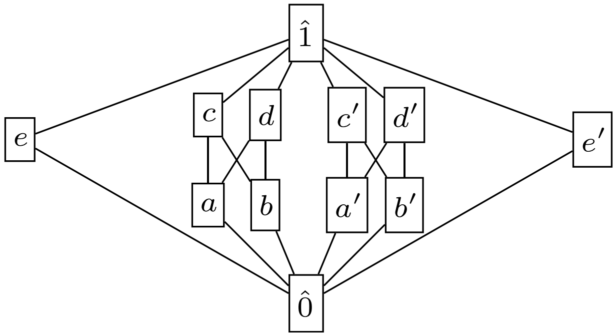

Figure 4.1. The poset $\mathcal{P}_0$.

Figure 4.2. A graph I and the interval $\mathcal{I}$.

Theorem 4.2. *Let G be a connected non-simple graph in $\mathcal{G}^*$. Then G is one of the graphs in Figure 1.1.*

Before starting the proof, recall that we often drop the braces and commas to denote a subset of $\mathcal{C}_G$.

Proof. Suppose that G is a connected non-simple graph in $\mathcal{G}^*$. If $|V(G)| = 2$, then $G = \widetilde{P}_{2,m}$ in Figure 1.1 for some m. Assume that $|V(G)| \geq 3$ and G has a bundle B whose endpoints are 1 and 2. $\square$

Claim 4.3. *The graph G has exactly one bundle B.*

Proof of Claim 4.3. Suppose that G has a bundle B' other than B. Take a shortest path in G whose starting vertex is an endpoint of B and whose terminal vertex is an endpoint of B'. We denote the path by Q. Note that Q does not contain a multiple edge. Let $Q := (v_1, \ldots, v_k)$, where $k \geq 1$, and let $v_1 = 2$ without loss of generality. Let H be a PI-graph of G such that $V(H) = V(Q) \cup \{1, 2\} \cup \{\text{endpoints of } B'\}$ and H has exactly two bundles B and B'. Let $a, b \in B$ and $a', b' \in B'$.

(Case 1) Suppose that $k = 1$. Then $|V(H)| = 3$, so we set $V(H) = \{1, 2, 3\}$. Then $A := 23aba'b'$ belongs to $\mathcal{A}(H)$. Setting $I = 123aba'b'$ (the dashed edge in Figure 4.2 is a simple edge or does not exist), we see $I \cap A = A$, and hence I is an element of $\mathcal{P}^{\text{even}}_{H,A}$. Let $I' = 1$ and consider the interval $\mathcal{I} = [I', I]$ of $\mathcal{P}^{\text{even}}_{H,A}$, see Figure 4.2. Then $\mathcal{I}$ is isomorphic to a subposet of $\mathcal{P}_0$ in Figure 4.1. By Lemma 4.1, $\mathcal{I}$ is not shellable, a contradiction to (1) of Theorem 2.1.

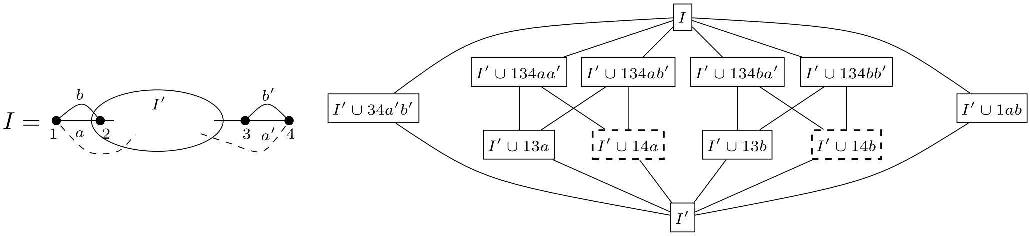

Figure 4.3. A graph I and the interval $\mathcal{I}$ where the dashed boxes may be in $\mathcal{I}$.

(Case 2) Suppose that $k \geq 2$. Let the endpoints of B' be labelled by 3 and 4, and $v_k = 3$. Let

$$A = \begin{cases} (V(H) \setminus \{1\}) \cup aba'b' & \text{if } k \text{ is odd;} \\ (V(H) \setminus \{1,2\}) \cup aba'b' & \text{if } k \text{ is even.} \end{cases}$$

Note that $A \in \mathcal{A}(H)$. Let $I' = V(Q) \setminus \{v_k\}$ and $I = I' \cup 134aba'b'$. Then $I' \cap A = \{v_1, \ldots, v_{k-1}\}$ (if k is odd) or $I' \cap A = \{v_2, \ldots, v_{k-1}\}$ (if k is even). Then they have the form in Figure 4.3 (the dashed edges are simple edges or do not exist), and both I' and I are elements of $\mathcal{P}^{\text{even}}_{H,A}$. Consider the interval $\mathcal{I} = [I', I]$ in $\mathcal{P}^{\text{even}}_{H,A}$, see Figure 4.3. Thus, $\mathcal{I}$ is isomorphic to a subposet of $\mathcal{P}_0$ in Figure 4.1. Note that $I' \cup 134aa'$, $I' \cup 134ab'$, $I' \cup 134ba'$, $I' \cup 134bb'$ are elements in $\mathcal{I}$, and both $I' \cup 13a$ and $I' \cup 13b$ are also elements in $\mathcal{I}$. The elements $I' \cup 14a$ and $I' \cup 14b$ in the dashed boxes of Figure 4.3 are in $\mathcal{I}$ if there is an edge between the vertex 4 and a vertex in I'. By Lemma 4.1, $\mathcal{I}$ is not shellable, a contradiction to (1) of Theorem 2.1. □

Hence, G has only one bundle B. If $|V(G)| = 3$, then clearly G is one of the graphs in Figure 1.1. Now assume that $|V(G)| \geq 4$. For each vertex i, we let $N^*(i) = N_G(i) \setminus \{1,2\}$, where $N_G(i)$ is the set of vertices which are adjacent to i in G.

Claim 4.4. $|N^*(1) \cup N^*(2)| = 1$.

Proof of Claim 4.4 Since $|V(G)| \geq 3$ and G is connected, $|N^*(1) \cup N^*(2)| \geq 1$. Suppose that $|N^*(1) \cup N^*(2)| \geq 2$, and $3, 4 \in N^*(1) \cup N^*(2)$. Let H be a PI-graph of G such that $V(H) = \{1,2,3,4\}$ and H has the bundle B. Let $A = 1234ab$ for some $a, b \in B$. Note that $A \in \mathcal{A}(H)$. Let $I = 1234ab$, and consider the interval $\mathcal{I} = [\emptyset, I]$ in $\mathcal{P}^{\text{even}}_{H,A}$, see Figure 4.4. Then I is a subgraph of a complete graph of four vertices with exactly one bundle of size two, and $\mathcal{I}$ is isomorphic to a subposet of $\mathcal{P}_0$. Note $123a$, $123b$, $124a$ and $124b$ are elements of $\mathcal{I}$. Since the vertex 3 is a neighbour of 1 or 2, at least one of 13 and 23 is an element of $\mathcal{I}$ (the elements 13 and 23 are drawn in a dotted box in Figure 4.4). Similarly, since the vertex 4 is also a neighbour of 1 or 2, at least one of 14 and 24 is an element of $\mathcal{I}$ (the elements 14 and 24 are drawn in a dotted box in Figure 4.4). By Lemma 4.1, $\mathcal{I}$ is not shellable, a contradiction to (1) of Theorem 2.1. □

From now on, we set $N^*(1) \cup N^*(2) = N^*(2) = \{3\}$.

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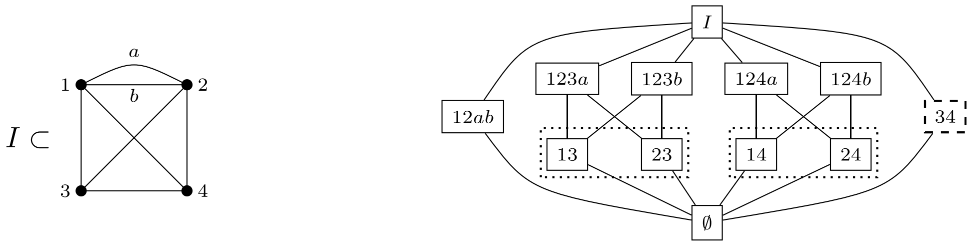

Figure 4.4. A graph containing I and the interval $\mathcal{I}$, where at least one of the elements in each dotted box is in $\mathcal{I}$, and the dashed box 34 may be in $\mathcal{I}$.

Claim 4.5. *For each vertex i other than 1 or 2, let Q_i be the shortest path of G from 3 to i. Then*

$$|N^*(i) \setminus V(Q_i)| \leq 2,$$

where the equality holds if and only if $|V(Q_i)|$ is odd and $V(G) = V(Q_i) \cup \{1, 2\} \cup N^*(i)$.

Proof of Claim 4.5. Suppose that there is a vertex $i \in V(G) \setminus \{1, 2\}$ satisfying one of the following:

(1) $|N^*(i) \setminus V(Q_i)| \geq 3$,
(2) $|N^*(i) \setminus V(Q_i)| = 2$ and $|V(Q_i)|$ is even, and
(3) $|N^*(i) \setminus V(Q_i)| = 2, |V(Q_i)|$ is odd, and $V(G) \neq V(Q_i) \cup \{1, 2\} \cup N^*(i)$.

If $|V(Q_i)|$ is even, then it is the case of (1) or (2), so we set $I' = Q_i$ and take two vertices x and y in $N^*(i) \setminus V(Q_i)$. Suppose that $|V(Q_i)|$ is odd. Then it is the case of (1) or (3). In case (1), we take three vertices $w, x, y \in N^*(i) \setminus V(Q_i)$ and set $I' = Q_i \cup w$. Otherwise, we take a vertex $w \in N^*(i) \setminus V(Q_i)$ and a vertex $y \in V(G) \setminus (V(Q_i) \cup \{1, 2\} \cup N^*(i))$ so that $Q_i \cup wy$ is connected. Then we set $I' = Q_i \cup w$ and take x as a vertex in $N^*(i) \setminus V(Q_i)$ other than w. For any case, note that $3 \in I', I' \cap \{1, 2\} = \emptyset$, $|I'|$ is even, and each of I', $I' \cup x$ and $I' \cup y$ is a connected subgraph of G.

Let H be a PI-graph such that $V(H) = I' \cup 12xy$ and B is the bundle of H. Let $A = V(H) \cup ab$ and $I = A$ for some $a, b \in B$. Note that $A \in \mathcal{A}(H)$ and I is the graph in the left of Figure 5.1 (the dashed edges are simple edges or do not exist). Consider the interval $\mathcal{I} = [I', I]$ in $\mathcal{P}^{\text{even}}_{H,A}$, see Figure 5.1. Then $\mathcal{I}$ is isomorphic to a subposet of $\mathcal{P}_0$. Note that $I' \cup 12xa$, $I' \cup 12xb$, $I' \cup 12ya$ and $I' \cup 12yb$ are elements in $\mathcal{I}$. Moreover, both $I' \cup 2x$ and $I' \cup 2y$ are in $\mathcal{I}$. The elements $I' \cup 1x$ and $I' \cup 1y$ in the dashed boxes of Figure 5.1 are in $\mathcal{I}$ if there is an edge between the vertex 1 and a vertex in I'. By Lemma 4.1, $\mathcal{I}$ is not shellable, a contradiction to (1) of Theorem 2.1. □

Proof. Since $|V(G)| \geq 4$, we have $|N^*(3)| \geq 1$. Since $N^*(3) \setminus V(Q_3) = N^*(3)$, we see $|N^*(3)| \leq 2$ by Claim 4.5. If $|N^*(3)| = 2$, then the equality part of Claim 4.5 says that G is one of $\widetilde{S}_{5,m}$, $\widetilde{S}'_{5,m}$, $\widetilde{T}_{5,m}$ and $\widetilde{T}'_{5,m}$ in Figure 1.1 for some m. Suppose that $|N^*(3)| = 1$, and let $N^*(3) = \{4\}$. Since $N^*(4) \setminus V(Q_4) = N^*(4) \setminus \{3\}$, we see $|N^*(4) \setminus \{3\}| \leq 1$ by Claim 4.5. If $|N^*(4) \setminus \{3\}| = 0$, then G is one of $\widetilde{P}_{4,m}$, and $\widetilde{P}'_{4,m}$ in Figure 1.1 for some m.

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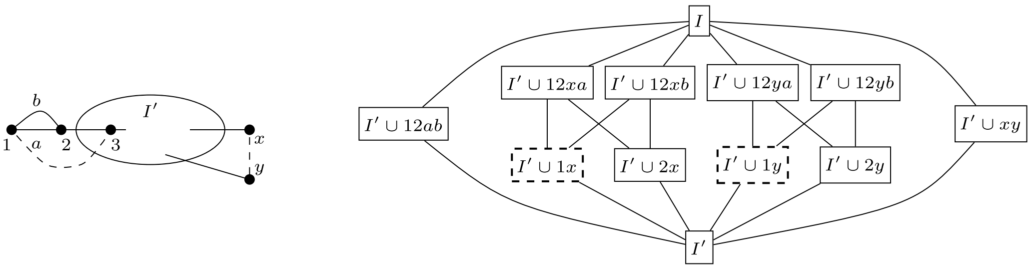

Figure 5.1. A graph containing I and the poset containing $\mathcal{I}$, where the dashed boxes may be in $\mathcal{I}$.

Suppose that $|N^*(4) \setminus \{3\}| = 1$, and let $N^*(4) \setminus \{3\} = \{5\}$. Then consider $N^*(5) \setminus V(Q_5)$. Repeating the argument through the vertices one by one completes the proof. $\square$

5. CL-shellability of $\mathcal{P}_{G,A}^{\text{even}}$

In this section, we show that the poset $\mathcal{P}_{H,A}^{\text{even}}$ is CL-shellable for every $(H, A) \in \mathcal{A}^*(G)$ if G is a graph in Figure 1.1. Note that a connected PI-graph of G in Figure 1.1 is a simple graph or a graph in Figure 1.1. Thus, it is sufficient to show that when G is a graph in Figure 1.1, $\mathcal{P}_{G,A}^{\text{even}}$ is shellable for every $A \in \mathcal{A}(G)$. From now on, throughout this section, we fix a graph G with n vertices and m multiple edges in Figure 1.1, and an admissible collection $A \in \mathcal{A}(G)$.

5.1. Definition of an ordering $\prec_{\text{atm}}^{I}$ for the atoms of $[I, G]$

We let $V = \{1, 2, \dots, n\}$ $(n \geq 2)$ be the set of vertices of G, and 1 and 2 be the endpoints of the bundle B. By the definition of an admissible collection, note that $\{3, \dots, n\} \subset A$, $A \cap B \neq \emptyset$ and $|A \cap B|$ is even, so we let $B \cap A = \{a_1, \dots, a_{2m}\}$ $(m \geq 1)$ and $B \setminus A = \{b_1, \dots, b_\ell\}$. Here, $B \setminus A$ may be the empty set. There are three cases:

- $|V|$ is odd and $V \cap A = V \setminus \{w\}$ for some $w \in \{1, 2\}$;
- $|V|$ is even and $V \cap A = V \setminus \{1, 2\}$;
- $|V|$ is even and $V \cap A = V$.

We label the vertices that are not the endpoints of B so that for each $i \in \{3, \dots, n\}$, the vertex i is closest to the vertex $i-1$. We relabel the endpoints of B so that $1 \notin A$ if $|V|$ is odd and 13 is an edge if $|V|$ is even. See (i) of Figure 5.2 for all the possible labellings when $|V|$ is odd. We illustrate all the possible labellings when $|V|$ is even in (ii) of Figure 5.2. See Figure 5.3 for examples of $\mathcal{P}_{G,A}^{\text{even}}$ under this labelling. We also assume that there is a total ordering between the vertices: $1 \prec 2 \prec \cdots \prec n$. Thus, for $I \subset V$, the minimum of I, denoted by $\min(I)$, means the frontmost one in the ordering.

Note that given a cover $I \lessdot J$ in $\mathcal{P}_{G,A}^{\text{even}}$, if $(J \setminus I) \cap B \neq \emptyset$, then either $(J \setminus I) \cap (B \cap A) = \emptyset$ or $(J \setminus I) \cap (B \setminus A) = \emptyset$. Suppose not, that is, $(J \setminus I) \cap (B \cap A) \neq \emptyset$ and $(J \setminus I) \cap (B \setminus A) \neq \emptyset$. Then $K := J \setminus \{(J \setminus I) \cap (B \setminus A)\}$ satisfies that $I < K < J$ and $|(K \setminus I) \cap A| \equiv |(J \setminus I) \cap A|$,

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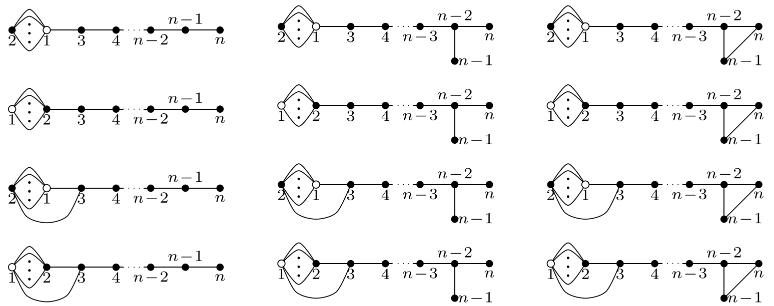

(i) Labeling of the vertices, where the hollow vertex does not belong to A, when n is odd.

(ii) Labeling of the vertices, where the hollow vertices do not belong to A, when n is even.

Figure 5.2. Labelling of the vertices.

Table 1. Types of $I \lessdot J$ in $\mathcal{P}^{\text{even}}_{G,A}$, where a, $a' \in B \cap A$, $b \in (B \setminus A)$, c, $c' \in A$, $v = \min(V \setminus (I \cup \{1,2\}))$

									(E3′)	
$I \lessdot J$	Type		(E1)	(E2)	(E3)	(E4)	(E1′)	(E2′)	(E3′-1)	(E3′-2)
$J \setminus I$	V:odd	$A \cap \{1,2\} = \{2\}$	1	cc'	$1ac$	–	b	$1b$	$2vb$	–
	V:even	$A \cap \{1,2\} = \emptyset$	1 or 2	cc'	$1ac$ or $2ac$	–	b	$1b$ or $2b$	–	–
		$\{1,2\} \subset A$	–	cc'	–	$12aa'$	b	–	$1vb$ or $2vb$	$12b$

For the case of (E3), $c \in B \cap A$ or $c = \min(V \setminus (I \cup \{1,2\}))$.

a contradiction to $I \lessdot J$. Hence, we can define the type of a cover $I \lessdot J$ in $\mathcal{P}^{\text{even}}_{G,A}$ according to the size of $J \setminus I$ and the intersection with $B \setminus A$. A cover $I \lessdot J$ has type (Ei) if $|J \setminus I| = i$ and $J \setminus I$ has no element of $B \setminus A$; $I \lessdot J$ has type (Ei') if $|J \setminus I| = i$ and $J \setminus I$ contains some elements of $B \setminus A$. Hence, (Ei') can occur only when $B \setminus A \neq \emptyset$.

Lemma 5.1. *Let $I \lessdot J$ be a cover in $\mathcal{P}^{\text{even}}_{G,A}$. Then $J \setminus I$ is one of the sets represented in Table 1.*

Proof. It follows from the fact that for a cover $I \lessdot J$ in $\mathcal{P}^{\text{even}}_{G,A}$, each of $|I \cap A|$, $|J \cap A|$ and $|(J \setminus I) \cap A|$ is even, and $J \setminus I$ satisfies the following condition, which we will call (†).

(†) The elements in $J \setminus I$ belong to the same connected component of J. □

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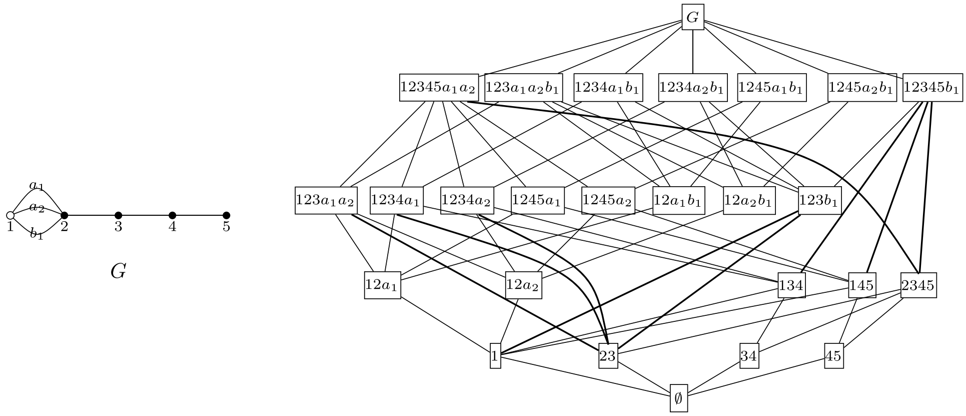
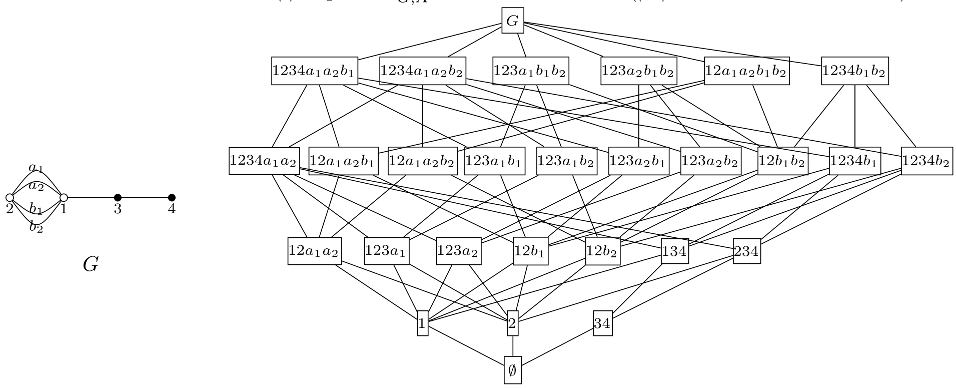
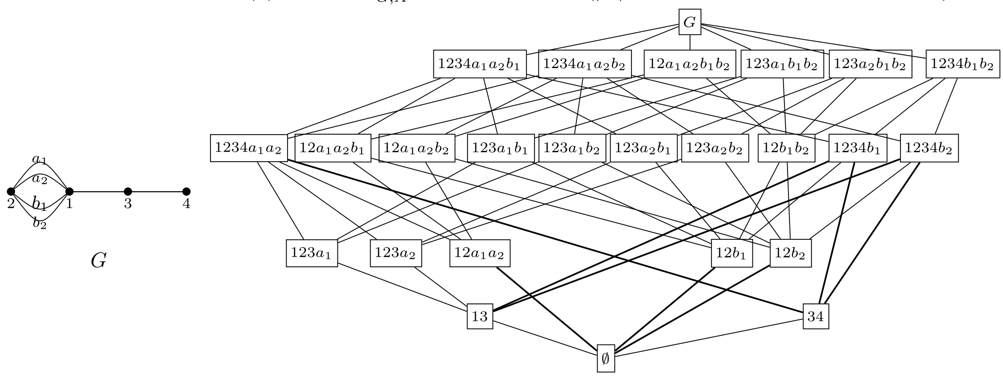


(i) A poset $\mathcal{P}^{\text{even}}_{G,A}$ when $A = 2345a_1a_2$ ($|V|$ is odd and 1 is not in A.)



(ii) A poset $\mathcal{P}^{\text{even}}_{G,A}$ when $A = 34a_1a_2$ ($|V|$ is even, and 1 and 2 are not in A.)



(iii) A poset $\mathcal{P}^{\text{even}}_{G,A}$ when $A = 1234a_1a_2$ ($|V|$ is even)

Figure 5.3. Examples of posets $\mathcal{P}^{\text{even}}_{G,A}$.

When $I \lessdot J$ is of (E3′), as in Table 1, we divide the type (E3′) into two subtypes according to the size of $(J \setminus I) \cap \{1, 2\}$:

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- $I \lessdot J$ has type (E3′-1) if $I \lessdot J$ has type (E3′) and $|(J \setminus I) \cap \{1,2\}| = 1$;
- $I \lessdot J$ has type (E3′-2) if $I \lessdot J$ has type (E3′) and $|(J \setminus I) \cap \{1,2\}| = 2$.

We can also show that when $I \lessdot J$ is of (E3′-1), $J \setminus I$ contains the vertex $\min(V \setminus (I \cup \{1,2\}))$.

Proposition 5.2. *The lengths of maximal chains of $\mathcal{P}^{\text{even}}_{G,A}$ are*

$$\begin{cases} \frac{|A|}{2} + |B \setminus A| + 1 \text{ or } \frac{|A|}{2} + |B \setminus A| & \text{if } |V| \text{is odd,} \\ \frac{|A|}{2} + |B \setminus A| + 1 & \text{if } |V| \text{is even and } A \cap V \neq V, \text{and} \\ \frac{|A|}{2} + |B \setminus A| \text{ or } \frac{|A|}{2} + |B \setminus A| - 1 & \text{if } |V| \text{is even and } A \cap V = V. \end{cases}$$

Moreover, if $|V|$ is odd, 2 and 3 are not adjacent in G and $B \subset A$, then $\mathcal{P}^{\text{even}}_{G,A}$ is pure and its length is $\frac{|A|}{2} + 1$.

Proof. Recall that $|V| = n, |B \cap A| = 2m$ and $|B \setminus A| = \ell$. Note that $2m + n \geq 4$. Let $\sigma \colon I_0 \lessdot I_1 \lessdot \cdots \lessdot I_p$ be a maximal chain of $\mathcal{P}^{\text{even}}_{G,A}$. Note that $\{I_i \setminus I_{i-1} \mid i = 1, \ldots, p\}$ is a partition of $V \cup B$.

Let k be the smallest index such that $I_k \setminus I_{k-1}$ contains an element in B, that is, I_k is the first element of σ containing a multiple edge. Then $\{1,2\} \subset I_k$ and $\{1,2\} \not\subset I_{k-1}$. Together with Table 1, we see that for each cover $I_{i-1} \lessdot I_i$ of σ, except the cover $I_{k-1} \lessdot I_k$, it holds that $|I_i \setminus I_{i-1}| = 1$ or 2. For each $j \in \{1,2\}$, let t_j be the number of covers $I_{i-1} \lessdot I_i$ of σ, except the cover $I_{k-1} \lessdot I_k$, such that $|I_i \setminus I_{i-1}| = j$. Then the number of covers of σ, which is equal to $\ell(\sigma)$, is $1 + t_1 + t_2$. Since $\{I_i \setminus I_{i-1} \mid i = 1, \ldots, p\}$ is a partition of $V \cup B$, we have

$$t_1 + 2t_2 + |I_k \setminus I_{k-1}| = |I_p \setminus I_0| = n + 2m + \ell,$$

or $t_2 = \frac{(n+2m+\ell) - t_1 - |I_k \setminus I_{k-1}|}{2}$. Therefore,

$$\ell(\sigma) = 1 + t_1 + t_2 = \frac{(n + 2m + \ell) + 2 + t_1 - |I_k \setminus I_{k-1}|}{2}. \tag{5.1}$$

Note that σ has exactly $|B \setminus (A \cup I_k)|$ covers of (E1′) and at most one cover of (E1). In addition, σ has one cover of (E1) if and only if I_{k-1} contains a vertex in $\{1,2\} \setminus A$. Since t_1 is the sum of the number of covers of (E1) and the number of covers of (E1′), we get

$$t_1 = \begin{cases} 1 + |B \setminus (A \cup I_k)| & \text{if } I_{k-1} \text{ contains a vertex in } \{1,2\} \setminus A, \\ |B \setminus (A \cup I_k)| & \text{otherwise.} \end{cases} \tag{5.2}$$

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Suppose that $|V|$ is odd. Then $|A| = 2m+n-1$. By Table 1 again, $I_{k-1} \lessdot I_k$ has one of types (E2), (E3), (E2′) and (E3′-1). By Equations (5.1) and (5.2),

$$\ell(\sigma) = \begin{cases} \frac{(n+2m+\ell)+2+(1+\ell)-2}{2} = \ell+1+\frac{n+2m-1}{2} = \ell+1+\frac{|A|}{2} & \text{if } I_{k-1} \lessdot I_k \text{ is of (E2)}, \\ \frac{(n+2m+\ell)+2+\ell-3}{2} = \ell+\frac{n+2m-1}{2} = \ell+\frac{|A|}{2} & \text{if } I_{k-1} \lessdot I_k \text{ is of (E3)}, \\ \frac{(n+2m+\ell)+2+(\ell-1)-2}{2} = \ell+\frac{n+2m-1}{2} = \ell+\frac{|A|}{2} & \text{if } I_{k-1} \lessdot I_k \text{ is of (E2}'\text{)}, \\ \frac{(n+2m+\ell)+2+(1+(\ell-1))-3}{2} = \ell+\frac{n+2m-1}{2} = \ell+\frac{|A|}{2} & \text{if } I_{k-1} \lessdot I_k \text{ is of (E3}'\text{-1)}. \end{cases}$$

Hence, every maximal chain has a length of either $\frac{|A|}{2}+\ell+1$ or $\frac{|A|}{2}+\ell$, and hence the poset $\mathcal{P}^{\text{even}}_{G,A}$ is nonpure. Note that if 2 and 3 are not adjacent in G, then there is no cover of (E3), and if $B \subset A$, then $B \setminus A = \emptyset$ and then there is no cover of (E2′) or (E3′). Hence, if 2 and 3 are not adjacent in G and $B \subset A$, then $\mathcal{P}^{\text{even}}_{G,A}$ is a pure poset of length $\frac{|A|}{2}+1$. We summarize in the following table:

$I_{k-1} \lessdot I_k$	The types of the covers in σ	Length of σ
(E2)	one (E1), ℓ (E1′)s, $\frac{2m+n-1}{2}$ (E2)s	$\frac{2m+n+1}{2}+\ell$
(E3)	one (E3), ℓ (E1′)s, $\frac{2m+n-3}{2}$ (E2)s	$\frac{2m+n-1}{2}+\ell$
(E2′)	one (E2′), $(\ell-1)$ (E1′)s, $\frac{2m+n-1}{2}$ (E2)s	
(E3′-1)	one (E1), one (E3′-1), $(\ell-1)$ (E1′)s, $\frac{2m+n-3}{2}$ (E2)s	

Suppose that $|V|$ is even. When $A \cap \{1,2\} = \emptyset$, it holds that $|A| = 2m+n-2$ and $I_{k-1} \lessdot I_k$ is of (E3) or (E2′). By Equations (5.1) and (5.2),

$$\ell(\sigma) = \begin{cases} \frac{(n+2m+\ell)+2+(1+\ell)-3}{2} = \ell+1+\frac{n+2m-2}{2} = \ell+1+\frac{|A|}{2} & \text{if } I_{k-1} \lessdot I_k \text{ is of (E3)}, \\ \frac{(n+2m+\ell)+2+(1+(\ell-1))-2}{2} = \ell+1+\frac{n+2m-2}{2} = \ell+1+\frac{|A|}{2} & \text{if } I_{k-1} \lessdot I_k \text{ is of (E2}'\text{)}. \end{cases}$$

Hence, every maximal chain has length $\frac{|A|}{2}+\ell+1$.

When A contains $\{1,2\}$, it holds that $|A| = 2m+n$ and $I_{k-1} \lessdot I_k$ is one of (E2), (E4) and (E3′). By Equation (5.1) and (5.2),

$$\ell(\sigma) = \begin{cases} \frac{(n+2m+\ell)+2+\ell-2}{2} = \ell+\frac{n+2m}{2} = \ell+\frac{|A|}{2} & \text{if } I_{k-1} \lessdot I_k \text{ is of (E2)}, \\ \frac{(n+2m+\ell)+2+\ell-4}{2} = \ell-1+\frac{n+2m}{2} = \ell-1+\frac{|A|}{2} & \text{if } I_{k-1} \lessdot I_k \text{ is of (E4)}, \\ \frac{(n+2m+\ell)+2+(\ell-1)-3}{2} = \ell-1+\frac{n+2m}{2} = \ell-1+\frac{|A|}{2} & \text{if } I_{k-1} \lessdot I_k \text{ is of (E3}'\text{)}. \end{cases}$$

Hence, every maximal chain has length $\ell+\frac{|A|}{2}$ or $\ell-1+\frac{|A|}{2}$. We summarize in the following table: □

We shall show that $\mathcal{P}^{\text{even}}_{G,A}$ admits a recursive atom ordering. We first define the lexicographic order $\prec^I_{\text{lex}}$ on $V \cup B$ for each $I \in \mathcal{P}^{\text{even}}_{G,A}$ and then define the atom ordering $\prec^I_{\text{atm}}$ for $[I, G]$.

$A \cap \{1,2\}$	$I_{k-1} \lessdot I_k$	The types of the covers in σ	Length of σ
$\emptyset$	(E3)	one (E1), one (E3), ℓ (E1′)s, $\frac{2m+n-4}{2}$ (E2)s	$\frac{2m+n}{2} + \ell$
	(E2′)	one (E1), one (E2′), $(\ell-1)$ (E1′)s, $\frac{2m+n-2}{2}$ (E2)s	
$\{1,2\}$	(E2)	$\frac{2m+n}{2}$ (E2)s, ℓ (E1′)s	$\frac{2m+n}{2} + \ell$
	(E4)	one (E4), ℓ (E1′), $\frac{2m+n-4}{2}$ (E2)s	$\frac{2m+n-2}{2} + \ell$
	(E3′)	one (E3′), $(\ell-1)$ (E1′)s, $\frac{2m+n-2}{2}$ (E2)s	

Definition 5.3. *Let $I \in \mathcal{P}^{\text{even}}_{G,A}$. We define the lexicographic order $\prec^I_{\text{lex}}$ on $V \cup B$ as follows:*

- *If $B \cap I = \emptyset$, then*

$$\prec^I_{\text{lex}}:\ 1, 2, 3, \ldots, n, a_1, \ldots, a_{2m}, b_1, \ldots, b_\ell.$$

- *If $B \cap I \neq \emptyset$ and $(B \setminus A) \cap I = \emptyset$, then let $k := \max\{i \mid a_i \in B \cap A \cap I\}$ and*

$$\prec^I_{\text{lex}}:\ 1, 2, a_1, \ldots, a_k, 3, \ldots, n, a_{k+1}, \ldots, a_{2m}, b_1, \ldots, b_\ell.$$

- *If $(B \setminus A) \cap I \neq \emptyset$, then let $k := \max\{i \mid b_i \in (B \setminus A) \cap I\}$ and*

$$\prec^I_{\text{lex}}:\ 1, 2, a_1, \ldots, a_{2m}, b_1, \ldots, b_k, 3, \ldots, n, b_{k+1}, \ldots, b_\ell.$$

Then for two atoms J and J' of $[I, G]$, we define $J \prec^I_{\text{atm}} J'$ if

(O1) $|(J \setminus I) \cap \{1,2\}| = 1$ and $|(J' \setminus I) \cap \{1,2\}| = 2$, or
(O2) (O1) does not hold and $J \setminus I \prec^I_{\text{lex}} J' \setminus I$, where we compare lexicographic order induced by $\prec^I_{\text{lex}}$, that is, we sort the elements in each of $J \setminus I$ and $J' \setminus I$ in ascending order by $\prec^I_{\text{lex}}$ and compare them by the lexicographic order $\prec^I_{\text{lex}}$.

In the above, we check (O1) first, and if (O1) does not hold, we check whether (O2) holds. Note that (O1) is considered only when $\mathcal{P}^{\text{even}}_{G,A}$ admits a cover of (E4) or (E3′-2), that is, $|V|$ is even and A contains $\{1,2\}$.

Here is an example. Let G be the graph $\widetilde{P}_{6,5}$ in Figure 1.1. Suppose that $A = V \cup \{a_1, a_2, a_3, a_4\}$. Then the atoms of $\mathcal{P}^{\text{even}}_{G,A}$ are ordered as follows:

$$\prec^\emptyset_{\text{atm}}:\ 13,\ 12a_1a_2,\ 12a_1a_3,\ 12a_1a_4,\ 12a_2a_3,\ 12a_2a_4,\ 12a_3a_4,\ 12b_1,\ 34,\ 45,\ 56.$$

For $I = 12a_1a_3$, $\prec^I_{\text{lex}}$: $1, 2, a_1, \mathbf{a_2}, a_3, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{a_4}, \mathbf{b_1}$, and the atoms of $[I, G]$ are ordered as follows:

$$\prec^I_{\text{atm}}:\ 12\mathbf{3}a_1\mathbf{a_2}a_3,\ 12a_1\mathbf{a_2}a_3\mathbf{a_4},\ 12\mathbf{34a_1a_3},\ \mathbf{123a_1a_3a_4},\ \mathbf{1245a_1a_3},\ \mathbf{1256a_1a_3}, \mathbf{12a_1a_3b_1}$$

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because $a_2 3 \prec^I_{\text{lex}} a_2 a_4 \prec^I_{\text{lex}} 34 \prec^I_{\text{lex}} 3a_4 \prec^I_{\text{lex}} 45 \prec^I_{\text{lex}} 56 \prec^I_{\text{lex}} b_1$, where the bold letters indicate the elements not in I.

The following is the main theorem of this section, whose proof is given in §5.2.

Theorem 5.4. *Let G be a connected graph in Figure 1.1, and A be an admissible collection of G. Then the poset $\mathcal{P}^{\text{even}}_{G,A}$ admits a recursive atom ordering, and hence $\mathcal{P}^{\text{even}}_{G,A}$ is CL-shellable.*

Remark. We insist that the ordering $\prec^I_{\text{atm}}$ is essential. Suppose that we consider the lexicographic order $\prec^*$ given by $1, 2, a_1, a_2, \ldots, a_{2m}, 3, 4, \ldots, n, b_1, \ldots, b_\ell$ and define $\prec^*_{\text{atm}}$ by an ordering obtained by replacing $\prec^I_{\text{lex}}$ in (O2) of Definition 5.3 with the fixed ordering $\prec^*$. For the posets in Figure 5.3, $\prec^*_{\text{atm}}$ gives a recursive atom ordering. However, it fails to be a recursive atom ordering in general. For example, let G be a graph in Figure 1.1 with $|V| = 4$ and $|B| = 6$, and let $A = V \cup B$. Then $A \in \mathcal{A}(G)$. Let $I = 12a_1a_3$, and consider the atoms $J_1 = 12a_1a_3\mathbf{a_5a_6}$ and $J_2 = 12\mathbf{3}a_1a_3\mathbf{a_5}$ of $[I, G]$, where the bold letters indicate the elements not in I. Then the atoms of $[\emptyset, G]$ preceding I in $\prec^*_{\text{atm}}$ are 13 and $12a_1a_2$. However, $J_1 \prec^*_{\text{atm}} J_2$, J_2 contains the atom 13, and J_1 does not contain any atom of $[\emptyset, G]$ preceding I. Thus, (2) of Definition 2.3 fails.

5.2. Proof of Theorem 5.4

For a subset $X \subset V \cup B$, $\min^I(X)$ and $\max^I(X)$ denote the minimum and the maximum of X with respect to $\prec^I_{\text{lex}}$, respectively. We will show that the ordering $\prec^I_{\text{atm}}$ ($I \in \mathcal{P}^{\text{even}}_{G,A}$) is a recursive atom ordering. We first check Condition (2) of Definition 2.3.

Lemma 5.5. *Let I_i and I_j be atoms of $[I, G]$ such that $I_i \prec^I_{\text{atm}} I_j$. If there is an element K of $[I, G]$ such that $I_i, I_j < K$, then there exists an atom K_* of $[I_j, G]$ and an atom I_* of $[I, G]$ such that*

$$K_* \leq K, \quad K_* \in [I_*, G], \quad \textit{and} \quad I_* \prec^I_{\text{atm}} I_j.$$

Proof. Let $K_0 = I_i \cup I_j$ for simplicity. Note that $K_0 \subset K$, and one can check from Lemma 5.1 that there is no element $L \in \mathcal{P}^{\text{even}}_{G,A}$ such that $I_j \subsetneq L \subsetneq K_0$.

(Case 1) K_0 is not a semi-induced subgraph of G. Note that a subset of $V \cup B$ is not a semi-induced subgraph if and only if it contains $\{1, 2\}$ and has no element in B. Then $I_i \setminus I$ and $I_j \setminus I$ contain exactly one of the endpoints of B, not the same. More precisely, letting $v = \min(V \setminus (I \cup \{1, 2\}))$, one of the following holds:

$$\begin{array}{llll} I_i \setminus I = 1 & \text{and} & I_j \setminus I = 2v & \text{if } A \cap \{1,2\} = \{2\}, \\ I_i \setminus I = 1 & \text{and} & I_j \setminus I = 2 & \text{if } A \cap \{1,2\} = \emptyset, \\ I_i \setminus I = 1v & \text{and} & I_j \setminus I = 2v & \text{if } A \cap \{1,2\} = \{1,2\}. \end{array}$$

Note that $I_j \setminus I = 2v$ occurs only when 2 and 3 are adjacent in G. Moreover, if $I_i \setminus I = 1v_1$ and $I_j \setminus I = 2v_2$ hold, then I is a simple connected graph with even number of vertices, so $v_1 = v_2 = v$. Indeed, if G is $\tilde{P}'_{n,m}$, then $v_1 = v_2 = v$ by the structure of a path. If G is $\tilde{S}'_{n,m}$ or $\tilde{T}'_{n,m}$ ($n \geq 5$, odd), then $v_1 \neq v_2$ only when $I = V \setminus \{1, 2, n-1, n\}$.

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However, since n is odd, I has an odd number of vertices. This is a contradiction. In addition, $1 \notin A$ if and only if $|K_0 \cap A| \equiv |(K_0 \setminus I) \cap A| \equiv 0 \pmod 2$. Since K has both 1 and 2, it should have a multiple edge e. We find K_* and I_* according to the parity of $|A \cap \{1, e\}|$.

(Subcase 1) $|A \cap \{1, e\}| \equiv 0 \pmod 2$. In this case, $K_* = K_0 \cup e = I_j \cup 1e$ and $I_* = I_i$.

(Subcase 2) $|A \cap \{1, e\}| = 1$. We consider the connected component H of K containing e. Since $e \in H \setminus K_0$, $|K_0 \cap A| \equiv |H \cap K_0 \cap A| \equiv |(H \setminus K_0) \cap A|$, where the first equivalence follows from the definition of H and the second equivalence comes from $|H \cap A| \equiv 0 \pmod 2$. Hence, $1 \notin A$ if and only if $|(H \setminus K_0) \cap A|$ is even. Let $X = (H \setminus K_0) \cap A$ for simplicity. If $|X|$ is even, then $1 \notin A$ and $e \in A$, and therefore, $|X \setminus \{e\}| \geq 1$. If $|X|$ is odd, then $1 \in A$ and $e \notin A$, and therefore, $|X \setminus \{e\}| = |X| \geq 1$. Hence, in any case, we can take an element $c \in X \setminus \{e\}$ so that $K_* = K_0 \cup ce = I_j \cup 1ce$ and $I_* = I_i$. More precisely, either c is the vertex $\min(V \setminus (I_j \cup \{1, 2\}))$ or belongs to $(B \cap A) \setminus e$.

(Case 2) K_0 is a semi-induced subgraph of G. Note that $(I_i \setminus I) \cap (I_j \setminus I) \cap A$ is nonempty and has at most three elements.

(Subcase 1) $|(I_i \setminus I) \cap (I_j \setminus I) \cap A|$ is even. In this case,

$$|K_0 \cap A| = |I_i \cap A| + |I_j \cap A| - |(I_i \cap I_j) \cap A| = |I_i \cap A| + |I_j \cap A| - |I \cap A| - |(I_i \setminus I) \cap (I_j \setminus I) \cap A|,$$

and therefore $|K_0 \cap A|$ is even. By (†) in the proof of Lemma 5.1, $K_0 = I_j \cup (I_i \setminus I_j)$ is an atom of $[I_j, G]$, so $K_* = K_0$ and $I_* = I_i$.

(Subcase 2) $|(I_i \setminus I) \cap (I_j \setminus I) \cap A|$ is odd. Then $|(I_i \setminus I) \cap (I_j \setminus I) \cap A|$ is one or three. Since $(I_i \setminus I) \cap (I_j \setminus I) \cap A \neq \emptyset$, the elements in $K_0 \setminus I$ lie on the same connected component of K_0 by (†). Thus, K_0 has exactly one connected component H_0 such that $|H_0 \cap A|$ is odd. Note that for the connected component H of K containing H_0, we have $|(H \setminus H_0) \cap A| \geq 1$ since $|H \cap A|$ is even.

(i) If H_0 contains a multiple edge, then there exists an element $c \in (H \setminus H_0) \cap A$ such that $K_* = K_0 \cup c$ and $I_* = I_i$. More precisely, if $(H \setminus H_0) \cap B \cap A \neq \emptyset$, then $c \in B \cap A$; otherwise, c is the vertex $\min(V \setminus H_0)$.

(ii) Suppose that H_0 has no multiple edges. Then both $I_i \setminus I$ and $I_j \setminus I$ consist of two vertices in A, and $|(I_i \setminus I) \cap (I_j \setminus I) \cap A| = 1$. Since H_0 is a semi-induced subgraph of G and $|H_0 \cap A|$ is odd, $(H \setminus H_0) \cap V \neq \emptyset$. If $(H \setminus H_0) \cap V$ has a vertex in $\{3, \ldots, n\}$, then by the structure of G, it is easy to see that there is a vertex v in $(H \setminus H_0) \cap \{3, \ldots, n\}$ such that $K_* = K_0 \cup v$ and $I_* = I_i$. Hence, we only need to consider the case in which $(H \setminus H_0) \cap V \subset \{1, 2\}$. If $(H \setminus H_0) \cap V = \{1, 2\}$, then

$$\begin{cases} K_* = I_j \cup 1 \text{ and } I_* = I \cup 1 & \text{if } 1 \notin A \\ K_* = K_0 \cup 1 \text{ and } I_* = I_i & \text{if } 1 \in A. \end{cases}$$

It remains to consider the case where $(H \setminus H_0) \cap V = \{1\}$ or $\{2\}$. Let $(H \setminus H_0) \cap V = \{w_1\}$, and let w_2 be the other vertex in $\{1, 2\}$. If H_0 does not contain w_2, then $H = H_0 \cup w_1$ and $w_1 \in A$ (and therefore, w_1 must be a neighbour of 3 since H is an element of $\mathcal{P}^{\text{even}}_{G,A}$), and hence $K_* = K_0 \cup w_1$ and $I_* = I_i$. The remaining case is that H_0 contains w_2. Then H contains a multiple edge, that is, $(H \setminus H_0) \cap B = H \cap B \neq \emptyset$. Hence,

$$1 \leq |H \cap B| = |H \cap B \cap A| + |H \cap (B \setminus A)|.$$

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Since $|(H \setminus H_0) \cap A|$ is odd and

$$|(H\setminus H_0)\cap A| = |(H\setminus H_0)\cap V\cap A| + |(H\setminus H_0)\cap B\cap A| = |\{w_1\}\cap A| + |(H\setminus H_0)\cap B\cap A|,$$

we see

$$\begin{cases} |H\cap B\cap A| \geq 1 & \text{if } w_1 \notin A, \\ |H\cap B\cap A| \geq 2 & \text{if } w_1 \in A \text{ and } H\cap (B\setminus A) = \emptyset \\ |H\cap (B\setminus A)| \geq 1 & \text{otherwise.} \end{cases}$$

Now there are two possibilities: (a) $w_1 \notin A$ and (b) $w_1 \in A$. In (a), it easily follows that $K_* = K_0 \cup w_1 a$ and $I_* = I_i$ for some $a \in H \cap B \cap A$. In (b), we prove it by dividing two subcases whether $w_2 \in I$ or not. If $w_2 \in I$, then

$$\begin{cases} K_* = I_j \cup w_1 a a' \text{ and } I_* = I \cup w_1 a \text{ for some } a, a' \in H\cap B\cap A & \text{if } H\cap (B\setminus A) = \emptyset, \\ K_* = K_0 \cup w_1 b \text{ and } I_* = I_i \text{ for some } b \in H \cap (B\setminus A) & \text{if } H\cap (B\setminus A) \neq \emptyset. \end{cases}$$

If $w_2 \notin I$, then $w_2 \in (I_i \setminus I) \cup (I_j \setminus I)$. Since $I_i \prec^I_{\text{atm}} I_j, w_2 \in I_i$. Moreover, by the structure of G, $I_i \setminus I = w_2 v$ and $I_j \setminus I = vv'$ for $v = \min(V \setminus (I \cup \{1,2\}))$ and $v' = \min(V \setminus (I \cup \{1,2,v\}))$. Hence,

$$\begin{cases} K_* = I_j \cup 12aa' \text{ and } I_* = I \cup 12aa' \text{ for some } a, a' \in H\cap B\cap A & \text{if } H\cap (B\setminus A) = \emptyset, \\ K_* = I_j \cup 12b \text{ and } I_* = I \cup 12b \text{ for some } b \in H \cap (B\setminus A) & \text{if } H\cap (B\setminus A) \neq \emptyset. \end{cases}$$

This completes the proof. □

For an element I of $\mathcal{P}^{\text{even}}_{G,A}$, a multiple edge e is called a *big* (respectively, *small*) edge of I if $e \succ^I_{\text{lex}} n$ (respectively, $e \preceq^I_{\text{lex}} n$). Note that if e is a small edge of I, then $e \prec^I_{\text{lex}} 3$. Now we check that $\prec^I_{\text{atm}}$ satisfies Condition (1) of Definition 2.3.

For an atom I_j of $[I,G]$, suppose that an atom J of $[I_j,G]$ belongs to $[I_*,G]$ for some atom I_* of $[I,G]$ with $I_* \prec^I_{\text{atm}} I_j$ if and only if $\min^{I_j}(J\setminus I_j) \prec^{I_j}_{\text{lex}} z$ for some $z \in V\cup B$. Then the atoms J belonging to $[I_*,G]$ for some atom I_* of $[I,G]$ with $I_* \prec^I_{\text{atm}} I_j$ come first in the order $\prec^{I_j}_{\text{atm}}$. Hence, the following lemma says that $\prec^I_{\text{atm}}$ satisfies Condition (1) of Definition 2.3.

Lemma 5.6. *Let I_j be an atom of $[I,G]$, not the first in $\prec^I_{\text{atm}}$. Then an atom J of $[I_j,G]$ belongs to $[I_*,G]$ for some atom I_* of $[I,G]$ with $I_* \prec^I_{\text{atm}} I_j$ if and only if $\min^{I_j}(J\setminus I_j)$ satisfies one of the following:*

(1) $\min^{I_j}(J\setminus I_j) \prec^{I_j}_{\text{lex}} 2$ *if $I_j \setminus I \subset V$ and $(I_j\setminus I)\cap\{1,2\} \neq \emptyset$;*

(2) $\min^{I_j}(J\setminus I_j) \prec^{I_j}_{\text{lex}} \min^{I_j}\{v, b_1\}$ *if $I_j \setminus I = va$ for $v \in V$ and a small edge a of I in $B\cap A$;*

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(3) $\min^{I_j}(J \setminus I_j) \preceq^{I_j}_{\text{lex}} n$ *if* $I_j \setminus I$ *consists of only big edges of* I*;*

(4) $\min^{I_j}(J \setminus I_j) \preceq^{I_j}_{\text{lex}} \min(V \setminus I_j)$ *if* $V \setminus I_j \neq \emptyset$, $I_j \setminus I$ *has an element* $c \preceq^{I}_{\text{lex}} n$ *and a big edge of* I*, and* $|(I_j \setminus I) \cap \{1,2\}| \equiv 0 \pmod 2$*; and*

(5) $\min^{I_j}(J \setminus I_j) \prec^{I_j}_{\text{lex}} \max^{I_j}(I_j \setminus I)$ *otherwise.*

Proof. In each case, when we show the 'if' part, we show that there exists an atom I_* $[I, G]$ such that $J \in [I_*, G]$ and $I_* \prec^{I}_{\text{atm}} I_j$. On the other hand, we prove the 'only if' part by contradiction. Set $x = \min^{I_j}(J \setminus I_j)$.

(1) Suppose that $I_j \setminus I \subset V$ and $(I_j \setminus I) \cap \{1,2\} \neq \emptyset$. Note that

$$\prec^{I}_{\text{lex}}\colon\ 1, 2, \ldots, n, a_1, \ldots, a_{2m}, b_1, \ldots, b_\ell.$$

Since we assumed I_j is not the first atom of $[I, G]$, we have $(I_j \setminus I) \cap \{1,2\} = \{2\}$. Suppose that $x \prec^{I_j}_{\text{lex}} 2$. Then $x = 1$, and

$$J \setminus I_j = \begin{cases} 1ac \text{ or } 1b, & \text{if } 1 \notin A, \\ 1a \text{ or } 1vb & \text{if } 1 \in A, \end{cases}$$

where $a \in B \cap A$, $c \in A$, $b \in B \setminus A$ and $v = \min(V \setminus (I_j \cup \{1\}))$. Then I_* is either $I \cup 1$ or $I \cup 1v$, which proves the 'if' part.

Suppose that $x \succeq^{I_j}_{\text{lex}} 2$. Then $1 \notin J \setminus I_j$, so $J \setminus I_j$ cannot have a multiple edge. Thus, $J \setminus I_j$ consists of vertices greater than $\max^{I}(I_j \setminus I)$ by (†) in the proof of Lemma 5.1. Then I_j is the first atom of $[I, G]$, which is a contradiction to the assumption.

(2) Suppose $I_j \setminus I = va$ for $v \in V$ and a small edge a of I in $B \cap A$. The existence of a small edge of I implies $\{1,2\} \subset I$ and $v = \min(V \setminus I)$. Hence, $\prec^{I}_{\text{lex}} = \prec^{I_j}_{\text{lex}}$, and they are either

$$\prec^{I}_{\text{lex}} = \prec^{I_j}_{\text{lex}}\colon\ 1, 2, a_1, a_2, \ldots, a_k, 3, \ldots, n, a_{k+1}, \ldots, a_{2m}, b_1, \ldots, b_\ell, \quad \text{or}$$
$$\prec^{I}_{\text{lex}} = \prec^{I_j}_{\text{lex}}\colon\ 1, 2, a_1, a_2, \ldots, a_{2m}, b_1, \ldots, b_k, 3, \ldots, n, b_{k+1}, \ldots, b_\ell.$$

Suppose $x \prec^{I_j}_{\text{lex}} \min^{I_j}\{v, b_1\}$. Then $J \setminus I_j$ has an element $a' \prec^{I_j}_{\text{lex}} \min^{I_j}\{v, b_1\}$. Hence, $a' \in B \cap A$ and $I_* = I \cup aa'$, which proves the 'if' part.

Suppose $x \succeq^{I_j}_{\text{lex}} \min^{I_j}\{v, b_1\}$. Then $I_j \lessdot J$ is of type (E1′) or (E2). If $J \setminus I_j = b$ for some $b \in B \setminus A$, then $[I, J]$ has only two atoms I_j and $I \cup b$, so I_j is the first. If $J \setminus I_j = cc' \subset A$ for some $c, c' \succeq^{I_j}_{\text{lex}} v$, then $I_j \setminus I$ consists of the first two smallest elements of $J \setminus I$, so I_j is the first. Then, in any case, I_j is the first atom of $[I, J]$, which is a contradiction to the assumption.

(3) Suppose that $I_j \setminus I$ consists of only big edges of I. Then $I \cap B \neq \emptyset$ and either $I_j \setminus I = aa'$ or $I_j \setminus I = b$, where $a, a' \in B \cap A$ and $b \in B \setminus A$.

<u>(Case 1) $I_j \setminus I = aa'$</u>. The existence of a big edge of I in $B \cap A$ implies that $I \cap (B \setminus A) = \emptyset$, so the lexicographic orders $\prec^{I}_{\text{lex}}$ and $\prec^{I_j}_{\text{lex}}$ are as follows:

$$\prec^{I}_{\text{lex}}\colon 1, 2, a_1, \ldots, a_k, 3, \ldots, n, a_{k+1}, \ldots, a, \ldots, a'(= a_t), \ldots, a_{2m}, b_1, \ldots, b_\ell$$

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$$\prec_{\text{lex}}^{I_j}: 1, 2, a_1, \dots, a_k, \dots, a, \dots, a'(= a_t), 3, \dots, n, a_{t+1}, \dots, a_{2m}, b_1, \dots, b_\ell.$$

Suppose that $x \preceq_{\text{lex}}^{I_j} n$. Then $x \prec_{\text{lex}}^{I} a'$ and $I_j \lessdot J$ is of (E2). Hence, we can set $J \setminus I_j = xx' \subset A$, where $x \prec_{\text{lex}}^{I_j} x'$. If $x \preceq_{\text{lex}}^{I_j} \min(V \setminus I)$, then $I_* = I \cup ax$. If $\min(V \setminus I) \prec_{\text{lex}}^{I_j} x$, then both x and x' are vertices and $I_* = I \cup xx'$. This proves the 'if' part.

Suppose that $x \succ_{\text{lex}}^{I_j} n$. Then either $J \setminus I_j = xx' \subset B \cap A$ or $J \setminus I_j = x \in B \setminus A$. Since $a \prec_{\text{lex}}^{I} a' \prec_{\text{lex}}^{I} x$, I_j is the first atom of $[I, J]$ in $\prec_{\text{atm}}^{I}$. This is a contradiction to the assumption.

<u>(Case 2)</u> <u>$I_j \setminus I = b$</u>. In this case, the lexicographic order $\prec_{\text{lex}}^{I_j}$ is given as follows:

$$\prec_{\text{lex}}^{I_j}: 1, 2, a_1, a_2, \dots, a_{2m}, b_1, \dots, b(= b_k), 3, 4, \dots, n, b_{k+1}, \dots, b_\ell.$$

Suppose that $J \setminus I_j$ contains an element x with $x \preceq_{\text{lex}}^{I_j} n$. Note that $x \prec_{\text{lex}}^{I} b$. If x is a vertex, then $J \setminus I_j$ consists of two vertices, and $I_* = I \cup (J \setminus I_j)$. Now let x be a multiple edge e. If $e \in B \setminus A$, then $I_* = I_j \cup e$. If $e \in B \cap A$, then $J \setminus I_j = ec$ for some $c \in A$, which implies that $I_* = I \cup ec$. This proves the 'if' part.

If $x \succeq_{\text{lex}}^{I_j} n$, then $J \setminus I_j = \{b'\}$ for some $b' \in B \setminus A$ with $b \prec_{\text{lex}}^{I_j} b'$, and hence $[I, J]$ has only two atoms I_j and $I \cup b'$, where I_j is the first atom in $\prec_{\text{atm}}^{I}$. This is a contradiction to the assumption. □

In order to show **(4)** and **(5)**, we need to show the following claim.

Claim 5.7. *Suppose that $I \cap (B \setminus A)$ is empty, $I_j \setminus I$ has both an element $c \preceq_{\text{lex}}^{I} n$ and a big edge of I, and $|(I_j \setminus I) \cap \{1, 2\}| \equiv 0 \pmod 2$. Then an atom J of $[I_j, G]$ belongs to $[I_*, G]$ for some atom I_* of $[I, G]$ with $I_* \prec_{\text{atm}}^{I} I_j$ if and only if one of the following holds:*

(i) $x \preceq_{\text{lex}}^{I_j} \min(V \setminus I_j)$ *if* $V \setminus I_j \neq \emptyset$,

(ii) $x \prec_{\text{lex}}^{I_j} \max^{I_j}(I_j \setminus I)$ *if* $V \setminus I_j = \emptyset$.

Proof of Claim 5.7. From the hypotheses, we need to consider the following four cases ①∼④ in the table below, where $a, a' \in B \cap A$ with $a' \prec_{\text{lex}}^{I} a$, $b \in B \setminus A$, and $v = \min(V \setminus I)$:

Note that cases ① and ② occur only when $\{1, 2\} \subset A$. Let $v_* := \min(V \setminus I_j)$, provided $V \setminus I_j \neq \emptyset$. In cases ①∼④, if $v_* \in J \setminus I_j$, then I_*'s are $I \cup 1v_*$, $I \cup 1v_*$, $I \cup vv_*$ and $I \cup a'v_*$, respectively. Note that when $v_* \notin J \setminus I_j$, it holds that $x \preceq_{\text{lex}}^{I_j} v_*$ if and only if $x \prec_{\text{lex}}^{I_j} \max^{I_j}(I_j \setminus I)$. Now we assume that $v_* \notin J \setminus I_j$ and $x \prec_{\text{lex}}^{I_j} \max^{I_j}(I_j \setminus I)$. Since x is a small edge of I_j, x is also a multiple edge, and hence $J \setminus I_j$ consists of multiple edges. In case ②, $I_* = I \cup 12 \cup (J \setminus I_j)$. For the other cases, $x \prec_{\text{lex}}^{I} a$ and $x, a \in B \cap A$. Hence, I_* is obtained from I_j by replacing a with x. This proves the 'if' part.

To prove the 'only if' part, first suppose that $V \setminus I_j \neq \emptyset$ and $x \succ_{\text{lex}}^{I_j} v_*$. Then x is either a vertex greater than v_* or a big edge of I_j. Hence, $J \setminus I_j$ consists of either two vertices greater than v_* or only big edges of I_j. Note that a big edge of I_j is also a big edge of I.

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$I_j \setminus I$	The lexicographic orders
① $12a'a$	$\prec_{\text{lex}}^{I}$: $1,2,\dots,n,a_1,\dots,a',\dots,a(=a_k),\dots,a_{2m},b_1,\dots,b_\ell$
	$\prec_{\text{lex}}^{I_j}$: $1,2,a_1,\dots,a',\dots,a(=a_k),3,\dots,n,a_{k+1},\dots,a_{2m},b_1,\dots,b_\ell$
② $12b$	$\prec_{\text{lex}}^{I}$: $1,2,\dots,n,a_1,\dots,a_{2m},b_1,\dots,b(=b_k),\dots,b_\ell$
	$\prec_{\text{lex}}^{I_j}$: $1,2,a_1,\dots,a_{2m},b_1,\dots,b(=b_k),3,\dots,n,b_{k+1},\dots,b_\ell$
③ va	$\prec_{\text{lex}}^{I}$: $1,2,a_1,\dots,a_j,3,\dots,v,\dots,n,a_{j+1},\dots,a(=a_k),\dots,a_{2m},b_1,\dots,b_\ell$
	$\prec_{\text{lex}}^{I_j}$: $1,2,a_1,\dots,a'(=a_k),3,\dots,v,\dots,n,a_{k+1},\dots,a_{2m},b_1,\dots,b_\ell$
④ $a'a$	$\prec_{\text{lex}}^{I}$: $1,2,a_1,\dots,a',\dots,a_j,3,\dots,n,a_{j+1},\dots,a(=a_k),\dots,a_{2m},b_1,\dots,b_\ell$
	$\prec_{\text{lex}}^{I_j}$: $1,2,a_1,\dots,a',\dots,a(=a_k),3,\dots,n,a_{k+1},\dots,a_{2m},b_1,\dots,b_\ell$

Then I_j is the first atom of $[I,J]$ in $\prec_{\text{atm}}^{I}$, which is a contradiction to the assumption. If $V \setminus I_j = \emptyset$ and $x \succ_{\text{lex}}^{I_j} \max^{I_j}(I_j \setminus I)$, then x is a big edge of I_j, and hence $I_j \setminus I$ consists of only big edges of I_j. Then I_j is the first atom of $[I,J]$ in $\prec_{\text{atm}}^{I}$, which is a contradiction to the assumption. □

From Claim 5.7, **(4)** follows and **(5)** partially follows. We exclude the cases of **(1)**–**(4)** and the case shown by Claim 5.7. We divide the remaining part into two cases according to the existence of a big edge of I in $I_j \setminus I$.
(Case 1) $I_j \setminus I$ has no big edge of I. By excluding **(1)** and **(2)**, we get one of the following:

① $I_j \setminus I = b$, where $b \in B \setminus A$ and b is a small edge of I;
② $I_j \setminus I = aa'$, where both $a, a' \in B \cap A$ are small edges of I; or
③ $I_j \setminus I = vv'$, where $v, v' \in V \setminus \{1,2\}$.

In each case, the 'only if' part easily follows, that is, if $x \succ_{\text{lex}}^{I_j} \max^{I_j}(I_j \setminus I)$, then $I_j \setminus I$ has the first $|I_j \setminus I|$ smallest elements of $J \setminus I$ (in $\prec_{\text{lex}}^{I}$), so I_j is the first atom of $[I,J]$ in $\prec_{\text{atm}}^{I}$. Let us prove the 'if' part of each case. We assume that $x \prec_{\text{lex}}^{I_j} \max^{I_j}(I_j \setminus I)$ and note that $\prec_{\text{lex}}^{I} = \prec_{\text{lex}}^{I_j}$.
① From the existence of a small edge in $B \setminus A$, it follows that $I \cap (B \setminus A) \neq \emptyset$ and

$$\prec_{\text{lex}}^{I} = \prec_{\text{lex}}^{I_j}: 1,2,a_1,a_2,\dots,a_{2m},b_1,\dots,b,\dots,b_k,3,\dots,n,b_{k+1},\dots,b_\ell.$$

If $x \prec_{\text{lex}}^{I_j} b$, then $I_* = I \cup (J \setminus I_j)$.
② Assume $a \prec_{\text{lex}}^{I_j} a'$. Then $a' = \max^{I_j}(J \setminus I_j)$. The assumption $x \prec_{\text{lex}}^{I_j} a'$ implies that $J \setminus I$ contains a multiple edge a'' with $a'' \prec_{\text{lex}}^{I} a'$, so $I_* = I \cup aa''$.
③ Assume $v \prec_{\text{lex}}^{I_j} v'$. Then $v' = \max^{I_j}(I_j \setminus I)$, so $x \prec_{\text{lex}}^{I_j} v'$. If $J \setminus I_j \subset V$, then $J \setminus I \subset V$ and $I_* = I \cup xy$, where x and y are the first two smallest elements of $J \setminus I$. When $J \setminus I_j \not\subset V$, Table 2 shows how to obtain I_*:

Table 2. I_*, *where* $b \in B \setminus A, a, a' \in A \cap B, v'' = \min V \setminus (I_j \cup x)$, $v_* = \min\{v, v''\}$ *and* $c_* = \min_{I_j}\{v, c\}$

$J \setminus I_j$	$1ac$ or $2ac$	$1v''a$ or $2v''b$	$v''a$	other cases $(b, 1b, 2b, 12b, aa', 12aa', 1a, 2a)$
I_*	$I \cup 1ac_*$ or $I \cup 2ac_*$	$I \cup 1v_*b$ or $I \cup 2v_*b$	$I \cup v$ or $I \cup v_*a$	$I \cup (J \setminus I_j)$

(Case 2) $I_j \setminus I$ has a big edge of I. Excluding **(3)** and **(4)**, we get the following five cases ①∼⑤ in the table, where $w \in \{1, 2\}, a, a' \in B \cap A$ with $a' \prec_{\text{lex}}^{I_j} a$, $b \in B \setminus A$, and $v = \min(V \setminus (I \cup \{1, 2\}))$:

	$I_j \setminus I$	The lexicographic order $\prec_{\text{lex}}^{I_j}$	$\max^{I_j}(I_j \setminus I)$
$w \notin A$	① $wa'a$	$\prec_{\text{lex}}^{I_j}$: $1, 2, a_1, \ldots, a', \ldots, a(= a_k)$, $3, \ldots, n, a_{k+1}, \ldots, a_{2m}, b_1, \ldots, b_\ell$	a
	② wva	$\prec_{\text{lex}}^{I_j}$: $1, 2, a_1, \ldots, a(= a_k)$, $3, \ldots, v, \ldots, n, a_{k+1}, \ldots, a_{2m}, b_1, \ldots, b_\ell$	v
	③ wb	$\prec_{\text{lex}}^{I_j}$: $1, 2, a_1, \ldots, a_{2m}, b_1, \ldots, b(= b_k)$, $3, \ldots, n, b_{k+1}, \ldots, b_\ell$	b
$w \in A$	④ wa	$\prec_{\text{lex}}^{I_j}$: $1, 2, a_1, \ldots, a(= a_k)$, $3, \ldots, n, a_{k+1}, \ldots, a_{2m}, b_1, \ldots, b_\ell$	a
	⑤ wvb	$\prec_{\text{lex}}^{I_j}$: $1, 2, a_1, \ldots, a_{2m}, b_1, \ldots, b(= b_k)$, $3, \ldots, v, \ldots, n, b_{k+1}, \ldots, b_\ell$	v

Note that, in any case, the lexicographic ordering $\prec_{\text{lex}}^{I}$ on $V \cup B$ is given by

$$\prec_{\text{lex}}^{I}\colon\ 1, 2, 3, \ldots, n, a_1, \ldots, \ldots, \ldots, a_{2m}, b_1, \ldots, b_\ell,$$

and any atom of $[I, J]$ containing the element w has a multiple edge. If $x \succ_{\text{lex}}^{I_j} \max^{I_j}(I_j \setminus I)$, then $J \setminus I_j$ cannot have a multiple edge less than $\max^I(B \cap (I_j \setminus I))$ in $\prec_{\text{lex}}^{I}$. Then I_j is the first atom of $[I, J]$ in $\prec_{\text{atm}}^{I}$, which is a contradiction to the assumption.

Proof. Suppose that $x \prec_{\text{lex}}^{I_j} \max^{I_j}(I_j \setminus I)$. In cases ①, ② and ④, $J \setminus I_j$ contains a multiple edge a'' with $a'' \prec_{\text{lex}}^{I_j} a$, which implies that I_* can be obtained from I_j by replacing a with a''. In cases ③ and ⑤, $I_j \setminus I$ contains a multiple edge $e \prec_{\text{lex}}^{I_j} b$. If $e \in A$, then $J \setminus I_j$ is ec for some $c \in A$, and hence I_*'s are $I \cup wec$ and $I \cup we$, respectively. If $e \notin A$, then I_*'s are $I \cup we$ and $I \cup wve$, respectively. This proves the 'if' part. $\square$

Lemmas 5.5 and 5.6 imply that the ordering $\prec_{\text{atm}}^{I}$ satisfies Definition 2.3. This proves Theorem 5.4.

6. Falling chains of $\mathcal{P}_{H,A}^{\text{even}}$

In this section, for a graph H in Figure 1.1 and its admissible collection A, we study the falling chains of our CL-shelling on $\mathcal{P}_{H,A}^{\text{even}}$, which let us know the homotopy type of the order complex of the proper part $\Delta(\overline{\mathcal{P}_{H,A}^{\text{even}}})$. Throughout the section, the labelling of the vertices follows the way in Figure 5.2, so that the labels of the endpoints of the bundle are changed according to A. We always let V and B be the vertex set and the bundle of G, respectively.

Recall that if a bounded poset $\mathcal{P}$ admits a recursive atom ordering, then we can find a CL-labelling ρ as in the sketch of the proof of Theorem 2.4. Furthermore, the ith reduced Betti number of the order complex $\Delta(\overline{\mathcal{P}})$ equals the number of falling chains of length $i+2$ from Theorem 2.2. For a graph $G \in \mathcal{G}^*$, if G is simple, then the homotopy type of $\Delta(\overline{\mathcal{P}_G^{\text{even}}})$ is completely determined by the length $\ell(\mathcal{P}_G^{\text{even}})$ and the Möbius invariant $\mu(\mathcal{P}_G^{\text{even}})$, see [9]. If G is a graph in Figure 1.1, then as we saw in §5, the order $\prec_{\text{atm}}^{I}$ in Definition 5.3 gives a recursive atom ordering of $\mathcal{P}_{G,A}^{\text{even}}$ for every $A \in \mathcal{A}(G)$. Hence, we can determine the homotopy type of $\Delta(\overline{\mathcal{P}_{G,A}^{\text{even}}})$ by considering the CL-labelling ρ obtained from the recursive atom order on $\mathcal{P}_{G,A}^{\text{even}}$.

Proposition 6.1. *Let (G, A) be a pair of a graph G and its admissible collection A illustrated in Figure 5.2. Then $\mathcal{P}_{G,A}^{\text{even}}$ has a falling chain if and only if one of the following holds: (a) 2 and 3 are adjacent, and (b) $|V|$ is even.*

Proof. We show the 'only if' part first. Let $\sigma : I_0 \lessdot I_1 \lessdot \cdots \lessdot I_p$ be a falling chain of $\mathcal{P}_{G,A}^{\text{even}}$ and let $I_{k-1} \lessdot I_k$ be the cover such that $I_k \setminus I_{k-1}$ contains the vertex 1. Suppose that the vertices 2 and 3 are not adjacent and $|V|$ is odd. Then $A \cap V \neq V$ and $1 \notin A$. Note that there is no cover $I \lessdot J$ such that $J \setminus I$ contains $\{1, 2\}$, and therefore, $2 \notin I_k \setminus I_{k-1}$. Since $1 \notin I_{k-1}$ and (a) fails, it follows that $2 \notin I_{k-1}$, so $2 \notin I_k$. Thus, $I_k = I_{k-1} \cup 1$ and hence $I_{k-1} \lessdot I_k$ is the first atom of $[I_{k-1}, G]$, which implies that σ cannot be a falling chain, a contradiction.

To show the 'if' part, recall that $A = \{a_1, \ldots, a_m\}$ and $B \setminus A = \{b_1, b_2, \ldots, b_\ell\}$ as long as $B \setminus A \neq \emptyset$. We fix a falling chain σ_E of an interval of $\mathcal{P}_{G,A}^{\text{even}}$ as follows. If $B \setminus A \neq \emptyset$, then let σ_E be a falling chain of $[V \cup b_\ell, G]$ defined by

$$V \cup b_\ell \lessdot V \cup b_{\ell-1} b_\ell \lessdot \cdots \lessdot V \cup b_1 \cdots b_\ell \lessdot \cdots \lessdot V \cup a_{m-1} a_m b_1 \cdots b_\ell \lessdot \cdots \lessdot V \cup a_1 \cdots b_\ell = G.$$

If $B \subset A$, then let σ_E be a falling chain of $[V \cup a_{m-1} a_m, G]$ defined by

$$V \cup a_{m-1} a_m \lessdot V \cup a_{m-3} a_{m-2} a_{m-1} a_m \lessdot \cdots \lessdot \cdots \lessdot V \cup a_1 \cdots a_m = G.$$

Note that there is a falling chain σ_* of $[\emptyset, I]$, where

$$I = \begin{cases} 34 \cdots n & \text{if } n \text{ is even;} \\ 4 \cdots n & \text{if } n \text{ is odd and } n \geq 5; \\ \emptyset & \text{if } n = 3. \end{cases}$$

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Suppose that (a) or (b) is true. We will show that σ_* and σ_E are extended to a falling chain of $\mathcal{P}^{\text{even}}_{G,A}$. When $\{1,2\} \subset A$, the chain σ obtained by concatenating σ_*, σ_0 and σ_E is a falling chain of $\mathcal{P}^{\text{even}}_{G,A}$, where

$$\sigma_0 = \begin{cases} I \lessdot I \cup 12b_\ell & \text{if } B \setminus A \neq \emptyset, \\ I \lessdot I \cup 12a_{m-1}a_m & \text{if } B \subset A. \end{cases}$$

When $A \cap \{1,2\} = \emptyset$, the chain σ obtained by concatenating $\sigma_*, I \lessdot I \cup 2 \lessdot I \cup 1b_\ell$ and σ_E is a falling chain of $\mathcal{P}^{\text{even}}_{G,A}$. If $A \cap \{1,2\} = \{2\}$, then (a) is true, and hence the chain σ obtained by concatenating σ_*, σ_0 and σ_E is a falling chain of $\mathcal{P}^{\text{even}}_{G,A}$, where

$$\sigma_0 = \begin{cases} I \lessdot I \cup 23 \lessdot 1b_\ell & \text{if } B \setminus A \neq \emptyset, \\ I \lessdot I \cup 23 \lessdot 1a_{m-1}a_m & \text{if } B \subset A. \end{cases}$$

$\square$

Proposition 6.2. *Let (G, A) be a pair of a graph G and its admissible collection A illustrated in Figure 5.2. Let $\sigma : I_0 \lessdot I_1 \lessdot \cdots \lessdot I_{p+1}$ be a falling chain of $\mathcal{P}^{\text{even}}_{G,A}$, and $I_{k-1} \lessdot I_k$ be the cover satisfying $1 \in I_k \setminus I_{k-1}$. Then the possible values of length $\ell(\sigma)$ and the set $I_k \setminus I_{k-1}$ are represented in Table 3.*

Proof. We first prove the case where $A \cap V \neq V$. Then $1 \notin A$ by the way of labelling. If $I_k = I_{k-1} \cup 1$, then $I_{k-1} \cup 1$ is the first atom of $[I_{k-1}, G]$, so the chain σ cannot be a falling chain. Hence, $I_k \neq I_{k-1} \cup 1$ and $2 \in I_{k-1}$. Then the vertices 2 and 3 are adjacent. By Proposition 5.2, if $|V|$ is even, then $\mathcal{P}^{\text{even}}_{G,A}$ is pure and $\ell(\sigma) = \frac{|A|}{2} + |B \setminus A| + 1$; if $|V|$ is odd, then σ is not a longest chain, so $\ell(\sigma) = \frac{|A|}{2} + |B \setminus A|$. If $B \setminus A \neq \emptyset$, then we let q be the first index such that $I_q \setminus I_{q-1}$ contains an element b in $B \setminus A$. Since $I_{q-2} \lessdot I_{q-1} \lessdot I_q$ is falling, we get $k = q$, that is, $I_k \setminus I_{k-1} = 1b$. If $B \subset A$, then $I_k \setminus I_{k-1} = 1aa'$ or $1av$ for some $a, a' \in B \cap A$ and $v \in V \setminus \{1,2\}$.

Now we assume $A \cap V = V$. Then $|V|$ is even. If $2 \notin I_k$, then $I_k = I_{k-1} \cup 1v$ for some $v \in V \setminus \{1,2\}$. Then σ cannot be a falling chain since $I_{k-1} \cup 1v$ is the first atom of $[I_{k-1}, G]$. Thus, $2 \in I_k$.

Suppose that $B \setminus A \neq \emptyset$. We let q be the first index such that $I_q \setminus I_{q-1}$ contains an element b in $B \setminus A$. Since $I_{q-2} \lessdot I_{q-1} \lessdot I_q$ is a falling chain of $[I_{q-2}, I_q]$, we have $k = q$, that is, $b \in I_k$. Then

$$I_k \setminus I_{k-1} = \begin{cases} 1vb & \text{if } 2 \in I_{k-1}; \\ 12b & \text{if } 2 \in I_k \setminus I_{k-1}. \end{cases}$$

Thus, σ is not a longest chain, so $\ell(\sigma) = \frac{|A|}{2} + |B \setminus A| - 1$ by Proposition 5.2.

Suppose that $B \subset A$. Suppose that the vertices 2 and 3 are not adjacent. Then $2 \notin I_{k-1}$, so $2 \in I_{k-1}$. Thus, $I_k \setminus I_{k-1} = 12aa'$ for some $a, a' \in B$, which implies that σ is not a longest chain. Hence, $\ell(\sigma) = \frac{|A|}{2} + |B \setminus A| - 1$ by Proposition 5.2. If the vertices 2 and 3

Table 3. Each cell represents $\ell(\sigma)$ and $I_k \setminus I_{k-1}$, where $a, a' \in A \cap B$, $b \in B \setminus A$ and $v \in V \setminus \{1, 2\}$.

$\ell(\sigma), I_k \setminus I_{k-1}$		$\lvert V\rvert$ is even		$\lvert V\rvert$ is odd (2 and 3 are adjacent by Proposition 6.1)
		$\widetilde{P}_{n,m}, \widetilde{S}_{n,m}, \widetilde{T}_{n,m}$	$\widetilde{P}'_{n,m}, \widetilde{S}'_{n,m}, \widetilde{T}'_{n,m}$	
$A \cap V = V$	$B \setminus A \neq \emptyset$	$\frac{\lvert A\rvert}{2} + \lvert B \setminus A\rvert - 1, 12b$	$\frac{\lvert A\rvert}{2} + \lvert B \setminus A\rvert - 1$, $12b$ or $1vb$	N/A
	$B \subset A$	$\frac{\lvert A\rvert}{2} - 1$, $12aa'$	$\frac{\lvert A\rvert}{2} - 1$ or $\frac{\lvert A\rvert}{2}$, $12aa'$ or $1a$	
$A \cap V \neq V$ $(1 \notin A)$	$B \setminus A \neq \emptyset$	$\frac{\lvert A\rvert}{2} + \lvert B \setminus A\rvert + 1$, $1b$		$\frac{\lvert A\rvert}{2} + \lvert B \setminus A\rvert$, $1b$
	$B \subset A$	$\frac{\lvert A\rvert}{2} + 1$, $1aa'$ or $1av$		$\frac{\lvert A\rvert}{2}$, $1aa'$ or $1av$

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are adjacent, then by Proposition 5.2, $\ell(\sigma) = \frac{|A|}{2} - 1$ or $\frac{|A|}{2}$, and it holds that

$$I_k \setminus I_{k-1} = \begin{cases} 1a & \text{if } 2 \in I_{k-1}; \\ 12aa' & \text{if } 2 \in I_k \setminus I_{k-1}. \end{cases}$$

□

By Theorem 2.2 and Propositions 6.1 and 6.2, the following hold:

Corollary 6.3. *Let (G, A) be a pair of a graph G and its admissible collection A illustrated in Figure 5.2. If neither (a) nor (b) of Proposition 6.1 holds, then $\Delta(\overline{\mathcal{P}^{\text{even}}_{G,A}})$ is contractible. If not, the order complex $\Delta(\overline{\mathcal{P}^{\text{even}}_{G,A}})$ is homotopy equivalent to a wedge of spheres of dimensions*

$$\begin{cases} \frac{|A|}{2} + |B \setminus A| - 2 & \textit{if } |V| \textit{is odd}, \\ \frac{|A|}{2} + |B \setminus A| - 1 & \textit{if } |V| \textit{is even and } A \cap V \neq V, \\ \frac{|A|}{2} - 1 \textit{ or } \frac{|A|}{2} & \textit{if } A = V \cup B \textit{ and the vertices 2 and 3 are adjacent, and} \\ \frac{|A|}{2} + |B \setminus A| - 3 & \textit{otherwise.} \end{cases}$$

Example 6.4. See the posets $\mathcal{P}^{\text{even}}_{G,A}$ in Figure 5.3. The posets in (i) and (iii) are nonpure but none of the longest maximal chains of (i) and (iii) are falling chains. In (i), (ii) and (iii), there are four, three and four falling chains, respectively:

$$\begin{array}{ll}
(i) & \emptyset < \mathbf{23} < \mathbf{123}\boldsymbol{b_1} < 1234\boldsymbol{a_1}b_1 < 1234\mathbf{5}a_1\boldsymbol{a_2}b_1 \\
 & \emptyset < \mathbf{23} < \mathbf{123}\boldsymbol{b_1} < 1234\boldsymbol{a_2}b_1 < 1234\mathbf{5}\boldsymbol{a_1}a_2b_1 \\
 & \emptyset < \mathbf{34} < \mathbf{2}345 < \mathbf{1}2345\boldsymbol{b_1} < 12345\boldsymbol{a_1}\boldsymbol{a_2}b_1 \\
 & \emptyset < \mathbf{45} < \mathbf{2}345 < \mathbf{1}2345\boldsymbol{b_1} < 12345\boldsymbol{a_1}\boldsymbol{a_2}b_1 \\
(ii) & \emptyset < \mathbf{2} < \mathbf{12}\boldsymbol{b_2} < 12\boldsymbol{b_1}b_2 < 12\mathbf{3}\boldsymbol{a_1}b_1b_2 < 1234a_1\boldsymbol{a_2}b_1b_2 \\
 & \emptyset < \mathbf{2} < \mathbf{12}\boldsymbol{b_2} < 12\boldsymbol{b_1}b_2 < 12\mathbf{3}\boldsymbol{a_2}b_1b_2 < 1234\boldsymbol{a_1}a_2b_1b_2 \\
 & \emptyset < \mathbf{34} < \mathbf{2}34 < \mathbf{1}234\boldsymbol{b_2} < 1234\boldsymbol{b_1}b_2 < 1234\boldsymbol{a_1}\boldsymbol{a_2}b_1b_2 \\
(iii) & \emptyset < \mathbf{12}\boldsymbol{b_2} < 12\boldsymbol{a_1}\boldsymbol{a_2}b_2 < 12a_1a_2\boldsymbol{b_1}b_2 < 12\mathbf{34}a_1a_2b_1b_2 \\
 & \emptyset < \mathbf{34} < \mathbf{12}34\boldsymbol{b_2} < 1234\boldsymbol{b_1}b_2 < 1234\boldsymbol{a_1}\boldsymbol{a_2}b_1b_2 \\
 & \emptyset < \mathbf{12}\boldsymbol{b_2} < 12\boldsymbol{b_1}b_2 < 12\mathbf{3}\boldsymbol{a_1}b_1b_2 < 1234a_1\boldsymbol{a_2}b_1b_2 \\
 & \emptyset < \mathbf{12}\boldsymbol{b_2} < 12\boldsymbol{b_1}b_2 < 12\mathbf{3}\boldsymbol{a_2}b_1b_2 < 1234\boldsymbol{a_1}a_2b_1b_2
\end{array}$$

Hence, the order complexes $\Delta(\overline{\mathcal{P}^{\text{even}}_{G,A}})$ of the proper parts of the posets in Figure 5.3 are homotopy equivalent to $\bigvee_4 S^2$, $\bigvee_3 S^3$ and $\bigvee_4 S^2$, respectively.

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7. Betti numbers of the real toric variety associated with $\widetilde{P}_{n,2}$

In this section, we consider the graph $G_n = \widetilde{P}_{n,2}$ in Figure 1.1. We count the number of falling chains of $\mathcal{P}^{\text{even}}_{G_n,A}$ for $A \in \mathcal{A}(G_n)$ and then compute the Betti numbers of the projective smooth real toric variety $X^{\mathbb{R}}_{G_n}$.

Proposition 7.1. *Let $G_n = \widetilde{P}_{n,2}$ in Figure 1.1. For $A \in \mathcal{A}(G_n)$, the number of falling chains of $\mathcal{P}^{\text{even}}_{G_n,A}$ is*

$$\begin{cases} C_k & \text{if } n = 2k \text{ for some } k \geq 1 \\ C_{k+1} - C_k & \text{if } n = 2k+1 \text{ for some } k \geq 1 \text{ and } A \cap V \text{ itself induces a connected graph,} \\ 0 & \text{otherwise,} \end{cases}$$

where C_k is the kth Catalan number.

Proof. Let V be the set of vertices and $B = \{a_1, a_2\}$ be the bundle of G_n. Recall that we follow the labelling of the vertices shown in Figure 5.2. Note that $A = V \cup B$ or $(V \cup B) \setminus 1$ or $(V \cup B) \setminus \{1, 2\}$.

Suppose that $A = V \cup B$. Then $|V|$ is even, and by the way of labelling, the vertices 2 and 3 are not adjacent. Let $\sigma : I_0 \lessdot I_1 \lessdot \cdots \lessdot I_{p+1}$ be a falling chain of $\mathcal{P}^{\text{even}}_{G_n,A}$, and $I_{k-1} \lessdot I_k$ be the cover such that $I_k \setminus I_{k-1}$ contains the vertex 1. By Proposition 6.2, $I_k \setminus I_{k-1} = 12a_1a_2$ and the number of falling chains of $\mathcal{P}^{\text{even}}_{G_n,A}$ is equal to

$$\sum_{I \subset V \setminus \{1,2\}} (\#\text{falling chains of } [I \cup 12a_1a_2, G_n]) \times (\#\text{falling chains of } [\emptyset, I]). \tag{7.1}$$

Since $|V|$ is even, we can set $|V| = 2k$ for some $k \geq 1$. If $k = 1$, then it is clear. Suppose that $k \geq 2$. From Equation (7.1), the number of falling chains of $\mathcal{P}^{\text{even}}_{G_n,A}$ is

$$\begin{aligned} &\sum_{q=1}^{2} (\#\text{falling chains of } \mathcal{P}_{2q} \text{ starting with } 12a_1a_2) \\ &\qquad \times \sum_{\substack{|I|=2k-2q \\ I \subset \{3,4,\ldots,2k\}}} (\#\text{falling chains of } [\emptyset, I]) \\ &= C_{k-1} + (\#\text{falling chains of } \mathcal{P}_4 \text{ starting with } 12a_1a_2) \\ &\qquad \times \sum_{\substack{|I|=2k-4 \\ I \subset V \setminus \{1,2\}}} (\#\text{falling chains of } [\emptyset, I]), \end{aligned}$$

where $\mathcal{P}_{2q}$ means the poset $\mathcal{P}^{\text{even}}_{G_{2q},G_{2q}}$ for $q \leq \lfloor \frac{n}{2} \rfloor$, and the second summation is over the vertices I of $\mathcal{P}^{\text{even}}_{G_n,A}$. Since the number of falling chains of $\mathcal{P}_4$ starting with $12a_1a_2$ is

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only one (see the second poset of Figure 3.2), the number of falling chains is $C_{k-1}+s$, where

$$s = \sum_{\substack{|I|=2k-4\\ I\subset V\setminus\{1,2\}}} (\#\text{falling chains of } [\emptyset, I]).$$

Let $I \subset \{3,4,\dots,2k\}$ be an element of $\mathcal{P}^{\text{even}}_{G_n,A}$ with $2k-4$ vertices. Then $V\setminus I = \{1,2,v_1,v_2\}$, where $v_1<v_2$. Since each connected component of I has an even number of vertices, v_1 is odd and v_2 is even. Thus, the number of falling chains of $[\emptyset, I]$ is $C_{\frac{v_1-3}{2}}C_{\frac{v_2-v_1-1}{2}}C_{\frac{2n-v_2}{2}}$. By a recursion of the Catalan numbers,

$$s = \sum_{\substack{v_1=3\\ v_1:\text{odd}}}^{2k-1}\ \sum_{\substack{v_2=v_1+1\\ v_2:\text{even}}}^{2k} C_{\frac{v_1-3}{2}}C_{\frac{v_2-v_1-1}{2}}C_{\frac{2k-v_2}{2}} = \sum_{\substack{v_1=3\\ v_1:\text{odd}}}^{2k-1} C_{\frac{v_1-3}{2}}C_{\frac{2k-v_1+1}{2}} = C_k - C_{k-1}. \quad (7.2)$$

Hence, the number of falling chains is $C_{k-1}+s=C_k$ when $n=2k$ $(k\geq 1)$.

Now we suppose that $A\cap V\neq V$. Then A is either $(V\cup B)\setminus 1$ or $(V\cup B)\setminus\{1,2\}$. Then $I_k\setminus I_{k-1} = 1a_1a_2$ or $1av$ by Proposition 6.2, where $a\in B$ and $v\in V\setminus\{1,2\}$. Thus, the number of falling chains of $\mathcal{P}^{\text{even}}_{G_n,A}$ is equal to

$$\sum_{\substack{I\subset V\setminus\{1\}\\ 2\in I}} (\#\text{falling chains of } [I\cup 1ac, G_n] \text{ for some } a\in B, c\in A) \times (\#\text{falling chains of } [\emptyset, I]). \quad (7.3)$$

Note that it follows from Proposition 6.2 that there is no falling chain of $\mathcal{P}^{\text{even}}_{G_n,A}$ if $|V|$ is odd and $A\cap V$ does not induce a connected graph. Hence, we need to consider the case where $|V|$ is even or $A\cap V$ induces a connected graph. In Equation (7.3), a falling chain of $[I\cup 1a_ic, G_n]$ for some $a_i\in B$ and $c\in A$ is either $I\lessdot I\cup 1a_2v\lessdot G_n$ $(v\in V)$ or $I\lessdot I\cup 1a_1a_2 = G_n$. In each of the cases, it is uniquely determined. Hence, the number of falling chains is equal to s_1+s_2, where

$$s_1 = (\#\text{falling chains of } [\emptyset, G_n\setminus(1\cup B)]), \qquad s_2 = \sum_{\substack{|I|=n-3\\ I\subset V\setminus\{1\},\ 2\in I}} (\#\text{falling chains of } [\emptyset, I]).$$

Let us compute s_1+s_2. First, suppose $n=2k$ and $k\geq 1$. Then s_1 is equal to C_{k-1}, the number of falling chains of $\mathcal{P}^{\text{even}}_{P_{2k-2}}$. If $k=1$, then $s_2=0$, so the number of falling chains is C_1 (since $C_0=C_1=1$). Suppose that $k\geq 2$. Let $I\subset\{2,3,\dots,2k\}$ be an element of $\mathcal{P}^{\text{even}}_{G_n,A}$ with $2k-3$ vertices containing the vertex 2. Then $V\setminus I=\{1,v_1,v_2\}$, where $2<v_1<v_2$. Since each connected component of I has an even number of vertices and $2\notin A$, v_1 is odd and v_2 is even. Since s_2 has the same equation as in Equation (7.2), $s_1+s_2 = C_{k-1}+(C_k-C_{k-1}) = C_k$. Hence, the number of falling chains is C_k if $n=2k$.

Suppose $n=2k+1$ and $k\geq 1$. Then s_1 is equal to C_k, the number of falling chains of $\mathcal{P}^{\text{even}}_{P_{2k}}$. If $k=1$, then $s_2=1$, so the number of falling chains is C_2-C_1 (note $C_2=2$ and

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Table 4. The homotopy types of $\overline{\mathcal{P}^{\text{even}}_{G_n,A}}$ for $A \in \mathcal{A}^*(G_n)$ and $G_n = \widetilde{P}_{n,2}$

		$\Delta(\overline{\mathcal{P}^{\text{even}}_{G_n,A}})$	
n	$A \in \mathcal{A}(G_n)$	Dimension	Homotopy type
$2k$	$[n] \setminus A = \emptyset$	$\frac{\lvert A\rvert}{2} - 3 = k-2$	$\bigvee_{C_k} S^{k-2}$
	$[n] \setminus A \neq \emptyset$	$\frac{\lvert A\rvert}{2} - 1 = k-1$	$\bigvee_{C_k} S^{k-1}$
$2k+1$	$[n] \setminus A \neq \emptyset$	$\frac{\lvert A\rvert}{2} - 2 = k-1$	$\bigvee_{C_{k+1}-C_k} S^{k-1}$

The last row of the table is true only when $A \cap V$ induces a connected graph.

$C_1 = 1$). Suppose $k \geq 2$. Let $I \subset \{2, 3, \ldots, 2k+1\}$ be an element of $\mathcal{P}^{\text{even}}_{G_n,A}$ with $2k-3$ vertices containing the vertex 2. Then $V \setminus I = \{1, v_1, v_2\}$, where $2 < v_1 < v_2$. Since each connected component of I has an even number of vertices and $2 \in A$, v_1 is even and v_2 is odd. Thus, $s_1 + s_2 = C_k + (C_{k+1} - 2C_k) = C_{k+1} - C_k$ since

$$s_2 = \sum_{\substack{v_1=4\\ v_1:\text{even}}}^{2k} \sum_{\substack{v_2=v_1+1\\ v_2:\text{odd}}}^{2k+1} C_{\frac{v_1-2}{2}} C_{\frac{v_2-v_1-1}{2}} C_{\frac{2k+1-v_2}{2}} = \sum_{\substack{v_1=4\\ v_1:\text{even}}}^{2k} C_{\frac{v_1-2}{2}} C_{\frac{2k-v_1+2}{2}} = C_{k+1} - 2C_k.$$

Thus, the number of falling chains is $C_{k+1} - C_k$. It completes the proof. $\square$

From Corollary 6.3 and Proposition 7.1, we can compute the homotopy types of $\Delta(\overline{\mathcal{P}^{\text{even}}_{G_n,A}})$ when $G_n = \widetilde{P}_{n,2}$ and A is an admissible collection of G_n, as in Table 4. One may formulate the number of falling chains of $\mathcal{P}^{\text{even}}_{G,A}$, when $G = \widetilde{P}_{n,m}$, in terms of the Catalan numbers (or the secant numbers), and it would be interesting to explain the formula by using other combinatorial objects.

Now we are ready to compute the Betti numbers of the projective smooth real toric variety $X^{\mathbb{R}}_{G_n}$ associated with the graph G_n. It was shown in [9, Theorem 2.5] that, for the simple path P_n with n vertices, $\Delta(\overline{\mathcal{P}^{\text{even}}_{P_n}})$ is homotopy equivalent to $\bigvee_{C_k} S^{k-1}$ for $n = 2k$ and it is contractible for an odd integer n. In addition, for an integer $n \geq 1$, we have

$$\beta^i(X^{\mathbb{R}}_{P_n}) = \begin{cases} \binom{n}{i} - \binom{n}{i-1} & \text{if } n \geq 2 \text{ and } 0 \leq i \leq \lfloor \frac{n}{2} \rfloor, \\ 1 & \text{if } n = 1 \text{ and } i = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{7.4}$$

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Table 5. The Betti numbers $\beta^i(X^{\mathbb{R}}_{\widetilde{P}_{n,2}})$ for small n

i/n	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	1	2	6	10	15	21	28	36	45	55	66	78	91	105
3	0	0	0	6	18	33	54	82	118	163	218	284	362	453
4	0	0	0	0	0	18	56	110	192	310	473	691	975	1337
5	0	0	0	0	0	0	0	56	180	372	682	1155	1846	2821
6	0	0	0	0	0	0	0	0	0	180	594	1276	2431	4277
7	0	0	0	0	0	0	0	0	0	0	0	594	2002	4433
8	0	0	0	0	0	0	0	0	0	0	0	0	0	2002

For a non-simple connected graph G_k $(k>0)$ in Table 4, since the pseudograph associahedron P_{G_k} is k-dimensional, it follows from Equation (3.3) that

$$\sum_{A\in\mathcal{A}(G_k)} \widetilde{\beta}^i\left(\Delta\left(\overline{\mathcal{P}^{\mathrm{odd}}_{G_k,A}}\right)\right) = \begin{cases} C_{\frac{k}{2}} & \text{if } i=\frac{k}{2}\text{or } \frac{k}{2}-1 \text{ for even } k \\ C_{\frac{k+1}{2}} - C_{\frac{k-1}{2}} & \text{if } i=\frac{k-1}{2} \text{ for odd } k \\ 0 & \text{otherwise.} \end{cases} \tag{7.5}$$

Note that for a connected graph G_n and $(H,A)\in\mathcal{A}^*(G_n)$, if H_1 is a connected component of H and $A_1 = A\cap\mathcal{C}_{H_1}$ for $A\in\mathcal{A}(H)$, then $\mathcal{P}^{\mathrm{odd}}_{H,A}$ is isomorphic to the join $\mathcal{P}^{\mathrm{odd}}_{H_1,A_1} * \mathcal{P}^{\mathrm{odd}}_{H_2,A_2}$, where $H_2 = H\setminus H_1$ and $A_2 = A\setminus A_1$, see [8, Lemma 4.5], and therefore the following holds:

$$\widetilde{\beta}^{i-1}\left(\Delta\left(\overline{\mathcal{P}^{\mathrm{odd}}_{H,C}}\right)\right) = \sum_{\ell} \widetilde{\beta}^{\ell}\left(\Delta\left(\overline{\mathcal{P}^{\mathrm{odd}}_{H_1,C_1}}\right)\right) \times \widetilde{\beta}^{i-\ell-2}\left(\Delta\left(\overline{\mathcal{P}^{\mathrm{odd}}_{H_2,C_2}}\right)\right). \tag{7.6}$$

Now we are ready to explain how to compute $\beta^i(X^{\mathbb{R}}_{G_n})$ from Equations (7.4), (7.5) and (7.6). Assume that $i>0$. Let $\mathcal{H}_1$ be the set of all simple PI-graphs of G_n and $\mathcal{H}_2$ the set of all non-simple PI-graphs of G_n. By Proposition 3.4, $\beta^i(X^{\mathbb{R}}_{G_n}) = s_1^i(G_n) + s_2^i(G_n)$, where

$$s_1^i(G_n) = \sum_{H\in\mathcal{H}_1}\sum_{A\in\mathcal{A}(H)} \widetilde{\beta}^{i-1}(\Delta(\overline{\mathcal{P}^{\mathrm{odd}}_{H,A}})) \quad \text{and}$$
$$s_2^i(G_n) = \sum_{H\in\mathcal{H}_2}\sum_{A\in\mathcal{A}(H)} \widetilde{\beta}^{i-1}(\Delta(\overline{\mathcal{P}^{\mathrm{odd}}_{H,A}})).$$

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As $\mathcal{H}_1$ is the set of PI-graphs of the simple graph $P_n, s_1^i(G_n) = \beta^i(X^{\mathbb{R}}_{P_n})$. By Proposition 3.4 and Equation (7.6),

$$s_2^i(G_n) = \sum_{m=2}^{n-1} \sum_{\ell=0}^{\lfloor \frac{m}{2} \rfloor} \sum_{A \in \mathcal{A}(G_m)} \widetilde{\beta}^{\ell}\left(\Delta\left(\overline{\mathcal{P}^{\text{odd}}_{G_m,A}}\right)\right) \times \beta^{i-\ell-1}\left(X^{\mathbb{R}}_{P_{n-m-1}}\right) + \sum_{A \in \mathcal{A}(G_n)} \widetilde{\beta}^{i-1}\left(\Delta\left(\overline{\mathcal{P}^{\text{odd}}_{G_n,A}}\right)\right).$$

Note that

$$\beta^{i-1}(X^{\mathbb{R}}_{P_n}) + \beta^i(X^{\mathbb{R}}_{P_n}) = \beta^i(X^{\mathbb{R}}_{P_{n+1}}) \quad \text{for } 1 \le i < \frac{n}{2}, \tag{7.7}$$

that is, $s_1^{i-1}(G_n) + s_1^i(G_n) = s_1^i(G_{n+1})$ for $1 \le i < \frac{n}{2}$. Using Equations (7.5) and (7.7), we also have $s_2^{i-1}(G_n) + s_2^i(G_n) = s_2^i(G_{n+1})$ for $1 \le i < \frac{n}{2}$. Hence, for $1 \le i < \frac{n}{2}$, we have

$$\beta^{i-1}(X^{\mathbb{R}}_{G_n}) + \beta^i(X^{\mathbb{R}}_{G_n}) = \beta^i(X^{\mathbb{R}}_{G_{n+1}}). \tag{7.8}$$

For $k = \lfloor \frac{n}{2} \rfloor$, plugging Equations (7.4) and (7.5) into $s_1^k(G_n) + s_2^k(G_n)$, we see that

$$\beta^k\left(X^{\mathbb{R}}_{G_{2k}}\right) = \beta^{k+1}\left(X^{\mathbb{R}}_{G_{2k+1}}\right) = \frac{6k}{k+2} C_k, \tag{7.9}$$

which is known as the total number of nonempty subtrees over all binary trees having $k+1$ internal vertices, see[22, A071721]. Table 5 shows the Betti numbers of $X^{\mathbb{R}}_{G_n}$ for some small integers n.

On the other hand, it is not difficult to check $\{\beta^i(X^{\mathbb{R}}_{P_n})\}_{i \ge 0}$ in Equation (7.4) is log-concave, and hence unimodal. For $\widetilde{P}_{n,2}$, we can also see that $\{\beta^i(X^{\mathbb{R}}_{\widetilde{P}_{n,2}})\}_{i \ge 0}$ is unimodal from Equations (7.7) and (7.9). We remain the unimodality of $\beta^i(X^{\mathbb{R}}_{\widetilde{P}_{n,m}})$ for general n and m, as an open problem.

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Appendix 1. Appendix: Proof of Proposition 3.4

In this appendix, we give a sketch of Proposition 3.4. Since the notation given in [8] is not the same as this paper, we give the definitions used in [8].

Proof of Proposition 3.4. Let G be a graph and

$$2^{\mathcal{C}_G}_{\text{even}} = \{C \subset \mathcal{C}_G \mid |C \cap V| \text{is even, } |C \cap B| \text{is even for every bundle } B \text{ of } G\}.$$

Section 3 of [8] is devoted to show that for each subset S in the set $\{1,2,\ldots,\dim P_G\}$, there exists $C \in 2^{\mathcal{C}_G}_{\text{even}}$ such that $K_{P_G,S} = K^{\text{odd}}_{C,G}$. Hence, Equation (3.2) becomes

$$\beta^i(X^{\mathbb{R}}_G) = \sum_{C \in 2^{\mathcal{C}_G}_{\text{even}}} \tilde{\beta}^{i-1}(K^{\text{odd}}_{C,G}), \tag{A.1}$$

where $K^{\text{odd}}_{C,G}$ is the dual complex of the union of all facets F_I of P_G such that $|I \cap c$ is odd. Then, for $C \in 2^{\mathcal{C}_G}_{\text{even}}$, we define $\widetilde{\Gamma}_G(C)$ as the subgraph of G induced by the set

$$(V \cap C) \cup \{v \mid v \text{ is an endpoint of some edge } e \in C\}.$$

$\square$

Lemma 1.1. ([8, Lemmas 4.3 and 4.4]). *Let $K''_{C,G}$ be the subcomplex of $K^{\text{odd}}_{C,G}$ whose vertex set consists of vertices I satisfying the following:*

(1) *$I \subset \widetilde{\Gamma}_G(C)$ and $|I \cap C|$ is odd.*
(2) *For each bundle B of $\widetilde{\Gamma}_G(C)$ such that $B \cap C = \emptyset$, if the endpoints of B are in I, then $B \subset I$.*

Then $K^{\text{odd}}_{C,G}$ is homotopy equivalent to $K''_{C,G}$.

Then we get the following formula from Equation (A.1),

$$\beta^i(X^{\mathbb{R}}_G) = \sum_{C \in 2^{\mathcal{C}_G}_{\text{even}}} \tilde{\beta}^{i-1}(K''_{C,G}). \tag{A.2}$$

Lemma 1.2. ([8, Lemma 4.5(i)]). *For $C \in 2^{\mathcal{C}_G}_{\text{even}}$, if $|I \cap C|$ is odd for some connected component I of $\widetilde{\Gamma}_G(C)$, then $K''_{C,G}$ is contractible.*

The above lemma implies that, to compute $\beta^i(X^{\mathbb{R}}_G)$, it is enough to consider the collection C in $2^{\mathcal{C}_G}_{\text{even}}$ such that the intersection of C and each of the connected components of $\widetilde{\Gamma}_G(C)$ belongs to $2^{\mathcal{C}_G}_{\text{even}}$. Thus, we define

$$2^{\mathcal{C}_G}_{\text{even}*} := \{C \in 2^{\mathcal{C}_G}_{\text{even}} \mid \text{each connected component of } \widetilde{\Gamma}_G(C) \text{ is even with respect to } C\}.$$

Note that for a collection C not in $2^{\mathcal{C}_G}_{\text{even}*}$, $K^{\text{odd}}_{C,G}$ is contractible. For $C \in 2^{\mathcal{C}_G}_{\text{even}*}$, we let $\Gamma_G(C)$ be the graph from $\widetilde{\Gamma}_G(C)$ by replacing each bundle B of $\widetilde{\Gamma}_G(C)$, satisfying $C \cap B = \emptyset$ with an (unlabelled) simple edge.

Proposition 1.3. **([8, Proposition 4.7]).** *For* $C \in 2^{\mathcal{C}_G}_{\text{even}*}$, $K^{\text{odd}}_{C,G}$ *is homotopy equivalent to* $K^{\text{odd}}_{C,\Gamma_G(C)}$, *and hence, the ith rational Betti number of* $X^{\mathbb{R}}_G$ *is*

$$\beta^i(X^{\mathbb{R}}_G) = \sum_{C \in 2^{\mathcal{C}_G}_{\text{even}*}} \tilde{\beta}^{i-1}\left(K^{\text{odd}}_{C,\Gamma_G(C)}\right), \tag{A.3}$$

Proof. Note that $\Gamma_G(C)$ is a PI-graph of G. Moreover, by definition, it holds that for a PI-graph H of G, $C \in \mathcal{A}(H)$ if and only if $H = \Gamma_G(C)$. Thus,

$$\beta^i(X^{\mathbb{R}}_G) = \sum_{\substack{H:\text{PI-graph} \\ \text{of } G}} \sum_{A \in \mathcal{A}(H)} \tilde{\beta}^{i-1}(K^{\text{odd}}_{A,H}).$$

It was also shown in [8, §5] that for each admissible collection $A \in \mathcal{A}(H)$, the simplicial complex $K^{\text{odd}}_{A,H}$ is homotopy equivalent to the order complex of the proper part of the poset $\mathcal{P}^{\text{odd}}_{H,A}$.[8] This proves Proposition 3.4. □

[8] In [8], the proper part of the poset $\mathcal{P}^{\text{odd}}_{H,A}$ is denoted by $S^{\text{odd}}_{H,A}$.

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