

On the Tucker Circles of a Spherical Triangle

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This paper contains merely a statement of theorems in the geometry of a spherical triangle analogous to the well-known results discovered within recent years for a plane triangle. The analogy is a very close one, so that it has not been thought necessary to state the corresponding plane theorems alongside those given.

In the paper as read proofs were given of most of the theorems. A fuller account than the present, with additional developments, I hope to contribute on another occasion.

1. Let ABC be an ordinary spherical triangle of great circles. Let QAR , RBP , PCQ be the great circles tangent to the circum-circle at A , B , C . Join AP , BQ , and CR by great circles cutting the sides at X , Y , Z .

Then AP , BQ , CR meet at a point K , *the symmedian point*, within the triangle.

A spheroconic can be described to touch the sides at X , Y , Z , and the points antipodal to these, and this conic, *the Brocard conic*, has double contact with the circumcircle.

2. When the sides of ABC are not great circles, but their planes meet at any point V , the great circles of the above construction are to be replaced by circles whose planes pass through V . The Brocard conic is the section of the sphere by a quadric cone with vertex V . Three other real cones pass through the spheroconic. Their vertices form with V a self-conjugate tetrahedron of the sphere. The bitangent circles of the spheroconic are the sections of the sphere by the tangent planes to these cones. Foci are the point bitangent circles. Four of these are real and lie on the major axis. By a linear transformation, which for points on the sphere amounts simply to an inversion of the sphere into itself, the general spheroconic can be transformed into an ordinary spheroconic, that is, the section of the sphere by a cone with vertex at the centre. Any

spheroconic inverts in general from a point on the sphere into a plane bicircular quartic. Hence accurate figures can be constructed in a plane without altering the angles between circles.

We return now to the case of a great circular triangle.

3. Two of the real foci of a Brocard conic S , S' lie within the triangle, the other two are S_1 antipodal to S , and S_1' antipodal to S' . The four may be considered to be the Brocard points.

4. Let A , B , C be joined to S by circles through S_1' , and to S' by circles through S_1 .

These circles are not great circles, but are orthogonal to the third circular axis of the spheroconic.

The focus to be marked S can be so chosen that the six angles SAB , SBC , SCA and $S'BA$, $S'CB$, $S'AC$ are all equal.

Their common value is ω the Brocard angle.

5. The symmedian point K is the pole of the chord of contact of the circumcircle with the Brocard conic.

Like O , the centre of the circumcircle, it lies on the minor axis, and the great circles SK , SO are at right angles.

6. A Tucker circle of the spherical triangle is defined to be a circle bitangent to the Brocard conic and of the same system as the circumcircle. Its centre (pole) lies on the minor axis. The sine of the radius is proportional to the sine of the distance of the centre from a focus.

7. Let a Tucker circle cut the sides BC , CA , AB at D , D' ; E , E' ; F , F' respectively.

These points may be so marked that the following properties hold, the lines indicated by two letters being all great circles.

8. DE' , $D'F$ as also DF' , $D'E$ intersect on AK ;

EF' , $E'D$ as also ED' , $E'F$ on BK ;

FD' , $F'E$ as also FE' , $F'D$ on CK .

9. EF' , $E'F$ intersect on YZ , at L say ;

FD' , $F'D$ on ZX at M ; and DE' , $D'E$ on XY at N .

The Pascal line LMN is the chord of contact of the Tucker circle and the Brocard conic.

10. If $A'B'C'$ are the second points of intersection of the sides then B, C, E', F lie on a circle; likewise C, A, F', D ; A, B, D', E ; and B', C', E, F' ; C', A', F, D' ; A', B', D, E' .

11. One of the six points is arbitrary, but the other five are uniquely defined by the last property when one is given.

The six divisions formed on the three sides by points of the six types are homographic. On the side $BCB'C'$, B and C' belong to the type D , and C, B' to the type D' . The double points on BC are X and X' , the points of contact of the Brocard conic.

The right lines BD', CD intersect on the right line XX' .

12. If $D_1D'_1E_1E'_1F_1F'_1$ be another Tucker circle with the points marked on the same system as before except for the suffixes, then E, F', E'_1, F_1 lie on a circle, as also E', F, E_1, F'_1 .

13. For the circumcircle the points E', F coincide at A , but E, F' are those which coincide for the second Tucker circle through A . Thus the circle AEF' touches the circumcircle at A , and the circle $AE'F$ touches the second Tucker circle through A . Similarly the circle $A'E'F$ touches the circumcircle $A'B'C'$, and $A'EF'$ touches the second Tucker circle through A' .

14. Let A, B, C and D, E, F be joined to S by circles through S'_1 ; and A, B, C, D', E', F' to S' by circles through S_1 . Then considering angles formed by these circles and the sides, the six angles SDD', SEE', SFF' and $S'D'D, S'E'E, S'F'F$ are all equal.

15. The difference between one of these angles and the Brocard angle is equal to the angle between the *great* circles through S (or S') orthogonal to the circumcircle and the Tucker circle.

16. For the *auxiliary circle*, which touches the Brocard ellipse at the ends of the major axis, the angle SDD' is a right angle.

Hence the angle between the major axis and the great circle SO is $\frac{\pi}{2} - \omega$, and (§5) the angle between the major axis and the great circle SK is ω .

With the help of this result a formula can be found for $\tan 2\omega$, rational in the sines and cosines of the angles of the triangle. This gives two values for ω differing by a right angle. The

explanation is that the four imaginary foci of the Brocard conic which lie on the third circular axis have similar properties to those of the four real Brocard conics. These four foci are the points where the sphere intersects the right lines conjugate with respect to the sphere to SS_1' and $S'S_1$, so that all circles through one pair are orthogonal to the circles through SS_1' , and those through the other pair to the circles through $S'S_1$.

17. SD makes the same angle with the Tucker circle that SB makes with the circumcircle.

Thus (art. 15) if θ be the angle between the great circles through S to O and the centre of the Tucker circle, then the side BC cuts the Tucker circle at an angle $\frac{\pi}{2} - \frac{1}{2}(A + B + C) + A - \theta$.

18. The six angles ASF , BSD , CSE , and $AS'E'$, $BS'F'$, $CS'D'$ are all equal, each being equal to the angle between the circles through S, S_1' orthogonal to the circumcircle and the Tucker circle.

19. Let a triangle DEF be formed by circles EFA' , FDB' , and DEC' . Similarly form a triangle $D'E'F'$ by circles $E'F'A'$, $F'D'B'$, and $D'E'C'$. Then these triangles are equiangular to one another and to ABC .

The angles SFD , SDE and SEF are equal to the Brocard angle of ABC .

20. The envelope of the circle EFA' is a degenerate sort of spheroconic (*a spherical parabola*), one of the ovals of which shrinks into a point at A' . It inverts from A' into a plane conic.

21. By a linear transformation which leaves points still on the sphere, the triangle DEF can be transformed into the triangle ABC , while the points S, S_1' remain unchanged. Hence these points are two of the Brocard points of the triangle DEF .

The triangles $DEF, D'E'F'$ have therefore the same Brocard angle as ABC (cf. art. 16).

22. Let the circles DSS_1' and $D'S'S_1$ meet at d and d' . Let the circle through S, S_1' orthogonal to the circumcircle meet the minor axis at O_1, O_1' , and the circle through S, S_1' orthogonal to this meet

the minor axis in K_1, K_1' . Then the circles $dd'O_1O_1'$ and $dd'K_1K_1'$ are at right angles to each other.

The points d, d' and those obtained similarly from the same Tucker circle and the other two sides (or any other great circular tangent to the Brocard conic) lie on a general spheroconic which passes through O_1, O_1', K_1, K_1' , through the four Brocard points, through the four foci which may be considered to lie on a circle at infinity, and through the points of contact with the Brocard conic of the Tucker circle, of the other Tucker circle orthogonal to SO_1 , and of the two Tucker circles orthogonal to SK_1 .

23. If $A_1B_1C_1$ be any other triangle inscribed in the circumcircle and circumscribed to the Brocard conic, it has the same symmedian point and Brocard angle as ABC . Also the various angles of arts. 14–18 for a given Tucker circle are the same in this triangle as in ABC .

