# PARTIAL GAUSSIAN SUMS III

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**1.** For integers $a, N$ and $H > 0$ write

$$T_a(N, H) = \sum_{x=N+1}^{N+H} \chi(x) e(ax/k),$$

where $\chi$ denotes a non-principal Dirichlet character modulo the positive integer $k$ and $e(y)$ denotes $e^{2\pi iy}$. By a well-known generalisation of the Pólya–Vinogradov inequality

$$|T_a(N, H)| \ll k^{1/2} \log k. \tag{1}$$

In [**4**] I showed that, for any integer $r \geqslant 2$ and any prime $k$,

$$|T_a(N, H)| \ll H^{1-1/r} k^{1/4(r-1)} \log^2 k. \tag{2}$$

For $r = 2$ this is a simple consequence of (1), but for each $r > 2$ (2) is stronger than (1) for some range of $H$. In [**5**] I showed (the case $r = 3$)

$$|T_a(N, H)| \ll H^{2/3} k^{1/8+\varepsilon}$$

holds for any positive integer $k$. In the present paper I prove (the case $r = 4$) the following theorem.

Theorem 1. *Let $k = p^\alpha$ be a power of a prime $p > 3$. Then for all integers $a, N$ and $H > 0$ and all non-principal characters $\chi$ modulo $k$ we have*

$$|T_a(N, H)| \ll H^{3/4} k^{1/12} \log^3 k,$$

*the implied constant being absolute.*

By arguments such as that of A. I. Vinogradov [**8**] the following theorem can be deduced.

Theorem 2. *With the conditions of Theorem* 1 *we have*

$$|T_a(N, H)| \ll H^{1/4} k^{1/3} \log^4 k.$$

This estimate is a new estimate even for pure character sums $T_0(N, H)$ for the range

$$k^{7/12} < H < k^{2/3}, \qquad N \neq 0.$$

By the argument of [**1**, §5] it suffices to assume that $\chi$ is a primitive character to the prime power modulus $k$ to prove Theorem 1. The latter result itself follows by the arguments of [**4**] from the following theorem.

Theorem 3. *Let $\chi$ be a primitive character to the modulus $k = p^\alpha (\alpha > 1)$ a power of the prime $p > 3$. Then*

$$\sum_{a=0}^{k-1} \sum_{n=0}^{k-1} |T_a(n, h)|^8 \ll \alpha^{12} k^2 h^4 \qquad (0 < h \leqslant k^{1/6}).$$

This theorem we shall prove in the remainder of this paper.

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**2.** Let $r$ be any fixed positive integer. Let $X$ denote an indeterminate. For vectors $\mathbf{m} \in \mathbb{Z}^{2r}$ with components satisfying

$$0 < m_i \leq h \qquad (0 < i \leq 2r)$$

put

$$f_1(X) = \prod_{i=1}^{r} (X + m_i), \qquad f_2(X) = \prod_{i=r+1}^{2r} (X + m_i).$$

Write

$$A_1 = \{\mathbf{m} : 0 < m_i \leq h, m_1 + \ldots + m_r = m_{r+1} + \ldots + m_{2r}\},$$
$$A_2(\alpha) = \{x : 0 \leq x < p^\alpha, p \nmid f_1(x)f_2(x)\}.$$

Then

$$\sum_{a=0}^{k-1} \sum_{n=0}^{k-1} |T_a(n,h)|^{2r} \leq k \sum_{\mathbf{m} \in A_1} \left| \sum_{x \in A_2(\alpha)} \chi\left(\frac{f_1}{f_2}(x)\right) \right|. \tag{3}$$

The principal task of this paper is to give, in the following propositions, some suitable estimates for the inner sum of (3). These estimates are then combined with the theorem of **[6]** to prove Theorem 3. To describe the required estimates we write

$$F(X) = f_1'(X)f_2(X) - f_2'(X)f_1(X).$$

Note that the degree of $F$ is $\leq 2r - 3$. Let $s \geq 0$ be a fixed integer. Let $\mathbf{d} \in \mathbb{Z}^{s+1}$ be of the form

$$\mathbf{d} = (d_0, d_1, \ldots, d_s), \qquad d_0 \geq d_1 \geq \ldots \geq d_s \geq 0,$$

and let $\gamma$ be a positive integer. Write†

$$A_3(\gamma, \mathbf{d}) = \{x \in A_2(\gamma) : p^{d_i} \mid F^{(i)}(x)(0 \leq i \leq s), \text{ if } s > 0\, p^{d_s} = (p^{d_{s-1}}, F^{(s)}(x))\},$$
$$A_4(\gamma, \mathbf{d}) = \{x \in A_3(\gamma, \mathbf{d}) : p^{d_i} = (p^{d_{i-1}}, F^{(i)}(x))(0 < i \leq s)\},$$
$$A_5(\gamma, \mathbf{d}) = \{x \in A_4(\gamma, \mathbf{d}) : p^{d_0} \,\|\, F(x)\}.$$

For $j = 3, 4$ and $5$ write

$$S_j(\gamma, \mathbf{d}) = \sum_{x \in A_j(\gamma, \mathbf{d})} \chi\left(\frac{f_1}{f_2}(x)\right).$$

In the next section we shall prove the following propositions.‡

PROPOSITION 1. *If*

$$d_0 < \lceil \alpha/2 \rceil, \qquad s = 0,$$

*then*

$$S_5(\alpha, \mathbf{d}) \ll p^{\alpha/2}.$$

PROPOSITION 2. *If*

$$d_0 = \lceil \alpha/2 \rceil, \qquad 0 \leq d_1 < \lceil \alpha/3 \rceil, \qquad s = 1,$$

† $p^c \,\|\, d$ denotes $p^c \mid d$ but $p^{c+1} \nmid d$.
‡ $\lceil x \rceil$ denotes the least integer greater than or equal to $x$.

*then*

$$S_3(\alpha, \mathbf{d}) \ll p^{(\alpha+d_1)/2}.$$

PROPOSITION 3. *If*

$$\lceil \alpha/2 \rceil \leq d_0 < [2\alpha/3], \qquad \lceil \alpha/3 \rceil \leq d_1, \qquad s = 1,$$

*then*

$$S_5(\alpha, \mathbf{d}) \ll \alpha^{2r-4} p^{[2\alpha/3]}.$$

PROPOSITION 4. *If*

$$d_0 = [2\alpha/3], \qquad \lceil \alpha/3 \rceil \leq d_1, \qquad s \geq 1,$$

*then for each $i$ with $0 < i \leq s$*

$$S_4(\alpha, \mathbf{d}) \ll p^{\alpha + d_i - d_{i-1}}.$$

It should be noted that in these propositions $p$ is not restricted to be greater than 3.

These propositions are sufficient to replace Theorems 1 to 6 of [**2**] in the proof of Theorem 2 of [**3**] and so to provide the proofs of the character and Gaussian sum estimate of [**3**] and [**5**] with $r = 3$.

**3.** In the proofs of these propositions we shall use the following lemma.

LEMMA 1. *Let $f$ be a polynomial of degree $n$ having integer coefficients. Let $p$ be a prime and $\phi$ and $\psi$ be non-negative integers. Then*

$$\#\{x \in \textit{a complete set of residues} \pmod{p^\phi} : p^{\phi+\psi} \mid f(x), p^\psi \| f'(x)\} \leq n.$$

*Proof.* This is a special case of Proposition 1 of [**6**].

Write

$$\beta = \lceil \alpha/2 \rceil, \qquad \gamma = \lceil \alpha/3 \rceil,$$

so that

$$[\alpha/2] = \alpha - \beta, \qquad [2\alpha/3] = \alpha - \gamma.$$

For any $x$ denote by $\lambda(x)$ the inverse of $f_1(x) f_2(x) \pmod{p^\alpha}$, when it is defined.

Since $\chi$ is a primitive character modulo $p^\alpha$ there exists a $c$, not divisible by $p$, such that

$$\chi(1 + p^\beta) = e(c/p^{\alpha-\beta}).$$

We shall require the $p$-adic Taylor expansion of $(f_1/f_2)(x + p^\varepsilon y) \pmod{p^\alpha}$ which can be written as

$$\begin{aligned} \frac{f_1}{f_2}(x + p^\varepsilon y) \equiv \frac{f_1}{f_2}(x)\Big\{1 &+ \frac{F(x)}{f_1(x) f_2(x)} p^\varepsilon y \\ &+ \frac{1}{2!}\left(\frac{F'(x)}{f_1(x) f_2(x)} + F(x) \frac{f_2(x)}{f_1(x)} \left(\frac{1}{f_2(x)^2}\right)'\right) p^{2\varepsilon} y^2 + \ldots\Big\} \pmod{p^\alpha}. \end{aligned}$$

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*Proof of Proposition* 1. For $p>2$ this is essentially the proof of Theorems 1, 2 and 3 of [**2**], but for completeness it is sketched here.

For all primes $p$ we have

$$\begin{aligned}
S_5(\alpha,(d_0)) &= \sum_{x\in A_5(\beta,(d_0))}\ \sum_{0\le y<p^{\alpha-\beta}} \chi\Big(\frac{f_1}{f_2}(x+yp^{\beta})\Big)\\
&= \sum_{x\in A_5(\beta,(d_0))} \chi\Big(\frac{f_1}{f_2}(x)\Big) \sum_{0\le y<p^{\alpha-\beta}} \mathrm{e}(c\lambda(x)yp^{\beta}F(x)/p^{\alpha})\\
&= \begin{cases} 0 & \text{if } d_0<\alpha-\beta,\\ p^{\alpha-\beta}S_5(\beta,(d_0)) & \text{if } \alpha \text{ odd and } d_0=\alpha-\beta.\end{cases}
\end{aligned}$$

In the latter case, for $p>2$ we have

$$\begin{aligned}
S_5(\beta,(d_0)) &= \sum_{x\in A_3(\beta-1,(\beta-1))}\ \sum_{\substack{0\le y<p\\ p^{\beta-1}\,\|\, F(x+yp^{\beta-1})}} \chi\Big(\frac{f_1}{f_2}(x+yp^{\beta-1})\Big)\\
&= \sum_{x\in A_3(\beta-1,(\beta-1,0))} \chi\Big(\frac{f_1}{f_2}(x)\Big) \sum_{0\le y<p} \mathrm{e}(c\lambda(x)(yp^{\beta-1}F(x)\\
&\quad + \tfrac{1}{2}y^2p^{2\beta-2}F'(x))/p^{\alpha})\\
&\quad + 0(\#A_3(\beta-1,(\beta-1,0))) + \sum_{\substack{x\in A_5(\beta-1,(\beta-1))\\ p\mid F'(x)}} \chi\Big(\frac{f_1}{f_2}(x)\Big)\\
&\quad \times \sum_{0\le y<p} \mathrm{e}(c\lambda(x)yp^{\beta-1}F(x)/p^{\alpha}).
\end{aligned}$$

The last sum vanishes. By Lemma 1

$$\#A_3(\beta-1,(\beta-1,0)) = O(1),$$

and for elements $x$ of this set

$$\sum_{0\le y<p} \mathrm{e}(c\lambda(x)(yp^{\beta-1}F(x)+\tfrac{1}{2}y^2p^{2\beta-2}F'(x))/p^{\alpha}) = O(\sqrt{p})$$

by Weil (see [7, II, Corollary 2F}). On the other hand for $p=2$ we may assume $\alpha\ge 5$ and then we have

$$\begin{aligned}
S_5(\beta,(d_0)) &= \sum_{\substack{x\in A_5(\beta,(\beta-1))\\ p^2\mid F'(x)}} \chi\Big(\frac{f_1}{f_2}(x)\Big) + O(\#A_5(\beta,(\beta-1,0))) + O(\#A_5(\beta,(\beta-1,1)))\\
&= \sum_{\substack{x\in A_5(\beta-1,(\beta-1))\\ p^2\mid F'(x)}}\ \sum_{0\le y<p} \chi\Big(\frac{f_1}{f_2}(x+yp^{\beta-1})\Big) + O(1)\\
&= \sum_{\substack{x\in A_5(\beta-1,(\beta-1))\\ p^2\mid F'(x)}} \chi\Big(\frac{f_1}{f_2}(x)\Big) \sum_{o\le y<p} \mathrm{e}(c\lambda(x)yp^{\beta-1}F(x)/p^{\alpha}) + O(1),
\end{aligned}$$

and the inner sum vanishes.

*Proof of Proposition* 2. For $p>3$ and $r=3$ this is essentially Theorem 4 of [**2**], but the proof presented here is shorter even in that case.

Write

$$\delta=\lceil(\alpha-d_1)/2\rceil, \tag{4}$$

so that

$$\beta\geq\delta>\alpha/3>d_1,\qquad \delta+d_1\geq d_0.$$

Then for all primes $p>3$ we have

$$\begin{aligned}S_3(\alpha,\mathbf{d})&=\sum_{x\in A_3(\delta,\mathbf{d})}\sum_{0\leq y<p^{\alpha-\delta}}\chi\left(\frac{f_1}{f_2}(x+yp^{\delta})\right)\\&=\sum_{x\in A_3(\delta,\mathbf{d})}\chi\left(\frac{f_1}{f_2}(x)\right)\sum_{0\leq y<p^{\alpha-\delta}}\mathrm{e}(c\lambda(x)yp^{\delta}F(x)/p^{\alpha})\\&=p^{\alpha-\delta}S_3(\delta,(\max(d_0,\alpha-\delta),d_1)).\end{aligned}$$

Thus by Lemma 1

$$\begin{aligned}S_3(\alpha,\mathbf{d})&\ll p^{\alpha-\delta}\#A_3(\delta,(\max(d_0,\alpha-\delta),d_1))\\&\ll p^{\alpha-\max(d_0,\alpha-\delta)+d_1}\\&\ll p^{\frac{1}{2}(\alpha+d_1)}\end{aligned}$$

if $\alpha-d_1$ is even or $d_1=0$. If $\alpha-d_1$ is odd and $d_1>0$ and if further $d_1<\delta-1$ then

$$\begin{aligned}S_3(\alpha,\mathbf{d})&=p^{\alpha-\delta}\sum_{x\in A_3(\delta,(\alpha-\delta,d_1))}\chi\left(\frac{f_1}{f_2}(x)\right)\\&=p^{\alpha-\delta}\sum_{x\in A_3(\delta-1,(\alpha-\delta,d_1))}\sum_{0\leq y<p}\chi\left(\frac{f_1}{f_2}(x+yp^{\delta-1})\right)\\&=p^{\alpha-\delta}\sum_{x\in A_3(\delta-1,(\alpha-\delta,d_1))}\chi\left(\frac{f_1}{f_2}(x)\right)\sum_{0\leq y<p}\mathrm{e}(c\lambda(x)(yp^{\delta-1}F(x)\\&\quad+\tfrac{1}{2}y^2p^{2\delta-2}F'(x))/p^{\alpha}).\end{aligned}$$

The inner sum is $\ll\sqrt{p}$ by Weil, so

$$\begin{aligned}S_3(\alpha,\mathbf{d})&\ll p^{\alpha-\delta+1/2}\#A_3(\delta-1,(\alpha-\delta,d_1))\\&\ll p^{(\alpha+d_1)/2}\end{aligned}$$

by Lemma 1. On the other hand if $\alpha-d_1$ is odd, $d_1>0$ and $d_1=\delta-1$ then $d_1=(\alpha-1)/3$ and $\delta=(\alpha+2)/3$ so

$$\begin{aligned}S_3(\alpha,\mathbf{d})&=p^{\alpha-\delta}\sum_{x\in A_3(\delta,(\alpha-\delta,d_1))}\chi\left(\frac{f_1}{f_2}(x)\right)\\&=p^{\alpha-\delta}\sum_{\substack{x\in A_3(d_1,(\alpha-\delta))\\ p^{d_1}|F'(x)}}\ \sum_{\substack{0\leq y<p\\ p^{d_1}\,\|\,F'(x+yp^{d_1})}}\chi\left(\frac{f_1}{f_2}(x+yp^{d_1})\right).\end{aligned}$$

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The inner sum

$$=\chi\left(\frac{f_1}{f_2}(x)\right)\sum_{\substack{0\leq y<p\\ p^{d_1}\,\|\,F'(x)+yp^{d_1}F''(x)}} e(c\lambda(x)(yp^{d_1}F(x)+\tfrac{1}{2}y^2p^{2d_1}F'(x)+\tfrac{1}{6}y^3p^{3d_1}F''(x))/p^{\alpha}).$$

This sum is non-empty only if $p^{d_1}\,\|\,F'(x)$ or $p\nmid F''(x)$ and in either case the sum is $\ll\sqrt{p}$ by Weil, so that

$$\begin{aligned}S_3(\alpha,\mathbf{d})&\ll p^{\alpha-\delta+1/2}\{\#A_3(d_1,(\alpha-\delta,d_1))+\#A_3(d_1,(\alpha-\delta,d_1,0)\}\\ &\ll p^{(\alpha+d_1)/2}\end{aligned}$$

by Lemma 1.

Finally if $p\leq 3$ we may assume $\alpha\geq 4$. Put

$$\delta=\lceil(\alpha-d_1+1)/2\rceil$$

so

$$\delta>d_1,\qquad 2\delta+d_1-1\geq\alpha.$$

Then

$$\begin{aligned}S_3(\alpha,\mathbf{d})&=\sum_{x\in A_3(\delta,\mathbf{d})}\sum_{0\leq y<p^{\alpha-\delta}}\chi\left(\frac{f_1}{f_2}(x+yp^{\delta})\right)\\ &=\sum_{x\in A_3(\delta,\mathbf{d})}\chi\left(\frac{f_1}{f_2}(x)\right)\sum_{0\leq y<p^{\alpha-\delta}}e(c\lambda(x)yp^{\delta}F(x)/p^{\alpha})\\ &\ll p^{\alpha-\delta}\#A_3(\delta,(\max(d_0,\alpha-\delta),d_1)\\ &\ll p^{(\alpha+d_1)/2}\end{aligned}$$

by Lemma 1.

*Proof of Proposition* 3.

$$S_5(\alpha,\mathbf{d})=\sum_{\mathbf{e}}S_5(\alpha,\mathbf{e}),$$

where $\mathbf{e}\in\mathbb{Z}^{2r-3}$ and $0\leq e_{2r-3}\leq e_{2r-4}\leq\ldots\leq e_1=d_1\leq e_0=d_0$. Denote by $\kappa$ the $p$-adic order of $(2r)!$ If there exists a $t$ such that

$$e_{t-1}\geq e_t+\gamma-\kappa$$

then

$$\begin{aligned}S_5(\alpha,\mathbf{e})&\ll\#A_5(\alpha,\mathbf{e})\\ &\ll p^{\alpha-e_{t-1}+e_t}\\ &\ll p^{\alpha-\gamma}\end{aligned}$$

by Lemma 1. If for all $t$

$$e_{t-1}<e_t+\gamma-\kappa \tag{5}$$

then

$$
\begin{aligned}
S_5(\alpha, \mathbf{e}) &= \sum_{x \in A_5(\gamma, \mathbf{e})} \sum_{0 \leq y < p^{\alpha-\gamma}} \chi\left(\frac{f_1}{f_2}(x + yp^{\gamma})\right) \\
&= \sum_{x \in A_5(\gamma, \mathbf{e})} \chi\left(\frac{f_1}{f_2}(x)\right) \sum_{0 \leq y < p^{\alpha-\gamma}} \mathrm{e}(c\lambda(x)(yp^{\gamma}F(x) + \tfrac{1}{2}y^2p^{2\gamma}F'(x))/p^{\alpha}) \\
&= 0
\end{aligned}
$$

if $p > 2$ or $d_1 \neq \alpha/3$. But if $p = 2$ and $d_1 = \alpha/3$ then by (5)

$$
e_0 < 2\alpha/3 - \kappa.
$$

Thus

$$
\begin{aligned}
S_5(\alpha, \mathbf{e}) &= \sum_{x \in A_5(\gamma+\kappa, \mathbf{e})} \sum_{0 \leq y < p^{2\gamma-\kappa}} \chi\left(\frac{f_1}{f_2}(x + yp^{\gamma+\kappa})\right) \\
&= \sum_{x \in A_5(\gamma+\kappa, \mathbf{e})} \chi\left(\frac{f_1}{f_2}(x)\right) \sum_{0 \leq y < p^{2\gamma-\kappa}} \mathrm{e}(c\lambda(x)yp^{\gamma+\kappa}F(x)/p^{\alpha}) \\
&= 0.
\end{aligned}
$$

*Proof of Proposition* 4.

$$
\begin{aligned}
S_4(\alpha, \mathbf{d}) &\ll \# A_4(\alpha, \mathbf{d}) \\
&\ll p^{\alpha - d_{i-1} + d_i}
\end{aligned}
$$

for any $i \in (0, s]$ by Lemma 1.

**4.** To deduce Theorem 3 from Propositions 1 to 4 we need the following Lemma which is the principal theorem of [**6**].

LEMMA 2. *Let $r = 4$. Let $p > 3$ be a prime and let $h$ be a positive integer. Then for $\mu, \nu, \xi$ satisfying $\mu \geq \nu \geq \xi \geq 0$ we have*

$$
\begin{aligned}
&\#\{\mathbf{m} \in A_1 : \exists x \text{ such that } p \nmid f_1(x)f_2(x), p^{\mu} \mid F(x), p^{\mu} \mid F'(x), \\
&p^{\nu} = (F''(x), p^{\mu}), p^{\xi} = (F'''(x), p^{\nu})\} \\
&\qquad \ll (\mu + 1)^4(h^4 + h^5p^{-\xi} + h^6p^{-\xi-\omega} + h^7p^{-\xi-\omega-\omega_0}),
\end{aligned}
$$

where

$$
\omega = \min(\lceil \mu/3 \rceil, \lceil (2\nu - \xi)/3 \rceil), \qquad \omega_0 = \lceil \mu/2 \rceil.
$$

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*Proof of Theorem* 3. From (3)

$$
\begin{aligned}
k^{-1}\sum_{a=0}^{k-1}\sum_{n=0}^{k-1}|T_a(n,h)|^{2r} &\leqslant \sum_{\mathbf{m}\in A_1}\left|\sum_{x\in A_2(\alpha)}\chi\left(\frac{f_1}{f_2}(x)\right)\right| \\
&\leqslant \sum_{0\leqslant d_0<\beta}\sum_{\mathbf{m}}|S_5(\alpha,(d_0))| + \sum_{0\leqslant d_1<\gamma}\sum_{\mathbf{m}}|S_3(\alpha,(\beta,d_1))| \\
&+ \sum_{\beta\leqslant d_0<\alpha-\gamma}\ \sum_{\gamma\leqslant d_1\leqslant d_0}\sum_{\mathbf{m}}|S_5(\alpha,(d_0,d_1))| \\
&+ \sum_{\gamma\leqslant d_1\leqslant\alpha-\gamma}\ \sum_{0\leqslant d_3\leqslant d_2\leqslant d_1}\sum_{\mathbf{m}} S_4(\alpha,(\alpha-\gamma,d_1,d_2,d_3)| \\
&= \sum_1+\sum_2+\sum_3+\sum_4
\end{aligned}
$$

say.

Trivially by Proposition 1

$$\sum_1 \ll \sum_{0\leqslant d_0\leqslant\beta} h^7 p^{\alpha/2} \ll \alpha h^4 k.$$

By Lemma 2 and by Proposition 2

$$
\begin{aligned}
\sum_2 &\ll \sum_{0\leqslant d_1<\gamma}\alpha^6(h^7/p^{\lceil d_1/2\rceil}+h^6)p^{(\alpha+d_1)/2} \\
&\ll \alpha^7 h^4 k.
\end{aligned}
$$

By Lemma 2 and Proposition 3

$$
\begin{aligned}
\sum_3 &\ll \sum_{\beta\leqslant d_0<\alpha-\gamma}\ \sum_{\gamma\leqslant\delta_1\leqslant d_0}\alpha^6(h^7/p^{\lceil d_1/2\rceil}+h^6)\alpha^4 p^{\alpha-\gamma} \\
&\ll \alpha^{12}h^4 k.
\end{aligned}
$$

Finally also by Lemma 2, writing

$$\omega=\min(\lceil d_1/3\rceil,\lceil(2d_2-d_3/3\rceil),\qquad \omega_0=\lceil d_1/2\rceil,$$

we have by Proposition 4

$$
\begin{aligned}
\sum_4 \ll \sum_{\gamma\leqslant d_1\leqslant\alpha-\gamma}\ \sum_{0\leqslant d_3\leqslant d_2\leqslant d_1}&\alpha^4(h^4+h^5p^{-d_3}+h^6p^{-d_3-\omega} \\
&+h^7p^{-d_3-\omega-\omega_0})p^{\min(d_1+\gamma,\alpha+d_2-d_1,\alpha+d_3-d_2)}.
\end{aligned}
$$

Thus to prove Theorem 3 it suffices to show that if

$$\gamma\leqslant d_1\leqslant\alpha-\gamma,\qquad 0\leqslant d_3\leqslant d_2\leqslant d_1$$

then

$$\alpha/6-d_3+\min(d_1+\gamma-\alpha,d_2-d_1,d_3-d_2)\leqslant 0, \tag{6}$$

$$\alpha/3-d_3-\omega+\min(d_1+\gamma-\alpha,d_2-d_1,d_3-d_2)\leqslant 0. \tag{7}$$

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The left-hand side of (6) is

$$\begin{aligned}&\leq \alpha/6 - d_3 + \tfrac{1}{3}(d_1 + \gamma - \alpha + d_2 - d_1 - d_3 - d_2)\\ &\leq (2\gamma - \alpha)/6.\end{aligned}$$

If $\alpha \geq 6$ then $\gamma \leq \alpha/3 + 1 \leq \alpha/2$. For $2 \leq \alpha \leq 5$ we still have $\gamma \leq \alpha/2$. Thus (6) is true.

Next if $\lceil d_1/3 \rceil < \lceil (2d_2 - d_3)/3 \rceil$ then the left-hand side of (7) is

$$\begin{aligned}&\leq \alpha/3 - d_3 - \lceil d_1/3 \rceil + \tfrac{1}{3}(d_1 + \gamma - \alpha + d_2 - d_1 + d_3 - d_2)\\ &\leq (\gamma - d_1)/3 \leq 0.\end{aligned}$$

Finally if $\lceil d_1/3 \rceil \geq \lceil (2d_2 - d_3)/3 \rceil$ then the left-hand side of (7) is

$$\begin{aligned}&\leq \alpha/3 - d_3 - \lceil (2d_2 - d_3)/3 \rceil + \tfrac{1}{2}(d_1 + \gamma - \alpha + d_2 - d_1)\\ &\leq \tfrac{1}{2} - \lceil 2d_2/3 \rceil + d_2/2.\end{aligned}$$

If $d_2 > 0$ by writing $d_2 = 3u + 1, 2$ or $3$ we see the last expression is $\leq 0$. If $d_2 = 0$ the left-hand side of (7) is

$$\leq \alpha/3 - d_1 \leq 0.$$

This completes the proof of Theorem 3.

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