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ON THE FECUNDITY OF MATHEMATICS FROM OMAR KHAYYAM TO G. SACCHERI

As an introduction to the study which is the subject of this article, we would like to provide some definitions and make a few elementary observations, in order to make it possible for the less informed reader to understand better the origins and the *raison d'être* of the propositions that are under examination.

By principles of mathematics we mean a body of propositions, which arise in mathematical demonstrations and which serve as premises for deduction. These principles are not proven.

In so-called classical mathematics, or more precisely, the classical method of explicating mathematics, which was in current use up until the end of the nineteenth century, three types of mathematical principles were distinguished: the definitions, the axioms and the postulates.

A definition was understood to be a proposition indicating *what* a given thing is. Thus, for instance, Euclid defined parallels as two straight lines which are located on the same plane, and

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which when extended to infinity in both directions do not meet in either direction. We will simply point out here that a definition in no way proves the existence of the defined object. Its existence is either assumed, as in the case of the point and the line, or demonstrated through the geometric construction of the defined object. Thus, in proposition XXI of Book I of the *Elements,* Euclid proves that parallels exist by demonstrating that one may draw through a point situated outside a straight line another straight line that does not cut the first.

An axiom was understood to be a proposition "evident in itself." For example: the whole is greater than any of its parts. An axiom, understood in this way, is not provable. The unverifiability of the axiom derives from its very definition. One cannot prove what is evident in itself and constitutes in a way a truth of reason. But it must be pointed out in this regard that the concept of "evidence" is extremely relative. Thus the proposition, which we have just made, was considered as being evident, up until the time when Galileo demonstrated that there were certain objects—later called infinite sets—of which a part was as great as the whole.

A postulate was understood to be a proposition, the truth of which was not evident and which the reader was required to accept. For instance, Euclid included among the postulates of his geometry the proposition affirming that a straight line could be drawn between one point and another.

One may see from this example that, contrary to the axiom, the object of the postulate is not a truth evident in itself. It may be challenged without touching upon the basic principles of reason. Its apparent evidence—it may, in fact, appear "evident" to certain people that a straight line may always be drawn between two points—arises from the fact that it generally refers to some operation which *experience* could confirm. But this type of empirical evidence is not a criterion of truth for a mathematician. Only a demonstration that transforms this empirical experience into rational evidence permits us to assert the truth of a proposition.

Why then were postulates not proven? The classical answer to this question was that postulates were not provable, and many philosophical theories have been put forward in order to explain this characteristic. In fact, postulates are simply autonomous. This meams that, given a particular body of postulates, accepted as forming the base of a given geometry, these postulates cannot be deduced one from the other. They are undemonstrable in this sense, that they cannot be proved with the aid of the only other accepted postulates and theorems which are derived from them.

But a doubt may always persist as to the knowledge of whether one of the postulates belonging to the body of given postulates may not rightly be deduced from the others. This doubt may lead the geometrician to attempt to prove this postulate—that is, to deduce it from the other postulates and theorems which depend on them—in order to assure himself of its autonomy.

But there is another reason that may lead the mathematician to attempt to prove a postulate. We have said that the predication of a postulate was generally related to certain operations, or more precisely to geometrical constructions, which could be confirmed by experience. When these operations are considered "simple" within common experience, this simplicity can endow them with an apparent evidence, and the possibility of these operations could be reasonably assumed without proof. This is the case, for example, for the postulate related to the possibility of connecting two points by a straight line. But when these operations are more complex, the mathematician then may be tempted legitimately to prove their possibility. This is the case of the postulate which serves as the point of departure for this study.

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It is known that non-Euclidian geometries were the result, in the nineteenth century, of the various and vain attempts made since the first century before our age to prove Euclid's fifth postulate. One of the last of these, in fact the most important, was the one made by G. Saccheri in 1733. This attempt, which introduced a number of non-Euclidian theorems, marks the end of what we call the pre-history of non-Euclidian geometries. We also know that Beltrami proved, in 1877, that the postulate was not provable, thus rendering final homage to Euclid's brilliant insight. This postulate, known to all secondary school students, is generally formulated in the following way: through a point located outside a straight line one may draw, within the plane formed by this straight line and this point, one and only one parallel line to the straight line. But this formulation of the postulate is not the one given by Euclid, and we cannot understand the pre-history of non-Euclidian geometries unless Euclid's own formulation is adopted: "... if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than the two right angles." [1]

It is precisely this formulation that geometricians have made futile efforts to prove throughout the pre-history of non-Euclidian geometries. But during the most important phase of this period, which began in the tenth century and ended with Saccheri, the leading geometricians who endeavored to prove the fifth postulate, in the form we have just quoted, used for this purpose a quadrilateral known as "Saccheri's quadrilateral." Wrongly attributed to the latter, it owes its origin, in fact, to the Egyptian mathematician ibn El-Haîtham (965-1039), who was the first to make use of it in his proof of the fifth postulate.[2] Omar Khayyam inherited it and used it in four theorems, which we find again in almost identical form with Saccheri.

The object of this discussion is to show that the first three propositions of Saccheri are similar and, in some points, identical with the four theorems proved seven centuries earlier by Omar Khayyam. It is enough to keep in mind that Saccheri was the great forerunner of non-Euclidian geometries, and that the theorems to which we refer constituted the point of departure for his geometry, in order to indicate the interest this study has for the history of mathematics.

We should point out here that, to our knowledge, the only comparative study which has been made on this subject is an article by D. E. Smith.[3] But the latter did not make use of Khayyam's manuscript for the purpose of his article, nor, it seems,

[1] Heath, *The Thirteen Books of Euclid's Elements*, 2nd. edition, Dover Publications Inc., New York, Vol. I, p. 155.

[2] Juschkewitsch, *Geschichte der Mathematik im Mittelalter*, Pfalz-Verlag, Basel, 1964, pp. 280-283.

[3] "Euclid, Khayyam and Saccheri," in *Scripta Mathematica*, Vol. II, No. 1, January 1935, pp. 5-10.

of Saccheri's work. His study, which, except for one variation, faithfully reproduces the two first propositions of our two authors, gives only some elements of Khayyam's third proposition, and does not refer to the fourth at all. However, the most important propositions for this comparison are precisely the third and fourth propositions of Khayyam, as well as Saccheri's third.

We have divided our study into three parts. In the first, we give the reader a brief introduction to Omar Khayyam and Saccheri. In the second, the most important part, we give a comparative description of their respective demonstrations. Finally, in the third, we pose the problem which such a strange coincidence raises for the historian of science.

Omar Khayyam, or, more precisely, Ali Al-Fath Omar ibn Ibrahim Al-Khayyami is unquestionably the most unrecognized mathematician in history. The famous author of the *Rubaiyat* has left to the public, even the cultivated public, only the recollection of a disillusioned poet, who drowned his bitterness with drink and women. But Khayyam wrote verses only for diversion; primarily an astronomer, a geometrician and algebraist, he was, in fact, the greatest mathematician of the Middle Ages.

He was born in about 1045 in Naishapour, in the Khorassan, in Persia, and he died there in about 1125. Thanks to his fellow-student and friend, Nizam Al-Moulk, he became vizir of the kings Seldjoukid Alp the Lion and of Malik Chah, and received a pension which made it possible for him to devote himself to mathematical and astronomical research. He revised the Persian calendar on the order of Malik Chah, making it more precise than the Julian calendar, but less precise than the Gregorian. As an algebraist he established a systematic classification of equations from the first to the third degree; he provided the geometrical solutions of third-degree equations and pointed out the difficulty, later solved by Cardan, of the algebraic solutions of these same equations.

But it is his work as a geometrician that interests us here. In his small treatise[4] entitled *Theses on the information of the prob-*

[4] Bibliothèque Nationale, Fonds Arabe, Ms. No. 4946, folio 38 sqq.; Library of the University of Leyden, Ms. No. 967. This manuscript has been published by Dr. Erani, Teheran, 1936.

lematical postulates of Euclid's book, in three parts, he subjects some of the basic concepts of Euclidian geometry to philosophical and mathematical criticism. In the first part he attempts, by making use of the quadrilateral of Ibn El-Haîtham, to establish a theory of parallels which would be independent of the fifth postulate. He then gives what he considers to be a proof of the latter. It is the first, second, third and fourth propositions of this part, as well as the ideas contained in the third proposition, which we shall meet in Saccheri; and which we will discuss further on. The work was completed toward the end of the first half of December, 1077.

Saccheri, a Jesuit, was an Italian mathematician. He was born in 1677 and died in August, 1733, one month after the publication, in Milan, of his work entitled *Euclides ab omni naevo vindicatus* (Euclid cleaned of any stain).[5] In the first part of this book, Saccheri endeavors to prove, through ingenious reasoning, Euclid's fifth postulate. In doing so, he establishes for the first time a sequence of theorems which in principle should lead to a proof of the postulate, but which constitute, in reality, a genuine non-Euclidian geometry. (He proves, for instance, that, by employing certain hypotheses, the sum of the angles of a triangle may be greater than two right angles). It is the first three propositions of this part that reiterate Khayyam's four first proposition.

Before giving a comparative illustration of Khayyam's and Saccheri's propositions, we would like to point out that the composition of the two works mentioned above is identical. Both are divided into three parts, dealing respectively with the same subjects. The first is devoted to the theory of parallels and to the proof of the fifth postulate; the second to the theory of proportional magnitudes; and the third to that of compounded ratios.

In view of situating Khayyam and Saccheri better within the prehistory of non-Euclidian geometries, the most important period of which starts with the use of Ibn El-Haîtham's quadrilateral, we will give a schematic table of the leading geometricians of this period. The reader will thus be able to see at a glance the "*élan vital*" of non-Euclidian geometries.

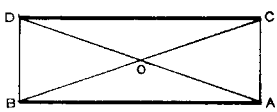
[5] For the exposition of Saccheri's propositions we have used the English translation by Halsted in the *American Mathematical Monthly*, 1894, Vol. I-V.

Ibn El-Haîtham (965-1039)
|
Omar Al-Khayyam (1045-1125) —— Vitale (1633-1711) —— Saccheri (1677-1733)
|
N. Al-Dine Al-Toussi (1201-1274) — Gauss, Bolyai, Lobatschewsky
|
J. Wallis (1616-1703) — Riemann

We will now pass on to a comparative study of Khayyam's and Saccheri's propositions. To make this study more clear, so that the similarity of these propositions will immediately emerge, we have set them forth in two columns under the headings of the respective two authors.

KHAYYAM

PROPOSITION I



Hypotheses:

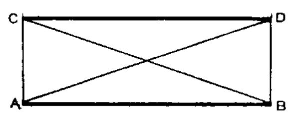
Quadrilateral $ABCD$
AC perpendicular to AB
BD perpendicular to AB
$AC = BD$

Conclusion:

$\hat{C} = \hat{D}$

SACCHERI

PROPOSITION I



Hypotheses:

Quadrilateral $ABCD$
$AC = BD$
$\hat{A} = \hat{B}$

Conclusion:

$\hat{C} = \hat{D}$

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Proof:

AC is parallel to *BD* (*Eucl.* I, 28).[6]
We join *CB* and *AD*.
We consider the triangles *ACB* and *ABD*:

$AC = BD$,

AB is common to both

$\hat{A} = \hat{B} = 1$ right angle.

Hence:

$AD = CB$,

and the homologous angles are equal.

Thus:

$O\hat{A}B = O\hat{B}A$.

In the triangle *OAB*:

$OB = OA$.

Therefore:

$OC = OD$

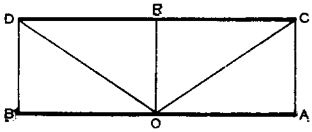
But in the triangle *DOC*:

$O\hat{C}D = O\hat{D}C$.

Therefore:

$A\hat{C}D = B\hat{D}C$.

PROPOSITION II



Proof:

We join *CB* and *AD*.
We consider the triangles: *ACB* and *ABD*

Then (*Eucl.* I, 4):

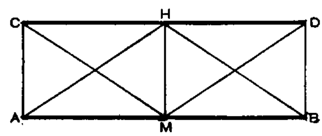
$AD = CB$

We take the triangles *ABC* and *BDC*;

Then (*Eucl.* I, 8):

$A\hat{C}D = B\hat{D}C$.

PROPOSITION II



[6] In the demonstrations that follow, we refer to the propositions of Euclid's *Elements*, placing in parentheses the abbreviation *Eucl.* followed by the number of the book in Roman numerals, and the proposition number in Arabic numerals.
For the sake of clarity, we have presented Khayyam's and Saccheri's demonstrations in modern form, whereas in their own works they are given in a literal form.

Hypotheses:

Same hypotheses as in Proposition I, to which the following are added:

$BO = OA$
$OR \perp AB$

Conclusions:

$CR = RD$
$OR \perp CD$

Proof:

We join DO and OC. We have:

$AC = BD$
$AO = OB$
$\hat{A} = \hat{B} = 1$ right angle.

Hence:

$DO = OC$,

and

$\widehat{AOC} = \widehat{BOD}$

Then $\widehat{DOR} = \widehat{ROC}$.

We take the triangles DOR and ROC:

$DO = OC$,
OR is common to both,
$\widehat{DOR} = \widehat{ROC}$.

These triangles are then equal, and it follows that:

$DR = RC$;
$\widehat{DRO} = \widehat{CRO} = 1$ right angle.

Hypotheses:

Same hypotheses as in Proposition I, to which the following are added:

$AM = MB$,
$CH = HD$.

Conclusions:

$CHM = DHM = 1$ right angle.
$AMH = BMH = 1$ right angle.

Proof:

We join AH, BH, CM, DM.
We have:

$\hat{A} = \hat{B}$
$\hat{C} = \hat{D}$ (Proposition I)

We take the triangles CAM and DBM

Then by (*Eucl.* I, 4) and taking into account the equality of the sides:

$CM = DM$.

We take the triangles CHM and DHM as well as the triangles AMH and BMH.
Then (*Eucl.* I, 8):

$\widehat{CHM} = \widehat{DHM} = 1$ right angle;
$\widehat{AMH} = \widehat{BMH} = 1$ right angle.

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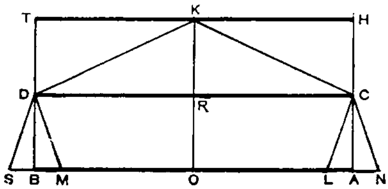
PROPOSITION III



Hypotheses:

Same hypotheses as in Proposition I, to which the following are added:

$\hat{C} = \hat{D}$

$OR \perp$ to AB (therefore also to CD by virtue of Proposition II)
We produce OR so that:

$OR = RK$

$HKT \perp OK$

Conclusion:

$\hat{C} = \hat{D} = 1$ right angle.

Proof:

(Inasmuch as we find in Saccheri, in his Proposition III, only the main idea, which regulates the demonstration of Proposition III of Khayyam, and considering its lenght, we vill only summarize it.)

First Khayyam proves that AC and BC produced HKT in H and T. He then shows, on the basis of the properties of triangles, that the angles H and T are equal,

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that $CH = DT$ and that $HK = KT$.

He then begins the proof of Proposition III, that is:
$\hat{C} = \hat{D} =$ 1 right angle. For that purpose he divides his proof into three parts:

a) if $\hat{C} = \hat{D} <$ 1 right angle, then

$\widehat{HCR} > \widehat{ACR}$

Folding the half-plane DH on the half-plane CB, he shows that:

$HT = NS > AB$

and concludes that AC and BD depart from the upper level of the half-plane determined by AB.

If the entire figure is turned over around AB, then AC and BD depart from the other side of the half-plane.

"There would then be two straight lines cutting a straight line at right angles, and whose distance would then increase on both sides of this straight line, which is impossible..."

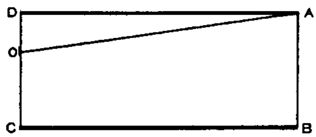
b) if $\hat{C} = \hat{D} >$ 1 straight line.

An identical but inverted demonstration: The straight lines will approach each other from both sides of AB, which is impossible.

c) Therefore:

$\hat{C} = \hat{D} =$ 1 right angle.

PROPOSITION IV



Hypotheses:

Quadrilateral $ABCD$,
$A = B = C = D = 1$ right angle.

Conclusions:

$CD = AB$

Proof:

If $AB \neq CD$
We assume $DC > AB$
Take $OC = AB$. Join AO.
Then:

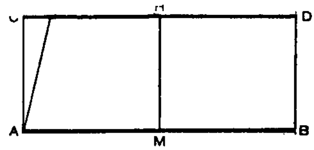
$B\hat{A}O = C\hat{O}A$ (Proposition III)
But:

$B\hat{A}O < 1$ right angle,

and $C\hat{O}A > 1$ right angle (angle exterior to the triangle OAD which already has a right angle).

Therefore $C\hat{O}A$, which is greater than a right angle, would be equal to $B\hat{A}O$, which is smaller than a right angle: which is impossible.

PROPOSITION III



Hypotheses:

Quadrilateral $ABCD$,
$AC = BD$,
AC and $BD \perp$ to AB

Conclusions:

a) If $\hat{C} = \hat{D} = 1$ right angle, $CD = AB$

b) If $\hat{C} = \hat{D} > 1$ right angle, $CD < AB$

c) If $\hat{C} = \hat{D} < 1$ right angle, $CD > AB$.

Proof:

First part:

$\hat{C} = \hat{D} = 1$ right angle.
If $AB \neq CD$
We assume $DC > AB$
Take $DK = AB$. Join AK
Then:

$B\hat{A}K = D\hat{K}A$ (Proposition I).
But:

$B\hat{A}K < 1$ right angle.

and $D\hat{A}K > 1$ right angle (angle exterior to the triangle ACK, whose angle $D\hat{C}A$ is a right angle).

It is also demonstrated that:

$AB > DC$,

which is impossible.

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Therefore:

$DC = AB.$

Therefore:

$DC = AB.$

Second part.

$\hat{C} = \hat{D} > 1$ right angle.

If we join the centers M and H of AB and CD,

we have:

$AM \perp MH$

$CH \perp MH$

$C \neq 1$ right angle.

Therefore (Proposition 1):

$CH \neq AM$

But the segment CH is not greater than the segment AM. In fact, let us assume that this is so and take:

$HK = AM$

Then (Proposition 1):

$M\hat{A}K = A\hat{K}H.$

But this is impossible, because:

$M\hat{A}K < 1$ right angle,

and (*Eucl.* I, 16):

$H\hat{K}A > A\hat{C}D > 1$ right angle.

Therefore:

$CH < AM$

and

$CD < AB$

Third part.

$\hat{C} = \hat{D} < 1$ right angle.

Saccheri proves, through reasoning analogous to that of the Second part, that:

$CD > AB.$

We would like to point out, with regard to the first proposition, the evident equivalence of the hypotheses and the similarity of the proof. The only difference which exists between the two propositions is the qualification of the angles at the base *A* and *B*. They are right angles for Khayyam and equal for Saccheri. In this sense it may be said that the latter's proposition is more general than Khayyam's. But, just as the right-angled base is an unnecessary restriction in Khayyam's first proposition, Saccheri's generalization does not add anything new. Moreover, we will see that in the proposition in which it is essential that the angles at the base be right angles, Saccheri introduces right angles. The first propositions of the two authors are therefore identical.

The same is the case for both second propositions.

So far as the third and fourth propositions of Khayyam on the one hand, and the third proposition of Saccheri on the other, are concerned, the similarity between the two authors is less apparent, but it is no less profound. If Khayyam's third proposition does not correspond exactly to one of Saccheri's propositions, still the central idea upon which it is based is an integral part of Saccheri's proposition III. Let us point out first of all that the three cases distinguished in this proposition—which were to become famous under the name of "Saccheri's three hypotheses," and which were to correspond to three different geometries[7]—are those that Khayyam distinguished in the three parts of his proof. Moreover, the essential relations, established by Saccheri in his third proposition between the sides of the quadrilateral and the obtuse and acute angles, are the same as those which Khayyam established in his proof of his third proposition. There is a difference, it is true. It is not the opposite side *CD* which in Khayyam is greater or smaller than the base, but the straight line *TH*. But the essential relationship between the sides and the angles is identical with both authors.

The brilliant idea of Saccheri, which opened the way to non-Euclidian geometries, consisted precisely here, in having made use

[7] The hypothesis of the right angle corresponds to Euclidian geometry, that of the acute angle to the geometry of Lobatshewski and Bolyai, and of the obtuse angle to the geometry of Riemann.

of these relationships in order to establish his proposition III, which is nothing else than a generalization of Khayyam's proposition IV. In fact, in the latter the Persian geometrician proves that if the quadrilateral is a rectangle, the opposite sides are equal. The Italian geometrician also proves this property in the first part of his proposition III and, keeping the right angles adjacent at the base,—which Khayyam had done in his three first propositions, as we have indicated above—he considers the cases in which the two angles adjacent to the opposite side are obtuse or acute. He then concludes that in the first case this side is smaller than the base and, in the second case, greater. The first part of Saccheri's proposition III and his proof are identical with Khayyam's proposition IV and its demonstration. But the two other parts of Saccheri's proposition III, which generalize the property established in the first part, do not seem to have homologies with Khayyam. But, in fact, they take ug again the relationships established by the latter in his proposition III and the process of the proof which is used is basically the same—with one variation—as that for the first part, that is, the same used by Khayyam in the proof of his proposition IV. In fact, so far as the relationships between the sides and the angles are concerned, we have already pointed out above that they had been established by Khayyam in his proposition III. With regard to the demonstrations of the second and third parts, Saccheri introduces a variant, which changes nothing in the process of proving the first part. The construction of the bisector *MH* has the effect only of limiting the proof to the left half of the diagram, thus permitting the use in the proof of but one angle at the vertex, the angle *C*. Except for this difference, it is the same proof as that of Khayyam.

Nonetheless, despite the similarities, the two authors diverged fundamentally in the goals which they proposed and, consequently, in the means they used to attain them. Khayyam hoped to substitute for proposition 29 of Book I of Euclid's *Elements* a body of theorems which would make it possible for him to establish a theory of parallels in which the fifth postulate would not intervene. In the eighth and last proposition of the first part of his work, he accomplishes the traditional task of "proving" his theory, but by making use implicitly of another postulate which is equivalent to the fifth postulate. Saccheri's goal was to prove the postulate. And curiously enough, the one whose aim was to do without it, remained dependent upon it—since he substituted another for it which is its equivalent,—and the one who intended to prove it went beyond it and opened the way to a new geometry.

This may be explained by the fact that the difference in the aims pursued involved a difference in method, and it was the method which gave Saccheri's geometry a fecundity which Khayyam's geometry did not have. In aiming to establish a theory of parallels that would be compatible with Euclidian geometry but independent of the postulate, Khayyam immediately tried to prove the impossibility of the obtuse and acute angles. For this purpose he used a construction which has no equivalent in Saccheri, and which permitted him to deduce from the hypothesis of the obtuse and acute angles the impossibility of the relationships between the sides. Saccheri, intending to prove the validity of the fifth postulate, took recourse to a method which is a "subtle elaboration of the *reductio ad absurdum.*"[8] Il Euclid's postulate is true, the angles at the vertices of the quadrilateral can be neither obtuse nor acute; they can only be right angles. Saccheri assumed that they are either obtuse or acute, and deduced from these hypothesis a sequence of propositions which had to lead him to the establishment of properties that are in contradiction with other properties of Euclidian space. In fact, it was through logical and philosophical, rather than geometrical, arguments that he concluded, in his proposition XIV, the impossibility of the obtuse angle, and, in proposition XXXIII, the impossibility of the acute angle. But, meanwhile, the propositions that he deduced from the hypothesis of the obtuse angle and from the hypothesis of the acute angle, and which had to lead to a contradiction with other geometrical properties, are in fact perfectly compatible with each other and constitute genuine non-Euclidian propositions. Saccheri's attempt to prove Euclid's postulate by a *reductio ad absurdum* failed, but "the method was to overturn the aim of its promoter: the hypothesis, which cannot be proved to

[8] Brunschvicg, *Les Etapes de la philosophie mathématique*, P.U.F., 1947, p. 315.

be contradictory, is worthy of being retained, as much as Euclid's thesis."[9]

Mention should be made of the problem posed by these similarities between Khayyam's four first propositions and Saccheri's three first propositions. In the article by Smith, quoted earlier, the author affirms without hesitation: "That this is the case is not very important since Saccheri was familiar with the work of John Wallis, and the latter mentions Nassir ad Dine at Toussi in his work. But Nassir ad Dine distinctly states that (this lemma) is due to Omar Khayyam, and from the text it seems clear that the latter was his inspirer." Smith then believes he is entitled to speak of "the influence of Omar Khayyam on the work of Saccheri." Both the explanation, advanced by Smith, as well as his affirmation of the influence exerted by Omar Khayyam do not seem to us to be sufficiently well founded. Saccheri, who, we stress, never referred to Khayyam anywhere in his work, certainly was acquainted with Wallis' *De postulato quinto*, since he quotes the English author and criticizes his principles of demonstration.[10] In the same passage Saccheri also criticizes the principles of demonstration of Nassir ad Dine at Toussi, reported by Wallis in his work. But the passages on Toussi's proof, given by Wallis,[11] contain absolutely no mention of Omar Khayyam.

In his small treatise devoted to the discussion of the fifth postulate[12] at Toussi quotes Khayyam's discussion almost in its entirety. But there is no mention of Khayyam in the passages quoted by Wallis. It is therefore not from Wallis' work that Saccheri could have known Khayyam.

We must, however, specify here that, among similar propositions established by Khayyam and Saccheri, an important relationship does exist, which Saccheri could have discovered through Toussi's text, quoted by Wallis: this is the relationship between the sides and the angles. The latter was established by Toussi, who was evidently inspired by Khayyam's proposition III and by a lemma which Toussi gives further on in his treatise. But

[9] *Ibid*, p. 318.

[10] Lib. I, prop. XXI, schol. III.

[11] Wallis, *Opera Math.*, Vol. II, pp. 669-673, Oxford, 1963.

[12] Bibliothèque Nationale, Fonds Arabe, Ms. No. 2467, fol. 73-89.

Saccheri's demonstration and his diagram are certainly those of Khayyam's proposition IV.

Toussi's text, as quoted by Wallis, is hence incapable of explaining the similarities that exist between Khayyam and Saccheri. These similarities are much to numerous and relate to too precise points to be attributed merely to coincidence.

Did Saccheri inherit—through Clavius, whom he quotes in scholium II of proposition XXI—a tradition which goes back to Khayyam, passing through Gerson's commentaries?[13] The affiliation seems to us to be too indirect and would require, in order to be established, a detailed study of texts of both authors.

Another hypothesis, a more pertinent one, is suggested by certain indications, which Wallis himself gives on the subject of the sources from which he drew his information about Arab mathematicians. Before presenting Toussi's proof, Wallis informs us that he had come to know it thanks to "the assistance of the illustrious Edward Pocock... a professor highly skilled in oriental languages and particularly in Arabic."[14] In the paragraph that follows the presentation of Toussi's proof, Wallis tells us that, again thanks to Pocock, he had become acquainted with two other Arab manuscripts, whose authors he does not mention. He then makes a brief criticism of these two manuscripts, quoting a postulate which is the one Khayyam used in his demonstrations. Would one of these manuscripts then have been Khayyam's, and could it be through a translation by Pocock, if the latter made one, that Saccherl could have become acquainted with the Persian mathematician? This is only an hypothesis, but it seems to us that the solution to this problem lies in this direction.

13 Juschkewitsch, *op. cit.*, p. 393.

14 Wallis, *op. cit.*, p. 669.