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POINTWISE CONVERGENCE OF ALTERNATING SEQUENCES

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1. Introduction. Let $1 < p < \infty$ and let L_p be the usual Banach Space of complex valued functions on a σ-finite measure space. Let (T_n), $n \geqq 1$, be a sequence of positive linear contractions on L_p. Hence $T_n L_p^+ \subset L_p^+$ and $\|T_n\| \leqq 1$, where L_p^+ is the part of L_p that consists of non-negative L_p functions. The adjoint of T_n is denoted by T_n^*, which is a positive linear contraction of L_q with $q = p/(p-1)$.

Our purpose in this paper is to show that the alternating sequences associated with (T_n), as introduced in **[2]**, converge almost everywhere. Complete definitions will be given later. When applied to a non negative function, however, this result is reduced to the following theorem.

(1.1) THEOREM. *If* (T_n) *is a sequence of positive contractions of* L_p *then*

(1.2) $\quad \lim_n T_1^* \dots T_n^*(T_n \dots T_1 f)^{p-1}$

exists a.e. *for all* $f \in L_p^+$.

This is related to the following theorems due to Stein **[9]** and Rota **[7]**; see also **[3]**, **[4]**, **[8]**, and **[5]** and **[10]**.

(1.3) THEOREM (Stein). *If T is a self adjoint positive contraction on* L_2 *then*

$$\lim_n T^{2n} f$$

exists a.e. *for each* $f \in L_2$.

(1.4) THEOREM (Rota). *If each* T_n *is a positive contraction of all* L_p*-Spaces,* $1 \leqq p \leqq \infty$, *simultaneously, and if* $T_n 1 = T_n^* 1 = 1$, *then*

(1.5) $\quad \lim_n T_1^* \dots T_n^* T_n \dots T_1 f$

exists a.e. *for each* $f \in L_p$, $\quad 1 < p$.

Theorem (1.1) generalizes Stein's Theorem. It also implies a part of Rota's Theorem, namely the existence of (1.5) first for $f \in L_2$, then, by an easy argument, for $f \in L_p$, $2 \leqq p$ (even if the measure is not finite). If $f \in L_p$, $1 < p < 2$, then the existence of (1.5) does not seem to follow

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from the existence of (1.2) in a direct way. It may be noticed, however, that, under the additional hypotheses of Rota's theorem, our proof can be modified in a trivial way to obtain (1.4). We omit this as the proof so obtained will not be very different from the existing proofs of (1.4).

The arguments of the present paper depend, beyond the basic Measure Theory, only on Doob's maximal theorems for martingales (as given in, e.g., **[10]**, pp. 89-94) and on a result in **[1]**. Our main result, in fact, will be obtained by adapting the arguments of **[1]** to the present case. Also, in Lemma (2.3) we seem to need the uniform convexity of L_p (see **[2]**). If one is interested only in $p = 2$ then this becomes, however, very simple.

In Section 2 we list a number of elementary results and observations that we will need later. In this section some of the simple proofs are omitted. In Section 3 we formulate our main result, Theorem (3.2), and reduce its proof to the proof of two maximal inequalities. We also notice that it will be enough to prove these inequalities only in the finite dimensional case. Finally in Section 4, which contains our main arguments, we give the proofs in the finite dimensional case.

2. Preliminaries. In this section we are dealing with a fixed (complex) L_p Space, $1 < p < \infty$, over a σ-finite measure space $(X, \mathscr{F}, \mu)$. The adjoint of L_p is identified with L_q, $q = p/(p - 1)$. The application of the functional $g \in L_q$ on $f \in L_p$ is given by

$$(f, g) = \int f\bar{g}d\mu.$$

Any statement made about L_p and/or L_q is also valid and will be used when the indices p and q are switched. The norm in L_p is denoted by $\|\cdot\|_p$ or simply by $\|\cdot\|$. Finally, all our statements are correct only up to sets of measure zero, where applicable.

The duality mapping $\psi_p:L_p \to L_q$ is defined by

$$(\psi_p f)(x) = \begin{cases} |f(x)|^p(\overline{f(x)})^{-1} & \text{if } f(x) \neq 0 \\ 0 \text{ otherwise.} \end{cases}$$

We also write f^* for $\psi_p f$. Note that $f^* \in L_q$ and

$$(f, f^*) = \|f\|_p^p = \|f\|_p\|f^*\|_q = \|f^*\|_q^q.$$

In fact, these equations define f^* uniquely. Also note that

$$\psi_q:L_q \to L_p$$

is the inverse of ψ_p. We also write $\psi_q g = g^*$, $g \in L_q$, when the distinction between ψ_p and ψ_q is clear from the context.

By a linear operator we always mean a linear bounded operator. A linear operator $V:L_p \to L_p$ is called a contraction if $\|Vf\| \leqq \|f\|$ for all $f \in L_p$ and is called positive if $VL_p^+ \subset L_p^+$, i.e., if $Vf \geqq 0$ whenever

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$f \geqq 0$ and $f \in L_p$. If $V:L_p \to L_p$ is a linear operator then the adjoint operator $V^*:L_q \to L_q$ is defined by the requirement that

$$(Vf, g) = (f, V^*g) \quad \text{for all } f \in L_p,\ g \in L_q.$$

If V is a positive linear contraction on L_p then V^* is a positive linear contraction on L_q.

(2.1) LEMMA. *If $V:L_p \to L_p$ is a positive linear operator then*

$$|Vf| \leqq V|f|, \quad f \in L_p.$$

Proof. If $\phi \in L_p$ then let $P(\phi)$ be the class of functions $h \in L_p^+$ such that

$$\left|\int \phi g d\mu\right| \leqq \int hgd\mu \quad \text{for all } g \in L_q^+.$$

Then $|\phi| \leqq h$ whenever $h \in P(\phi)$. But if $g \in L_q^+$ then

$$\left|\int Vf \cdot gd\mu\right| = \left|\int fV^*gd\mu\right|$$
$$\leqq \int |f|V^*gd\mu = \int V|f| \cdot gd\mu,$$

which shows that $V|f| \in P(Vf)$. Hence $|Vf| \leqq V|f|$.

(2.2) LEMMA. *Let $f_k \in L_p$, $1 \leqq k \leqq n$, and let $V:L_p \to L_p$ be a positive linear operator. Then*

$$\max|Vf_k| \leqq V(\max|f_k|),$$

and, consequently,

$$\|\max|Vf_k|\| \leqq \|V\| \cdot \|\max|f_k|\|.$$

(2.3) LEMMA. *For each $\epsilon > 0$ there is a $\delta > 0$ such that if $E:L_p \to L_p$ is a conditional expectation operator, $f \in L_p$, and if*

$$\|f\| - \|Ef\| < \delta\|f\|$$

then

$$\|f - Ef\| < \epsilon\|f\|.$$

Proof. We will use the uniform convexity of L_p, as stated in Lemma (2.2) of [**2**]: For each $\epsilon > 0$ there is an $\eta > 0$ such that $\|g - h\| < \epsilon$ whenever $g, h \in L_p$, $\|g\| \leqq 1$, $\|h\| \leqq 1$ and $\|g + h\| > 2 - \eta$. We choose a $\delta > 0$ such that

$$2(1 - \delta)^p > 2 - \eta.$$

Let $f \in L_p$, $\|f\| = 1$ and let

$$\|f\| - \|Ef\| < \delta.$$

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Then $1 - \delta < \|Ef\| \leqq 1$. Also,

$$\begin{aligned}(f + Ef, (Ef)^*) &= (2Ef, (Ef)^*) \\ &= 2\|Ef\|^p > 2(1 - \delta)^p,\end{aligned}$$

where the first equality follows from a basic property of E. Since $\|(Ef)^*\|_q \leqq 1$, we see that

$$\|f + Ef\| > 2(1 - \delta)^p > 2 - \eta.$$

Hence $\|f - Ef\| < \epsilon$. If $f \in L_p$ and $0 < \|f\| \neq 1$ then we apply the above argument to $f' = f/\|f\|$.

(2.4) LEMMA. *Let f_k, $k \geqq 0$, be a sequence in L_p such that*

$$F = (\sup|f_k|) \in L_p.$$

If λ_k is a sequence of complex numbers such that $|1 - \lambda_k| < \epsilon$ for each $k \geqq 0$ then

$$\|(\sup|f_k - f_0|) - (\sup|\lambda_k f_k - \lambda_0 f_0|)\| \leqq 4\epsilon\|F\|.$$

(2.5) LEMMA. *Let (f_{km}), $1 \leqq k \leqq n$, $1 \leqq m$, be n sequences in L_p such that $\lim_m\|f_{km} - f_k\| = 0$ for each k. Then*

$$\lim_m\| \max_{1 \leqq k \leqq n} |f_{km}| - \max_{1 \leqq k \leqq n} |f_k| \| = 0.$$

Proof. See Lemma (3.2) in [**1**].

(2.6) LEMMA (Uniform continuity of ψ_p). *Given $\epsilon > 0$ and $M > 0$ there is a $\delta > 0$ such that $\|f^* - g^*\|_q < \epsilon$ whenever $\|f - g\|_p < \delta$, $f \in L_p$, $g \in L_p$, and $\|f\|_p \leqq M$, $\|g\|_p \leqq M$.*

This lemma has been stated in [**2**]. As mentioned there, its proof follows directly from a result of Mazur [**6**]. It should be noted, however, it is also quite routine to give a direct elementary proof.

(2.7) LEMMA. *Given an $\epsilon > 0$ and an $M > 0$ there is a $\delta > 0$ with the following property. If f_k is a sequence in L_p such that*

$$\|\sup|f_k|\|_p \leqq M \quad \textit{and} \quad \|\sup|f_k - f_0|\|_p < \delta$$

then

$$\|\sup|f_n^* - f_0^*|\|_q < \epsilon.$$

Proof. Choose δ from the previous lemma corresponding to $(1/2)\epsilon$ and M. Given any $n \geqq 1$ there is a partition of X into n sets A_l, $1 \leqq l \leqq n$, such that

$$\max_{1 \leqq k \leqq n} |f_k^* - f_0^*| = \sum_l |f_l^* - f_0^*|\chi_{A_l},$$

where χ denotes the characteristic function of a set. Hence

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$$\max_{1 \leqq k \leqq n} |f_k^* - f_0^*| = |f^* - f_0^*|$$

with

$$f = \sum_l f_l \chi_{A_l}.$$

But $\|f\|_p \leqq M$ and $\|f_0\|_p \leqq M$ and

$$\|f - f_0\|_p \leqq \|\sup|f_k - f_0|\,\|.$$

Hence if this last norm is less than δ then

$$\|f^* - f_0^*\|_q < \frac{1}{2}\epsilon,$$

which completes the proof.

(2.8) LEMMA. *Let f_n be a sequence of functions in L_p such that*

$$(\sup|f_n|) \in L_p.$$

Then f_n converges a.e. *if and only if*

$$\lim_n \|\sup_{k \geqq n}|f_k - f_n|\,\| = 0.$$

Proof. Let

$$\theta_n = \sup_{k \geqq n}|f_k - f_n|.$$

First assume that f_n's are real valued. Let

$$\phi_n = \inf_{k \geqq n} f_k \quad \text{and} \quad \psi_n = \sup_{k \geqq n} f_k$$

$$f_n - \theta_n \leqq \phi_n \leqq f_k \leqq \psi_n \leqq f_n + \theta_n, \quad k \geqq n,$$

shows that

$$0 \leqq \theta_n \leqq \psi_n - \phi_n \leqq 2\theta_n.$$

Hence $\|\theta_n\| \to 0$ if and only if $\|\psi_n - \phi_n\| \to 0$. But f_n converges a.e. if and only if $\psi_n - \phi_n$ converges a.e. to zero, which happens if and only if $\|\psi_n - \phi_n\|$ converges to zero.

For the complex valued case, let

$$f_n = f_n' + if_n'',$$

where f_n' and f_n'' are real valued. Let θ_n be as before and let θ_n', θ_n'' be the similar functions associated with the sequences f_n' and f_n'', respectively. Then we see that $\theta_n' \leqq \theta_n$, $\theta_n'' \leqq \theta_n$, and $\theta_n \leqq \theta_n' + \theta_n''$. Hence $\|\theta_n\| \to 0$ if and only if $\|\theta_n'\| + \|\theta_n''\| \to 0$, which happens if and only if both f_n' and f_n'' converge a.e.

(2.9) COROLLARY. *Let f_n be a sequence in L_p such that* $(\sup|f_n|) \in L_p$. *Let V be a positive linear operator. If f_n converges* a.e. *then Vf_n also converges* a.e.

(2.10) *Definition*. A point transformation $\tau : X \to X$ will be called an *automorphism* if it is invertible and if both τ and τ^{-1} are measurable and non singular. An automorphism induces two measures $\mu\tau^{-1}$ and $\mu\tau$, both absolutely continuous with respect to μ. If ρ is the Radon-Nikodym derivative of $\mu\tau^{-1}$ with respect to μ then the operator $Q:L_p \to L_p$ defined by

$$Qf = \rho^{1/p} f\tau^{-1}, \quad f \in L_p$$

(or, more explicitly, by

$$(Qf)(x) = \rho(x)^{1/p} f(\tau^{-1}x), \quad f \in L_p)$$

is a positive invertible isometry of L_p. We will call Q the L_p-isometry induced by τ.

(2.11) LEMMA. *If Q is the L_p-isometry induced by an automorphism τ then Q^{-1} is the L_p-isometry induced by τ^{-1} and Q^* is the L_q-isometry induced by τ^{-1}.*

Proof. This follows directly from the definitions, if it is observed that the Radon-Nikodym derivative ρ' of $\mu\tau$ with respect to μ is given by

$$\rho'(x) = 1/\rho(\tau x).$$

(2.12) LEMMA. *If Q is the L_p-isometry induced by an automorphism τ and if $f \in L_p$ then*

$$Q^* f^* = (Q^{-1}f)^*.$$

Proof. This, again, follows directly from the definitions and from Lemma (2.11).

3. The main result. If T is a linear operator in L_p then we let, as in [**2**],

$$(3.1) \qquad M(T)f = [T^*(Tf)^*]^*,$$

where $f \in L_p$. Note that $M(T)$ maps L_p into L_p, but in general it is a non linear operator. The main result of this paper is the following theorem.

(3.2) THEOREM. *If T_n is a sequence of positive linear contractions on L_p, $1 < p$, then*

$$\lim M(T_n \ldots T_1)f$$

exists a.e. *for each $f \in L_p$.*

(3.3) *Remarks*. The pointwise convergence of

$$M(T_n \ldots T_1)f = [T_1^* \ldots T_n^*(T_n \ldots T_1 f)^*]^*$$

is equivalent to the pointwise convergence of

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$$T_1^* \ldots T_n^*(T_n \ldots T_1 f)^*.$$

If $f \in L_p^+$ then this last sequence simplifies to

$$T_1^* \ldots T_n^*(T_n \ldots T_1 f)^{p-1}.$$

Hence Theorem (3.2) contains Theorem (1.1) stated in the introduction. Also, if $p = 2$ then $M(T_n \ldots T_1)f$ becomes $T_1^* \ldots T_n^* T_n \ldots T_1 f$. Hence Theorem (3.2) implies that if T_n is a sequence of positive linear contractions in L_2 then

$$\lim T_1^* \ldots T_n^* T_n \ldots T_1 f$$

exists a.e. for each $f \in L_2$.

(3.4) *Notation*. If T_k, $k \geqq 1$, is a sequence of linear operators on L_p and if $f \in L_p$ then we let $V_0 = 1$ be the identity on L_p,

$$V_n = T_n \ldots T_1, \quad n \geqq 1,$$

and

$$f_n = M(V_n)f = [V_n^*(V_n f)^*]^*,$$
$$g_n = f_n^* = V_n^*(V_n f)^*, \quad n \geqq 0.$$

Note that f_n is a sequence in L_p and g_n is a sequence in L_q and that f_n converges a.e. if and only if g_n converges a.e. Also, $f_0 = f = g_0^*$.

(3.5) *Definition*. Let T_n be a sequence of linear operators on L_p. We say that Estimate A holds for T_n if

$$||\sup|f_n|\,|| \leqq q||f||$$

for all $f \in L_p$, with the notations of (3.4).

(3.6) *Definition*. Let T_n be a sequence of linear contractions on L_p. We say that Estimate B holds for T_n if for each $\epsilon > 0$ there is a $\delta > 0$ depending only upon ϵ and p, such that

$$||\sup|f_n - f_0|\,|| < \epsilon$$

whenever

$$||f|| - \lim_n ||V_n f|| < \delta ||f||$$

and

$$||f|| \leqq 1,$$

with the notations of (3.4).

These two estimates can also be given in terms of the sequence g_n. We state this in the following two lemmas.

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(3.7) LEMMA. *Estimate* A *holds for* T_n *if and only if*

$$\|\sup|g_n|\,\|_q \leqq (q\|f\|_p)^{p-1}, \quad \textit{for all } f \in L_p.$$

Proof. We see that

$$\sup|g_n| = \sup|f_n^*| = (\sup|f_n|\,)^*.$$

Hence

$$\|\sup|g_n|\,\|_q^q = \|\sup|f_n|\,\|_p^p.$$

Then the proof follows.

(3.8) LEMMA. *Estimate* B *holds for* T_n *if and only if for each* $\xi > 0$ *there exists an* $\eta > 0$ *such that*

$$\|\sup|g_n - g_0|\,\|_q < \xi$$

whenever

$$\|f\|_p - \lim\|V_n f\|_p < \eta\|f\|_p$$

and

$$\|f\|_p \leqq 1.$$

Proof. Assume that Estimate B holds. First find $\delta_1 > 0$ such that

$$\|\sup|f_n - f_0|\,\| < 1$$

whenever

(3.9) $\quad \|f\| - \lim\|V_n f\| < \delta_1\|f\|,$

and

(3.10) $\quad \|f\| \leqq 1.$

Then

(3.11) $$\|\sup|f_n|\,\| \leqq \|\sup|f_n - f_0|\,\| + \|f_0\| < 2,$$

whenever (3.9) and (3.10) hold, since $\|f_0\| = \|f\|$. Given $\xi > 0$ we use Lemma (2.7) with $\epsilon = \xi$ and $M = 2$ to find a $\delta_0 > 0$ such that

(3.12) $\quad \|\sup|g_n - g_0|\,\|_q < \xi$

whenever

(3.13) $\quad \|\sup|f_n - f_0|\,\|_p < \delta_2$

and

(3.14) $\quad \|\sup|f_n|\,\|_p \leqq 2.$

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Then find η, $0 < \eta < \delta_1$, using the Estimate B with $\epsilon = \delta_2$. Then we see that (3.12) holds whenever

$$\|f\| - \lim\|V_n f\| < \eta\|f\| \leqq \eta.$$

The proof in the other direction is similar.

(3.15) LEMMA. *Given a sequence of linear operators T_k, $k \geqq 1$, a function $f \in L_p$ and an integer $n \geqq 1$, define a new sequence $\tilde{T}_k = T_{n+k}$, a new function*

$$\tilde{f} = V_n f = T_n \dots T_1 f$$

and then define $\tilde{V}_k$, $\tilde{f}_k$, $\tilde{g}_k$ as in (3.4). *Then*

$$g_{n+k} = V_n^* \tilde{g}_k, \quad k \geqq 0.$$

Proof.

$$\begin{aligned} g_{n+k} &= V_{n+k}^*(V_{n+k}f)^* \\ &= V_n^*\tilde{V}_k^*(\tilde{V}_k V_n f)^* \\ &= V_n^*\tilde{V}_k^*(\tilde{V}_k\tilde{f}) = V_n^*\tilde{g}_k. \end{aligned}$$

The following lemma reduces the proof of the main result to the proof of Estimates A and B for a sequence of positive linear contractions.

(3.16) LEMMA. *Let T_n be a sequence of positive linear contractions on L_p. If Estimates* A *and* B *hold for T_n then f_n converges* a.e. *for each $f \in L_p$, with the notations of* (3.4).

Proof. It is enough to consider $f \in L_p$ with $\|f\| \leqq 1$. We will prove the convergence of g_n. First, since Estimate A holds,

$$\sup|g_n| \in L_q.$$

Hence we will show

$$\lim_n\|\sup_{k\geqq 0}|g_{n+k} - g_n|\,\| = 0$$

to complete the proof.

Let $\beta = \lim\|V_n f\|$. Distinguish two cases.

Case 1. $\beta = 0$. Given $\epsilon > 0$ find $n_0 \geqq 1$ such that

$$\|V_{n_0}f\| < \epsilon.$$

Choose and fix an $n \geqq n_0$ and define $\tilde{g}_k$ and $\tilde{f}$ as in Lemma (3.15). Then

$$\begin{aligned} \|\sup_{k\geqq 0}|g_{n+k} - g_n|\,\| &= \|\sup_{k\geqq 0}|V_n^*\tilde{g}_k - V_n^*\tilde{g}_0|\,\| \\ &\leqq \|\sup_{k\geqq 0}|\tilde{g}_k - \tilde{g}_0|\,\| \end{aligned}$$

$$\leqq 2\|\sup_{k\geqq 0}|\tilde{g}_k|\,\|$$
$$\leqq 2(q\|\tilde{f}\|)^{p-1} < 2(q\epsilon)^{p-1},$$

where the first inequality follows from Lemma (2.2) and the second inequality from Estimate A. Note that Estimate B is not needed in this first case.

Case 2. $\beta > 0$. Given $\xi > 0$ choose $\eta > 0$ from Lemma (3.8). Choose $n_0 \geqq 1$ such that

$$\|V_{n_0}f\| < (1 + \eta)\beta.$$

Then choose and fix an $n \geqq n_0$ and define $\tilde{g}_k$ and $\tilde{f}$ as before. Then we see that, as in the first case,

$$\|\sup_{k\geqq 0}|g_{n+k} - g_n|\,\| \leqq \|\sup_{k\geqq 0}|\tilde{g}_k - \tilde{g}_0|\,\| < \xi,$$

since

$$\begin{aligned}\|\tilde{f}\| - \lim_k\|\tilde{V}_k\tilde{f}\| &= \|V_nf\| - \beta \\ &< (1 + \eta)\beta - \beta = \eta\beta \leqq \eta\|\tilde{f}\| \leqq \eta.\end{aligned}$$

(3.17) Lemma. *If Estimate* A *holds for any sequence of positive linear contractions on a finite dimensional L_p Space then it also holds for a sequence of positive linear contractions on a general L_p Space.*

Proof. Assume the hypothesis of the theorem. Let T_n be a sequence of positive linear contractions on a general L_p Space. If Estimate A does not hold for T_n then there is an $f \in L_p$ such that

$$\|\sup|f_k|\,\| > q\|f\|.$$

Then there is an $n \geqq 1$ such that

$$\|\max_{0\leqq k\leqq n}|f_k|\,\| > q\|f\|.$$

Then we see that there is a finite partition $\mathscr{P}$ of the measure space X such that if E is the conditional expectation with respect to $\mathscr{P}$ and if $T'_k = ET_kE$, $f' = Ef$ and if $f'_k = M(V'_k)f'$ with $V'_k = T'_k \dots T'_1$ then

$$\|\max_{0\leqq k\leqq n}|f'_k|\,\| > q\|f'\|. \tag{3.18}$$

This essentially follows from Lemma (2.5). We recall that the conditional expectation is defined as zero on the atoms of $\mathscr{P}$ with infinite measures.

Now $T'_k = ET_kE$ can be considered as a positive linear contraction on a finite dimensional L_p Space. Hence (3.18) contradicts our hypothesis. Therefore Estimate A must be satisfied for T_k.

(3.19) LEMMA. *Assume that for each* $\xi > 0$ *there is an* $\eta > 0$, *depending only on* ξ *and* p, *such that*

(3.20) $\| \max_{0 \leqq k \leqq n} |g'_k - g'_0| \|_q < \xi$

whenever $n \geqq 1$ *is an integer,* $T'_1, \ldots, T'_n$ *are* n *positive linear contractions of a finite dimensional* L_p *Space* l_p *and* $f' \in l_p$ *such that*

(3.21) $\|f'\| - \|V'_n f'\| < \eta \|f'\| \leqq \eta$,

where, as before,

$$g'_k = (V'_k)^*(V'_k f')^*$$

$$V'_k = T'_k \ldots T'_1, \; V'_0 = 1.$$

Then Estimate B *holds for any sequence of positive linear contractions on a general* L_p *Space.*

Proof. Let $\xi > 0$ be given. Find $\eta > 0$ from the hypothesis. Let T_k be any sequence of positive linear contractions on an L_p Space. We would like to show that

$$\|\sup_k |g_k - g_0| \| < 2\xi$$

whenever

(3.22) $\|f\| - \lim \|V_k f\| < \eta \|f\| \leqq \eta$.

In fact, otherwise there is an $f \in L_p$ satisfying (3.22) for which

$$\|\sup_k |g_k - g_0| \| > \xi.$$

Then there is an $n \geqq 1$ such that

$$\|f\| - \|V_n f\| < \eta \|f\| \leqq \eta$$

and

$$\| \max_{0 \leqq k \leqq n} |g_k - g_0| \| > \xi.$$

Then, as before, we find a conditional expectation E with respect to a finite partition such that if $T'_k = ET_kE$, $1 \leqq k \leqq n$ and $f' = Ef$ then (3.21) is satisfied but (3.20) is violated.

Finally we will show that if the hypotheses of the Lemmas (3.17) and (3.18) are satisfied for a special type of positive linear contractions on a finite dimensional L_p-Space then they are also satisfied for all positive linear contractions on finite dimensional L_p-Spaces. For convenience, we will call these special contractions the admissible contractions.

(3.23) *Definition.* Let T be a positive linear contraction on a finite dimensional L_p-Space l_p. Then T is called an *admissible contraction* if $\|T\| = 1$ and if $Tf > 0$ at each point whenever $f \in l_p^+$ and $f \neq 0$.

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(3.24) LEMMA. *Let T be any positive linear contraction of l_p. Let $1/\lambda = \|T\| > 0$. Then for each $\epsilon > 0$ there is an admissible contraction T' of l_p such that*

$$\|\lambda T - T'\| < \epsilon.$$

Proof. Let T_0 be an arbitrary admissible contraction. Find an admissible T' of the form

$$T' = r\lambda T + sT_0$$

where $r < 1$, $0 < s$, and s and $1 - r$ are sufficiently small.

(3.24) LEMMA. *Assume that Estimate* A *is satisfied for any sequence of admissible contractions. Then it is also satisfied for any sequence of positive linear contractions on a finite dimensional L_p-Space.*

Proof. If the conclusion is not correct then there are finitely many positive linear contractions $T_1, \ldots, T_n$ on l_p and $f \in l_p$ such that

$$\frac{1}{\lambda_k} = \|T_k\| > 0 \quad \text{and}$$

$$\| \max_{0 \leqq k \leqq n} |f_n| \| > q\|f\|.$$

Let $T_k' = \lambda_k T_k$ and $f' = f$ and

$$f_k' = M(V_k')f'.$$

Then we see that

$$f_k' = (\lambda_k \ldots \lambda_1)^q f_k.$$

Hence

$$\| \max_{0 \leqq k \leqq n} |f_k'| \| > q\|f'\|.$$

Then one can replace each T_k' by an admissible T_k'' such that one would still have

$$\| \max_{0 \leqq k \leqq n} |f_k''| \| > q\|f''\|$$

with $f'' = f$, $f_k'' = M(V_k'')f''$.

(3.25) LEMMA. *Suppose that for each $\xi' > 0$ there exists an $\eta' > 0$, depending only on ξ' and p such that*

$$\| \max_{0 \leqq k \leqq n} |g_k - g_0| \| < \xi'$$

whenever $n \geqq 1$ is an integer, $T_1, \ldots, T_n$ are n admissible contractions of l_p and $f \in l_p$ such that

$$\|f\| - \|V_n f\| < \eta'\|f\| \leqq \eta'.$$

Also assume that Estimate A *is satisfied for any sequence of admissible contractions on l_p. Then the hypothesis of Lemma* (3.19) *is satisifed.*

Proof. Let $\xi > 0$. Using the hypothesis choose an $\eta' > 0$ corresponding to $\xi' = (1/10)\xi$. Then choose $\eta'' > 0$ such that

$$(1 + 2x)^q < 1 + \frac{1}{10}\xi q^{1-p}$$

whenever $0 \leqq x < \eta''$. Let

$$\eta = \min\left(\eta', \eta'', \frac{1}{4}\right).$$

Then let $T'_1, \ldots, T'_n$ be linear positive l_p contractions. Let $f' \in l_p$ be such that

$$\|f'\| - \|V'_n f'\| < \eta\|f'\| \leqq \eta.$$

Then, since Estimate A holds for T'_k,

$$(3.26) \quad \|\max|g'_k|\,\| \leqq (q\|f'\|\,)^{p-1} \leqq q^{p-1},$$

by Lemma (3.7).

We would like to show that

$$(3.27) \quad \|\max|g'_k - g'_0|\,\| > \frac{1}{2}\xi$$

leads to a contradiction.

If $T'_k = 0$ for some k, $1 \leqq k \leqq n$, then $V'_n f' = 0$ and we have

$$\|f'\| < \eta\|f'\| \leqq \frac{1}{4}\|f'\|$$

which is impossible. Hence

$$0 < \|T'_k\| = \frac{1}{\lambda_k} \leqq 1 \quad \text{for all } k.$$

Also, since $\eta \leqq 1/4$, and since

$$(1 - \eta)\,\|f'\| \leqq \|V'_n f'\| = \|T'_n \ldots T'_1 f'\|$$
$$\leqq \frac{1}{\lambda_n \ldots \lambda_1}\|f\|,$$

we see that

$$1 \leqq \lambda_1 \ldots \lambda_n \leqq \frac{1}{1 - \eta} \leqq 1 + 2\eta.$$

Hence,if $\mu_k = \lambda_1 \dots \lambda_k$ then

$$1 \leqq \mu_k \leqq 1 + 2\eta \leqq 1 + 2\eta'' \quad \text{for all } k, \quad 1 \leqq k \leqq n.$$

Now let $T_k'' = \lambda_k T_k'$ and $f'' = f'$ and define g_k'' in terms of f'' and T_k'' as before. Then

$$g_k'' = \mu_k^q g_k'$$

and

$$1 \leqq \mu_k^q \leqq 1 + \frac{1}{10}\xi q^{1-p}$$

by the choice of η''.

Then, by Lemma (2.4) and by (3.26)

$$\begin{aligned} \|\max|g_k'' - g_0''|\,\| &\geqq \|\max|g_k' - g_0'|\,\| - \frac{1}{10}\xi \\ &> \frac{4}{10}\xi. \end{aligned}$$

Also

$$\begin{aligned} \|f''\| - \|V_n''f''\| &\leqq \|f'\| - \|V_n'f'\| \\ &< \eta\|f'\| = \eta\|f''\| \\ &\leqq \eta'\|f''\|. \end{aligned}$$

Now we can find admissible T_k such that if $f = f' = f''$ then we still have

$$\|\max|g_k - g_0|\,\| > \frac{4}{10}\xi > \xi'$$

and

$$\|f\| - \|V_nf\| < \eta'\|f\| \leqq \|f\|.$$

This contradicts the choice of η'. Hence we see that

$$\|\max|g_k' - g_0'|\,\| < \xi$$

whenever

$$\|f'\| - \|V_nf'\| < \eta\|f'\| \leqq \|f'\|.$$

This completes the proof.

Hence to prove the main theorem, Theorem (3.2), it will be enough to prove the hypotheses of Lemmas (3.24) and (3.25). This will be done in the next section.

4. Admissible contractions of l_p. Let l_p be a d-dimensional L_p-Space. Hence the associated measure space consists of d points with non zero measures $m_i > 0$, $1 \leqq i \leqq d$. Let T be an admissible contraction of l_p. A result in [**1**] shows that there is a finite measure space $(Z, \mathcal{F}, \mu)$ of a very special type, a partition $\mathcal{P}$ of Z into d atoms of measure m_i and an automorphism τ of Z, again of a very special type, such that if Q is the L_p-isometry induced by τ and if E is the conditional expectation with respect to the partition $\mathcal{P}$ then T can be represented by (or isomorphic to)

$$EQE$$

in the obvious sense. We will use the properties of Z, τ, Q and E to obtain the estimates on admissible contraction from the corresponding estimates on martingales. We will first describe Z and τ. Again, for convenience, we will call the particular type of automorphisms of Z we are going to define the admissible automorphisms of Z and the induced isometries the admissible isometries of L_p. In this section L_p will always denote $L_p(Z)$ and l_p the finite dimensional L_p-Space associated with the finite partition $\mathcal{P}$. (Hence $l_p = EL_p$.)

(4.1) *Definition of* Z. Let d be an arbitrary integer, $d \geqq 1$. The indices i and j will range through the integers $\{1, \dots, d\}$. Let m_i be also fixed numbers, $m_i > 0$. Let I_i be disjoint intervals on the x-axis such that the length of I_i is m_i. Let J_i be disjoint intervals on the y-axis, each of unit length. Let $P_i = I_i \times J_i$ and let $Z = UP_i$. Hence Z be a subset of the xy plane. The measure μ will be the two dimensional Lebesgue measure on Z. The partition of Z into P_i will be denoted by $\mathcal{P}$. The points of Z are denoted by (x, y), as usual.

(4.2) *Admissible automorphisms of* Z. An automorphism τ of Z will be called an *admissible automorphism* if it is of the following type. Each I_j is partitioned into d intervals $(I_{ij})_i$ of non zero length and each J_i is partitioned into d intervals $(J_{ij})_j$ of non zero length. If $R_{ij} = I_i \times J_{ij}$ and $S_{ij} = I_{ij} \times J_j$ then τ maps each R_{ij} onto S_{ij} and the restriction of τ to each R_{ij} is of the form

$$\tau(x, y) = (ax + b, cy + d),$$

where a, b, c, d are four constants depending on R_{ij}. The L_p-isometry Q induced by an admissible automorphism will be called an *admissible isometry* of L_p.

(4.3) *Admissible contractions of* L_p. A result in [**1**] shows that if T is an admissible contraction of l_p then there is an admissible isometry Q of L_p such that $T = EQE$, where E is the conditional expectation with respect to $\mathcal{P}$. Here we identified, in an obvious way, the finite dimensional space l_p by EL_p. The converse is not true. If Q is an admissible isometry of L_p then

EQE may not be an admissible contraction of l_p, in the sense of the previous section (the reason is that $\|EQE\| < 1$ is possible). This is not important, however, as we are going to prove our estimates for any sequence of operators of the type EQE, with Q being an admissible isometry of L_p.

(4.4) *Further notations and definitions*. We have already denoted the conditional expectation operator with respect to $\mathcal{P}$ by E. In general, if $\mathcal{G}$ is a finite partition then $E(\cdot\,|\mathcal{G})$ will denote the conditional expectation with respect to $\mathcal{G}$. Hence $E = E(\cdot\,|\mathcal{P})$.

Let G be a set in the xy-plane. A subset F of G will be called a *vertical subset* of G if

$$F = (F' \times \mathbf{R}) \cap G$$

for some subset F' of the x-axis. Similarly, if

$$H = (\mathbf{R} \times H') \cap G$$

then H will be called a *horizontal subset* of G. Let ϕ be a function defined on a subset of the xy-plane. We will say that ϕ is *constant on vertical lines* if $\phi(x, y)$ depends only on the x-coordinate. Similarly, if $\phi(x, y)$ depends only on the y-coordinate then we will say that ϕ is constant on horizontal lines.

(4.5) LEMMA. *Let $\mathcal{G}$ be a finite partition of Z such that each atom of $\mathcal{G}$ is a vertical subset of one of P_i. Let f be an L_p function which is constant on vertical lines. Let τ be an admissible automorphism and let Q be the induced L_p isometry. Then*

$$QE(f|\mathcal{G}) = E(Qf|\mathcal{P} \vee \tau\mathcal{G}). \tag{4.6}$$

Proof. If G is a subset of Z then let $A(f, G)$ be the average value of f on G; i.e., let

$$A(f, G) = \begin{cases} (\mu(G))^{-1} \int_G f d\mu & \text{if } \mu(G) > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

If G is an atom of $\mathcal{G}$ then it is of the form $G = G' \times J_i$, for some i, where G' is a subset of I_i. Let $G_j = G \cap R_{ij}$ with this particular i, where R_{ij}'s are associated with τ as in (4.2). Then $G_j = G' \times J_{ij}$. Since f is constant on vertical lines,

$$A(f, G_j) = A(f, G)$$

for each j. Hence,

$$A(f, G)\chi_G = \sum A(f, G_j)\chi_{G_j}. \tag{4.8}$$

We now notice that τ maps R_{ij} onto S_{ij} and that τ transports the measure μ on R_{ij} to a constant multiple of μ on S_{ij} (i.e.,

$$\rho = \frac{d\mu\tau^{-1}}{d\mu}$$

is a constant ρ_{ij} on S_{ij}). Hence, if F is a subset of R_{ij}, then

$$A(f, F) = A(f\tau^{-1}, \tau F).$$

We then have that, since each G_j is contained in a single R_{ij},

$$\begin{aligned} (4.9) \qquad QA(f, G_j)\chi_{G_j} &= A(f, G_j)Q\chi_{G_j} \\ &= A(f\tau^{-1}, \tau G_j)\rho_{ij}^{1/p}\chi_{\tau G_j} \\ &= A(\rho_{ij}^{1/p}f\tau^{-1}, \tau G_j)\chi_{\tau G_j} \\ &= A(Qf, \tau G_j)\chi_{\tau G_j}. \end{aligned}$$

We now observe that

$$\tau G_j = (\tau G) \cap S_{ij} = (\tau G) \cap P_i,$$

which means that τG_j's are exactly the atoms of $\mathscr{P} \vee \tau G$ that are contained in τG. Hence we conclude the proof first by applying Q to the both sides of (4.8), then by transforming the second side by means of (4.9) and finally by summing the resulting equations over the atoms G of the partition $\mathscr{G}$.

(4.10) LEMMA. *Let f, f' be two functions in L_p that are constant on vertical lines. If $Ef = Ef'$ then also $EQf = EQf'$, where Q is any admissible L_p isometry.*

Proof. This was already proved in [**1**]. But it also follows directly from the previous lemma, which shows that

$$E(Qf|\mathscr{P} \vee \tau\mathscr{P}) = E(Qf'|\mathscr{P} \vee \tau\mathscr{P}),$$

as these two sides are equal to QEf and to QEf' respectively.

These results have obvious analogues for the inverse of an admissible isometry. We state only the following result and omit the proof.

(4.11) LEMMA. *Let f and f' be two functions in L_p that are constant on horizontal lines. If $Ef = Ef'$ then also*

$$EQ^{-1}f = EQ^{-1}f',$$

where Q is any admissible isometry.

(4.12) LEMMA. *If $f \in L_p$ is constant on vertical lines and if Q is an admissible isometry then Qf is also constant on vertical lines. Similarly, if $g \in L_p$ is constant on horizontal lines then $Q^{-1}g$ is also constant on horizontal lines.*

Proof. This follows directly from the definitions, if one observes that the image of a vertical line in Z under an admissible τ consists of d vertical lines and that

$$\rho = \frac{d\mu\tau^{-1}}{d\tau}$$

is constant on vertical lines.

(4.13) *Notation.* Let n be a fixed integer, $n \geqq 1$. The index k will range through the integers $\{0, 1, \dots, n\}$. If $1 \leqq k \leqq n$ then let Q_k be an admissible L_p isometry induced by an admissible automorphism τ_k. Let Q_0 be the identity operator on L_p and τ_0 the identity automorphism on Z. Then let

$$T_k = EQ_kE, \quad V_k = T_k \dots T_0, \quad W_k = Q_k \dots Q_0.$$

For a fixed $f \in L_p$ we let, as before,

$$f_k = [V_k^*(V_kf)^*]^*,$$
$$g_k = f_k^* = V_k^*(V_kf)^*,$$

and also

$$\phi_k = W_k^{-1}EW_kEf, \quad 0 \leqq k \leqq n.$$

Note that

$$f_0 = \phi_0 = Ef.$$

(4.14) LEMMA. $V_kf = EW_kEf$.

Proof. First observe that W_kEf is constant on vertical lines. Hence,

$$W_{k+1}Ef = Q_{k+1}W_kEf = Q_{k+1}EW_kEf$$

where the second equality follows from Lemma (4.10). Then an obvious induction argument completes the proof.

(4.15) LEMMA. $V_k^*(V_kf)^* = E(W_k^{-1}EW_kEf)^*$.

Proof. Let $(V_kf)^* = g$, which is equal to

$$g = (EW_kEf)^*$$

by Lemma (4.14). Hence $Eg = g$. Since $T_k^* = EQ_k^*E$, and since Q_k^* is the L_q isometry induced by τ_k^{-1}, an obvious analogue of Lemma (4.14) shows that

$$V_k^*g = EW_k^*Eg = EW_k^*g.$$

Then Lemma (2.12) gives that

$$\begin{aligned} W_k^*g &= (W_k^{-1}g^*)^* \\ &= (W_k^{-1}EW_kEf)^*, \end{aligned}$$

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which completes the proof.

(4.16) LEMMA. *There exists a monotone sequence of finite partitions* $\mathscr{G}_n < \ldots < \mathscr{G}_0$, *each as in Lemma* (4.5), *such that, with the notations of* (4.13),

$$\phi_k = W_n^{-1}E(W_nEf|\mathscr{G}_k).$$

Proof. First we can take $\mathscr{G}_n = \mathscr{P}$, since

$$\phi_n = W_n^{-1}EW_nEf = W_n^{-1}E(W_nEf|\mathscr{P}).$$

To obtain $\mathscr{G}_{n-1}$ we use Lemma (3.8).

$$\begin{aligned}\phi_{n-1} &= W_{n-1}^{-1}E(W_{n-1}Ef|\mathscr{P})\\ &= W_{n-1}^{-1}Q_n^{-1}E(Q_nW_{n-1}Ef|\mathscr{P} \vee \tau_n\mathscr{P})\\ &= W_n^{-1}E(W_nEf|\mathscr{P} \vee \tau_n\mathscr{P}).\end{aligned}$$

Hence we can take $\mathscr{G}_{n-1} = \mathscr{P} \vee \tau_n\mathscr{P}$. Continuing in this way we see that

$$\mathscr{G}_{n-k} = \mathscr{P} \vee \tau_n\mathscr{P} \vee \ldots \vee \tau_n \ldots \tau_{n-k+1}\mathscr{P}, \quad 1 \leqq k \leqq n.$$

For a more formal proof one can apply an induction over n. Since $\mathscr{G}_k$ depends on n we write, for the purpose of this proof, $\mathscr{G}_k^n$ to show this dependence. Then an application of Lemma (3.8) shows that

$$\begin{aligned}W_n^{-1}E(W_nEf|\mathscr{G}_k^n) &= W_nQ_{n+1}^{-1}E(Q_{n+1}W_nEf|\mathscr{P} \vee \tau_{n+1}\mathscr{G}_k^n)\\ &= W_{n+1}^{-1}E(W_{n+1}Ef|\mathscr{G}_k^{n+1}),\end{aligned}$$

i.e., that

$$\mathscr{G}_k^{n+1} = \mathscr{P} \vee \tau_{n+1}\mathscr{G}_k^n.$$

Since $\mathscr{G}_k^k = \mathscr{P}$, we obtain $\mathscr{G}_k^n$ as before. Since n is fixed we will now again write $\mathscr{G}_k$ instead of $\mathscr{G}_k^n$.

(4.17) *Notation.* Let

$$u_k = E(W_nEf|\mathscr{G}_k), \quad 0 \leqq k \leqq n,$$

where $\mathscr{G}_k$'s are the partitions obtained in Lemma (3.17). Hence

$$\phi_k = W_n^{-1}u_k.$$

(4.18) THEOREM. *The sequence* $(u_0, \ldots, u_n)$ *is a martingale in* L_p *and*

$$||\max|u_k|\,|| \leqq q||u_0|| \leqq q||Ef||,$$

$$||\max|u_k - u_n|\,|| \leqq q||u_0 - u_n||.$$

Proof. Since the partitions $\mathscr{G}_k$ form a monotone sequence, the sequence $(u_0, \ldots, u_n)$ is a martingale in L_p with

$$u_k = E(u_0|\mathscr{G}_k).$$

Hence the martingale inequality (see, e.g. **[10]**, p. 91, where we can take $A_p = q = p/(p - 1)$) shows that

$$\|\max|u_k|\,\| \leqq q\|u_0\|.$$

For the second inequality, we observe that

$$h_k = u_k - u_n, \quad 0 \leqq k \leqq n,$$

is also a martingale, since

$$h_k = E(u_0 - u_n|\mathcal{G}_k) = E(h_0|\mathcal{G}_k).$$

Then we use the same martingale inequality for h_k.

(4.19) LEMMA.

$$\|\max|\phi_k|\,\| \leqq q\|\phi_0\| \quad \textit{and}$$

$$\|\max|\phi_k - \phi_n|\,\| \leqq q\|\phi_0 - \phi_n\|.$$

Proof. Note that W_n^{-1} is a positive invertible isometry of L_p and it is induced by an automorphism of Z. Since $\phi_k = W_n^{-1}u_k$ we see that

$$|\phi_k| = W_n^{-1}|u_k| \quad \text{and}$$

$$|\phi_k - \phi_n| = W_n^{-1}|u_k - u_n|.$$

Hence

$$\max|\phi_k| = W_n^{-1}(\max|u_k|\,) \quad \text{and}$$

$$\max|\phi_k - \phi_n| = W_n^{-1}(\max|u_k - u_n|\,).$$

Then the proof follows from Theorem (4.18).

(4.20) COROLLARY. $\|\max|\phi_k - \phi_0|\,\| \leqq 2q\|\phi_n - \phi_0\|$.

(4.21) LEMMA. *Given $\epsilon > 0$ there is a $\delta > 0$, depending only on ϵ and p, such that*

$$\|f_0\| - \|V_nf_0\| < \delta\|f_0\|$$

implies that $\|\phi_n - \phi_0\| < \epsilon\|\phi_0\|$.

Proof. Recall that $f_0 = \phi_0 = Ef$. Given $\epsilon > 0$ pick a $\delta > 0$ as given by Lemma (2.3). Let

$$\|f_0\| - \|V_nf_0\| < \delta\|f_0\|.$$

Then

$$\|W_nf_0\| - \|EW_nf_0\| = \|f_0\| - \|V_nf_0\|$$
$$< \delta\|f_0\| = \delta\|W_nf_0\|,$$

where we have used Lemma (4.14) and the fact that W_n is an L_p isometry. Hence Lemma (2.3) shows that

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$$\|W_n f_0 - EW_n f_0\| < \epsilon \|W_n f_0\|,$$

which implies that

$$\|f_0 - W_n^{-1} EW_n f_0\| < \epsilon \|f_0\|,$$

or that

$$\|\phi_0 - \phi_n\| < \epsilon \|\phi_0\|.$$

This completes the proof.

We are now ready to prove the main estimates.

(4.21) THEOREM. *With the notations of* (4.13),

$$\| \max_{0 \leqq k \leqq n} |f_k| \| \leqq q \|f_0\| \quad (= \|Ef\| \leqq \|f\|).$$

Proof. By Lemma (4.15) we see that

$$f_k^* = g_k = E\phi_k^*.$$

Hence

$$\begin{aligned} \|\max|g_k|\,\|_q &\leqq \|\max|\phi_k^*|\,\|_q \\ &\leqq \|\,(\max|\phi_k|\,)^*\|_q \\ &= \|\max|\phi_k|\,\|_p^{p/q} \\ &\leqq (q\|\phi_0\|\,)^{p/q}, \end{aligned}$$

where the last inequality follows from Lemma (4.19). We then have that

$$\begin{aligned} \|\max|f_k|\,\|_p &= \|\max|g_k^*|\,\|_p \\ &= \|\,(\max|g_k|\,)^*\|_p \\ &= \|\max|g_k|\,\|_q^{q/p} \\ &\leqq q\|\phi_0\| = q\|f_0\|. \end{aligned}$$

(4.22) THEOREM. *Given* $\xi > 0$ *there exists an* $\eta > 0$, *depending only on* ξ *and* p, *such that with the notations of* (4.13),

$$\|\max|g_k - g_0|\,\| < \xi$$

whenever

$$\|f_0\| - \|V_n f_0\| < \eta \|f_0\| \leqq \eta.$$

Proof. Use Lemma (2.7) with $\epsilon = \xi$ and $M = q$ to find a $\lambda > 0$ such that, for any sequence h_k in L_p,

$$\|\sup|h_k^* - h_0^*|\,\|_q < \xi,$$

whenever

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$$||\sup|h_k|\,||_p \leqq q$$

and

$$||\sup|h_k - h_0|\,||_p < \lambda.$$

Then use Lemma (4.21) with $\epsilon = \lambda/4q$ to find an $\eta > 0$ such that

$$||\phi_n - \phi_0|| < \frac{\lambda}{4q}||\phi_0||$$

whenever

$$(4.23) \quad ||f_0|| - ||V_n f_0|| < \eta||f_0||.$$

Now assume that (4.23) is satisfied with an $f_0 \in EL_p$ such that $||f_0|| \leqq 1$. Then

$$||\phi_n - \phi_0|| < \frac{\lambda}{4q}.$$

This implies, by Corollary (4.20),

$$||\max|\phi_k - \phi_0|\,|| < \lambda,$$

and also, by Lemma (4.19),

$$||\max|\phi_k|\,|| < q.$$

Hence the choice of λ implies that

$$||\max|\phi_k^* - \phi_0^*|\,||_q < \xi.$$

Then

$$\begin{aligned} ||\max|g_k - g_0|\,||_q &= ||\max|E\phi_k^* - E\phi_0^*|\,||_q \\ &\leqq ||\max|\phi_k^* - \phi_0^*|\,||_q \\ &< \xi. \end{aligned}$$

This completes the proof of the present theorem. Hence the proof of the main theorem, Theorem (3.2), is also completed.

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