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An Infinite Order Whittaker Function

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Abstract. In this paper we construct a flat smooth section of an induced space $I(s, \eta)$ of $SL_2(\mathbb{R})$ so that the attached Whittaker function is not of finite order. An asymptotic method of classical analysis is used.

1 Introduction

Whittaker functions appear in the *generic* Fourier coefficients of cuspidally induced Eisenstein series. This is the early work of F. Shahidi (cf. [14, 16]) in connection with the Langlands–Shahidi method. Their holomorphy (in the complex aspect, for the K-finite "part" of principal series representations) was first established by Jacquet [6]. Schiffman [13] later extended holomorphy to smooth vectors for real rank one groups. Shahidi [15], using intertwining estimates and following ideas of Jacquet [6], extended this result to real groups. The most general result is due to Wallach (cf. chapter 15 of [22]).

One is usually interested in the Langlands–Shahidi method for the appearance of automorphic L-functions (cf. Langlands [9] and Shahidi [18]). Indeed, the functional equation of Eisenstein series (cf. Langlands [10]) led Shahidi to a functional equation for these L-functions; and a theory of "local coefficients" (cf. Shahidi [14–19]). Put succinctly, the theory of local coefficients puts Jacquet's functional equation of Whittaker functions into a much more encompassing representation theoretic framework. Involved in this are multiplicity one (cf. Shalika [20]) and intertwining operators (which naturally fit with Langlands [9, 10]).

The field of automorphic forms has seen some striking new examples of functoriality, (cf. Kim and Shahidi [8] and Kim [7]). Central to these proofs is a converse theorem of Cogdell and Piatetski–Shapiro [2]. To apply this, one has to verify certain analytic information about specific automorphic L-functions. One requirement is boundedness in vertical strips. Gelbart and Shahidi [5] prove this. Their paper uses the theory of Eisenstein series, along with Shahidi's computation (the appearance of Whittaker functions in non-constant Fourier coefficients). Their paper also uses the functional equation of these L-functions, and so the proof of the main result only required a finite order estimate of Whittaker functions in a half-plane. (Further, the archimedean functions are the ones in question.) It is here that the question arose as to whether all smooth Whittaker functions are of finite order globally.

Decades ago, it was expected that the analytic behavior of these functions could be quite complicated (cf. Shahidi [15, Introduction]). In this paper, we construct

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a Whittaker function (attached to a smooth section of a principal series of $SL_2(\mathbb{R})$) that is not of finite order. The analytic tool used is Laplace's method from asymptotic analysis. For this reason, this paper might best fit into the "classical analysis" category, but the motivation (and result is for) was from the automorphic realm. This is a result that differs from the K-finite theory (cf. McKee [11]). It is possible that the analytic properties of smooth Eisenstein series could be different than those of the K-finite theory.

Let us describe the smooth Whittaker functions in Shahidi [15]. Assume G is a real split semisimple algebraic group with (real) Lie algebra $\mathfrak{g}$. Let us recall some definitions. (We refer to Chapter 2 of Wallach [21] for reductive properties.) Suppose Θ is a Cartan involution of $\mathfrak{g}$. If B denotes the Killing form on $\mathfrak{g}$ given by $B(X,Y) = \operatorname{tr}(XY)$, we can define an inner product on $\mathfrak{g}$ by $\langle X, Y\rangle = -B(X,\Theta(Y))$. We can decompose $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and -1 eigenspaces, respectively, of Θ. Then $\mathfrak{k}$ is the Lie algebra of a maximal compact subgroup of G, which we will call K. Further, let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$. We can write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ for $\mathfrak{n}$ a nilpotent Lie subalgebra. Then $\mathfrak{n}$ decomposes into *root-spaces* under the action of $\mathfrak{a}$ by ad. Let A and N be the connected subgroups of G corresponding to $\mathfrak{a}$ and $\mathfrak{n}$. For both subgroups, the exponential map is surjective. Then N is unipotent and clearly A is self-adjoint and abelian. The decomposition $G = ANK$ is known as an *Iwasawa decomposition*. Of course, K depends on Θ, and A and N depend also on the particular maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$.

Suppose we have an Iwasawa decomposition ANK with all the properties above. Let us define $M = \mathcal{Z}_G(\mathfrak{a})$, the centralizer in G of $\mathfrak{a}$. Then M is a real reductive group with split component A. Let us define $M_0 = M \cap K$. Then it is known that $P = MN$ is a (minimal) parabolic subgroup. Further, we have $P = M_0AN$, which is known as the *Langlands decomposition* of P. Since P is minimal, N is a full unipotent subgroup. (In the automorphic literature, this situation frequently uses the notation U instead of N.) Further, we can write $G = PK$. We take a function f_ν in the induced space $I(\nu,\eta)$ (a principal series) and a *generic* character χ on N. Here $\eta \in \hat{M}_0$; *i.e.*, η is a unitary character of M_0. The function f_ν then satisfies $f_\nu(m_o ank) = \eta(m_0)a^{\nu+\rho} f_\nu(k)$ where ν is identified to be in the complex dual of $\mathfrak{a}$. Further, ρ is half the sum of the roots generating N.

Then the Whittaker function attached to f_ν and χ, evaluated at the point $g \in G$ is

$$W_{f_\nu}(g) = \int_N f_\nu({w_l}^{-1} ng)\chi(n)dn.$$

Here w_l is the longest element in the Weyl group. This integral converges absolutely for ν in the positive Weyl chamber (where it is holomorphic in ν), and otherwise is interpreted by holomorphic continuation.

Further, one assumes the function f_ν above is *smooth* in g and ν. This brings in the semi-norm topology of $I_\infty(\nu,\eta)$, the smooth vectors of $I(\nu,\eta)$. (For a description, see Schiffman [13], Shahidi [15], or most thoroughly Chapter 10 of Wallach [22].) We are interested in the analytic properties of these functions in ν when $g = e$ is fixed. For this situation, when one reads the proofs of holomorphic continuation due to Schiffman [13] and Shahidi [15], one comes to the following conclusion.

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Due to this topology (forced by intertwining estimates), the further "to the left" in ν one wants to holomorphically continue the integral, the more derivatives of f_ν are needed. This can be seen even more precisely by the recurrence relation of the Bernstein polynomial method of the *meromorphic* continuation of intertwining integrals (see Wallach [22, Theorem 10.1.5]).

A finite order estimate for the Whittaker function attached to f_ν would require a *finite order* control of these derivatives of f_ν depending on ν. Since smoothness of f_ν really comes down to the restriction of f_ν to K, it seems this is not necessary for f_ν to be smooth. More precisely, it might be possible for f_ν to be smooth, but the derivatives of f_ν (depending on ν), coming from the proof of holomorphy of the attached Whittaker function, are infinite order in ν. This is even more to the point if we consider a *flat section* f_ν, *i.e.*, where $f_\nu|_K$ does not depend on ν. The question we address in this paper is how to construct a counterexample. More specifically, for the group $SL_2(\mathbb{R})$, we construct a flat section of $I_\infty(\nu, \eta)$ so that the attached Whittaker function is of infinite order. (When referring to an entire function, we take *infinite order* to mean not of finite order.)

In Section 2, we set up the $SL_2(\mathbb{R})$ coordinates and variables. As an example, we let $W_m(s)$ denote the Whittaker function attached to the flat section $f_s \in I_\infty(s, \eta)$ with K-type m, and generic character $e^{2\pi i x}$ of N. A particular integral representation of $W_m(s)$ is computed.

In Section 3, we exhibit a very simple Mellin transform. As a function of $x \in \mathbb{R}$, we obtain effective asymptotics, with remainder, as $x \to \infty$, by Laplace's method, (see 3.1). This asymptotic is of infinite order in x.

In Section 4 we construct a flat section $f_s \in I_\infty(s, \eta)$ so that the Whittaker function $W(s)$, attached to f_s matches (with $x = -(s-1)/2$) the Mellin transform of Section 3 with little error. This gives our main result, Theorem 4.1: a smooth Whittaker function of infinite order. Specifically, we have the estimate $W(s) \sim \sqrt{2\pi} e^{-\frac{s+1}{4}} e^{e^{-\frac{s+1}{2}}}$ as $s \to -\infty$.

In Section 4.1, we give a couple of remarks about Theorem 4.1. We discuss some motivation, *i.e.,* the framework of ideas that leads to the construction of f_s in Theorem 4.1.

2 SL_2 Preliminaries

For the rest of this paper, we will only consider the group $G = SL_2(\mathbb{R})$. Then of course the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$ consists of real 2×2 matrices of trace 0. Let us take Θ to be the particular Cartan involution on $\mathfrak{g}$ defined by $\Theta(X) = -X^*$, where X^* denotes the transpose of $X \in \mathfrak{g}$. Relative to Θ, it is well known that $SL_2(\mathbb{R})$ has an Iwasawa decomposition ANK, where $A = \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) : a > 0 \right\}$ is the split component, $N = \left\{ \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) : x \in \mathbb{R} \right\}$ is the full unipotent subgroup, and

$$K = SO(2) = \left\{ \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} : \theta \in \mathbb{R} \right\}$$

is the maximal compact subgroup. For our representative of the nontrivial Weyl element, we will take $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Clearly, an element $k \in K$ corresponds to a unique angle $\theta \mod 2\pi$. Let us write this as $k = k_\theta$, so we can work with θ without having to write the above matrix. Further, we can identify any element $a \in A$ with its first row, first column entry. Let us write $U(\mathfrak{sl}_{2\mathbb{C}})$ for the universal enveloping algebra of the complexification of $\mathfrak{sl}_2$.

The Lie algebra of A, $\mathfrak{a}$ consists of the linear span of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. By definition, $M_0 = \mathcal{Z}_K(\mathfrak{a})$. In our case, $SL_2(\mathbb{R})$ is so small that we can easily see that M_0 consists of two elements, ± 1, where 1 denotes the identity matrix. Let $\eta \in \widehat{M}_0$. Since M_0 is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, so is $\widehat{M}_0$. Thus there are only two possibilities for η. We can identify each case with either $\epsilon = 0$ or $\epsilon = 1$ as follows. First, let us identify $\pm 1 \in SL_2(\mathbb{R})$ with $\pm 1 \in \mathbb{R}$. With this identification, each $\eta \in \widehat{M}_0$ satisfies $\eta(m_0) = (m_0)^\epsilon$. For our construction, η will be trivial.

Let us describe a basic example. Let $s \in \mathbb{C}$ and $m \in \mathbb{Z}$. Let $f_{s,m}$ be the function on $SL_2(\mathbb{R})$ defined, using the Iwasawa decomposition, by $f_{s,m}(ank_\theta) = a^{s+1}e^{im\theta}$. (Here we have used the identification of A mentioned above; a on the right hand side of this equation is a positive real number.) With this definition, s is identified to be in the complex dual of $\mathfrak{a}$. Further, the 1 in a^{s+1} above corresponds to ρ, which is half the positive root. Notice $f_{s,m}(-1) = (-1)^m$. It follows that $f_{s,m} \in I(s,\eta)$ for all m, so 2 divides $m - \epsilon$. So if η is trivial, then $I(s,\eta)$ contains all even K-types, and if η is not trivial, then $I(s,\eta)$ contains all odd K-types. Notice the function above is a flat section of the induced space. In other words, $f_{s,m}|_K$ does not depend on s.

Continuing this example, let us compute the integral representation of the Whittaker function associated with $f_{s,m}$. This will be mentioned later. We assume $s \in \mathbb{C}$ with $\Re s > 0$. We must take χ to be a generic character of N. Since N has only one variable (x), we just need χ to be nontrivial. Then the Whittaker function of $f_{s,m}$ corresponding to the character χ is, (from above) as a function of $g \in SL_2(\mathbb{R})$

$$W_{f_{s,m},\chi}(g) \;=\; \int_N f_{s,m}(w^{-1}ng)\chi(n)dn\,.$$

With the assumption $\Re s > 0$ this integral converges absolutely. Recall that we are only interested in the s variable, so we take $g = 1$, the identity. Throughout this paper, we will fix $\chi(n) = e^{2\pi i x}$. The above integral is then

$$\int_{\mathbb{R}} f_{s,m}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) e^{2\pi i x} dx\,.$$

Since g and χ are fixed, let us denote this integral by $W_m(s)$. Breaking down $w^{-1}n$ into Iwasawa coordinates, and using the definition of $f_{s,m}$, we have

$$W_m(s) = \int_{\mathbb{R}} \frac{1}{(x^2+1)^{\frac{s+1}{2}}} \left(\frac{x+i}{\sqrt{x^2+1}} \right)^m e^{2\pi i x} dx\,. \tag{1}$$

3 An Infinite Order Mellin Transform

In what follows, we will be essentially interested in the integral, for $x > 0$,

$$\int_{e^e}^{\infty} y^{x - \log\log y} \, \frac{dy}{y}. \tag{2}$$

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In this section, we obtain effective asymptotics (with remainder) as $x \to \infty$ of this integral by Laplace's method. A good reference for this classical method is Murray [12]. Other good classical references are Erdélyi [4] and de Bruijn [1]. We have tried not to give too many details, since this computation consists essentially of calculus estimates.

First, let us assume $x > 12$. Under the change of variables $\log y = e^{x-1}z$ this integral becomes

$$e^{x-1}\int_{e^{2-x}}^{\infty} e^{e^{x-1}(z-z\log z)}\,dz.$$

For convenience, let us call $\lambda = e^{x-1}$. By our assumption on x, we have $0 < e^{2-x} < \frac{1}{2}$. We will first consider only the integral over the interval $[\frac{1}{2}, 2]$:

$$\int_{1/2}^{2} e^{\lambda(z-z\log z)}dz. \tag{3}$$

Note that the function $z-z\log z$ has a maximum of 1 at $z = 1$ (in the interval $(0,\infty)$), since its first derivative is $-\log z$, and its second is $-\frac{1}{z}$. Thus it is strictly increasing from 0 to 1, and then strictly decreasing from 1 to ∞. We can thus implicitly define a new variable w by setting

$$\lambda(z - z\log z - 1) = -w^2, \tag{4}$$

for $z \in (\frac{1}{2}, 2)$. Easily, this transformation is a diffeomorphism from $z \in (\frac{1}{2}, 2)$ to $w \in (-d_1\sqrt{\lambda}, d_2\sqrt{\lambda})$, for $d_1 = \sqrt{\frac{1}{2}(1-\log 2)}$ and $d_2 = \sqrt{2\log 2 - 1}$.

We need an accurate estimate of $\frac{dz}{dw}$. Implicitly differentiating (4), we see

$$\frac{dz}{dw} = \frac{2}{\lambda}\frac{w}{\log z} \quad\text{and}\quad \left.\frac{dz}{dw}\right|_{w=0} = \sqrt{\frac{2}{\lambda}}. \tag{5}$$

Implicitly differentiating (5) and using (4) we have

$$\frac{d^2z}{dw^2} = \frac{2}{\lambda}\frac{z(\log z)^2 + 2(z - z\log z - 1)}{z(\log z)^3}.$$

For this section only, let us define $\psi(z) = z(\log z)^2 + 2(z - z\log z - 1)$; *i.e.*, the numerator of the fraction on the right. Notice $\psi(1) = 0$ and $\psi'(z) = \log^2 z$. For $z \in (\frac{1}{2}, 2)$, two applications of the mean value theorem (the first to $\psi(z)$ and the second to $\frac{z-1}{\log z}$) give

$$\frac{d^2z}{dw^2} = \frac{2}{\lambda}\Big(\frac{\log\theta_z}{\log z}\Big)^2\frac{\tilde{\theta}_z}{z},$$

for some θ_z and $\tilde{\theta}_z$ between 1 and z. Note that this is always positive and is uniformly bounded by $4/\lambda$ for $z \in (\frac{1}{2}, 2)$. Consequently, using (5) and a calculus estimate in the w variable, we have

$$\frac{dz}{dw} = \sqrt{\frac{2}{\lambda}} + O\Big(\frac{w}{\lambda}\Big),$$

for $w \in (-d_1\sqrt{\lambda}, d_2\sqrt{\lambda})$. Consequently, the integral (3) becomes

$$e^{\lambda}\int_{-d_1\sqrt{\lambda}}^{d_2\sqrt{\lambda}} e^{-w^2}\cdot\Big(\sqrt{\tfrac{2}{\lambda}}+\big|O\big(\tfrac{w}{\lambda}\big)\big|\Big)\,dw.$$

With our assumption on x, for the specific numbers d_1 and d_2, one can show the above expression (and so the integral (3)) is

$$e^{\lambda}\cdot\Big(\sqrt{\tfrac{2\pi}{\lambda}}+O\big(\tfrac{1}{\lambda}\big)\Big). \tag{6}$$

This is using the known values $\int_{\mathbb{R}} e^{-t^2}dt = \sqrt{\pi}$, $\int_{\mathbb{R}} |t|e^{-t^2}dt = 1$, and the estimates (for $I_\lambda = (-\infty, -d_1\sqrt{\lambda}) \cup (d_2\sqrt{\lambda}, \infty)$)

$$\int_{I_\lambda} e^{-t^2}dt < 2\int_{d_1\sqrt{\lambda}}^{\infty} e^{-t^2}dt < 2\int_{d_1\sqrt{\lambda}}^{\infty} e^{-t}dt = 2e^{-d_1\sqrt{\lambda}} < \frac{\sqrt{2}}{\sqrt{\lambda}}.$$

These estimates come from the specific numbers d_1 and d_2, and our assumption $x > 12$ with $\lambda = e^{x-2}$. (Obviously $\int e^{-t^2}dt \le \int e^{-t}dt$ on a domain contained in $\{t \ge 1\}$.)

One can trivially show, with our assumption on x, that for $I_x = [e^{2-x}, \frac{1}{2}] \cup [2, \infty]$,

$$\int_{I_x} e^{\lambda(z - z\log z)}dz < \int_{[0,1/2]\cup[2,\infty]} e^{\lambda(z-z\log z)}dz \;=\; O(e^{\frac{17}{20}\lambda}). \tag{7}$$

Using estimates (7) and (6), as well as the specific numbers d_1 and d_2, a crude estimate gives the integral of (2) is

$$\lambda\cdot e^{\lambda}\cdot\Big(\sqrt{\tfrac{2\pi}{\lambda}}+O\big(\tfrac{1}{\lambda}\big)\Big),$$

once again with our assumption on x. We have proven:

Lemma 3.1 *For $x \in (12, +\infty)$, we have*

$$\int_{e^e}^{\infty} y^{x-\log\log y}\,\frac{dy}{y} = e^{\frac{x-1}{2}}e^{e^{x-1}}\cdot\Big(\sqrt{2\pi}+O\Big(\frac{1}{e^{\frac{x-1}{2}}}\Big)\Big)$$

Let us note that the constant in this lemma contained in the 'O' term is bounded in absolute value by 64, and of course is independent of $x \in (12, +\infty)$.

We will use a slightly altered version of this lemma for our construction. Suppose φ is any real function of $x \in [e^e, +\infty)$ that satisfies the following properties.

- φ is smooth on this domain.
- $\varphi(x) = 1$ for $x \ge e^{e^2}$.
- $0 \le \varphi(x) \le 1$ for $x \in [e^e, e^{e^2}]$.

Then for $x \in (12, +\infty)$, the integral

$$\int_{e^e}^{\infty} \varphi(y)y^{x-\log\log y}\,\frac{dy}{y}$$

satisfies the same conclusions as Lemma 3.1. To see this, notice (since $x > 12$) $[e^{2-x}, e^{3-x}] \subset [0, \frac{1}{2}]$. In calculating an error term for this integral, since $0 \le \varphi(x) \le 1$ for $x \in [e^e, e^{e^2}]$, it follows that the error computation in estimate (7) is sufficient.

4 Main Construction

In this section we construct a smooth flat section f_s whose corresponding Whittaker function is of infinite order. This is our main result, Theorem 4.1. The main tool is Lemma 3.1.

For the rest of this section, let us take ϕ to be any real valued function defined on $[0, \pi]$ with the following properties.

- $0 \le \phi(\theta) \le 1$ for all θ in this interval.
- $$\phi(\theta) = \begin{cases} 1 & 0 \le \theta \le \alpha \\ 0 & \beta \le \theta \le \pi, \end{cases}$$
  where $\alpha = \arctan\left\{ (e^{e^2} - 1)^{-1/2} \right\}$ and $\beta = \arctan\left\{ (e^e - 1)^{-1/2} \right\}$.
- ϕ is smooth in the entire interval.

Notice that $0 < \alpha < \beta < \frac{\pi}{2}$.

Let us define

$$g(\theta) = 2\cot(\theta)(\sin^2(\theta))^{\log\log(\sin^{-2}(\theta))} e^{-2\pi i \cot\theta}$$

for $\theta \in (0, \pi/2)$. Let us extend the definition of g to be 0 for $\theta \in [\pi/2, \pi]$. For $\theta \in (0, \pi]$, let us define $h(\theta) = \phi(\theta) \cdot g(\theta)$. Finally, let us put

$$f(\theta) = \begin{cases} h(\theta) & 0 < \theta \le \pi \\ h(2\pi - \theta) & \pi \le \theta < 2\pi \\ 0 & \theta = 0 \end{cases}.$$

Then one can check that, as a function of θ, f is smooth, and of course f is even by construction. Now $\theta = 0$ is really the only questionable point for smoothness, and $e^{-2\pi i \cot\theta}$ is far from smooth at $\theta = 0$. (This is similar to the oscillation of the standard real analysis example, the function $\sin(1/x)$ at $x = 0$.) However, the decay of the $(\sin^2(\theta))^{\log\log(\sin^{-2}(\theta))}$ term in the definition of f kills this oscillation as $\theta \to 0$. Using the full definition of f, in particular the form of the decaying term, f can be shown to be infinitely differentiable at $\theta = 0$, with $\frac{d^n f}{dt^n}(0) = 0$ for all $n \in \mathbb{N}$. (This is somewhat similar to another standard example, as follows. If we define $\zeta(x) = e^{-1/x^2}$ for nonzero $x \in \mathbb{R}$, and $\zeta(0) = 0$, then $\frac{d^n \zeta}{dt^n}(0) = 0$ for all $n \in \mathbb{N}$.)

Let us consider the flat section $f_s \in I(s, \eta)$ defined by $f_s(ank_\theta) = a^{s+1} f(\theta)$. Then clearly, $f_s \in I_\infty(s, \eta)$ with trivial η. The Whittaker function, $W(s)$, associated with f_s is then

$$W(s) = \int_{\mathbb{R}} f_s\left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \right) e^{2\pi i x} dx = \int_{\mathbb{R}} \frac{1}{(x^2+1)^{\frac{s+1}{2}}} f(\theta_x) e^{2\pi i x} dx, \tag{8}$$

where θ_x denotes the angle θ depending on x in the Iwasawa decomposition of $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) = ank_\theta$. We see θ satisfies $\cos(\theta) = \frac{x}{\sqrt{x^2+1}}$, $\sin(\theta) = \frac{1}{\sqrt{x^2+1}}$, and so $\cot(\theta) = x$. By all of the properties of the definitions above, this integral becomes

$$\int_{\sqrt{e^e - 1}}^{\infty} (x^2+1)^{-\frac{s+1}{2}} \left\{ \phi(\arctan(1/x)) 2x (x^2+1)^{-\log\log(x^2+1)} e^{-2\pi i x} \right\} e^{2\pi i x}\, dx.$$

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By the properties of ϕ, this integral is

$$\int_{\sqrt{e^{e^2}-1}}^{\infty} (x^2+1)^{-\frac{s+1}{2}} 2x(x^2+1)^{-\log\log(x^2+1)}\,dx$$

plus an integral over $[\sqrt{e^e-1}, \sqrt{e^{e^2}-1}]$ of the same integrand multiplied by $\phi(\arctan(1/x))$. By a trivial change of variable, we finally conclude that the integral (8) is equal to

$$(9) \qquad \int_{e^{e^2}}^{\infty} u^{-\frac{s-1}{2}-\log\log(u)}\frac{du}{u} + \int_{e^e}^{e^{e^2}} \phi(\arcsin(1/\sqrt{u}))u^{-\frac{s-1}{2}-\log\log(u)}\frac{du}{u}.$$

By Lemma 3.1 (or more precisely the comments following the lemma), for $x = -(s-1)/2 > 12$ (specifically, for $s < -23$) we have that the above expression is, for $\lambda = e^{x-1}$,

$$\sqrt{\lambda}e^{\lambda}\Big(\sqrt{2\pi} + O\Big(\frac{1}{\sqrt{\lambda}}\Big)\Big).$$

Thus we have the following.

Theorem 4.1 *For f_s given as above, we have for $W(s)$, the Whittaker function attached to f_s (the integral (8)) satisfies*

$$W(s) \sim \sqrt{2\pi}e^{-\frac{s+1}{4}}e^{e^{-\frac{s+1}{2}}}$$

as $s \to -\infty$.

Clearly, this function is not of finite order.

4.1 Remarks and Motivation

We must remark here that no functional equation for $W(s)$ was ever needed. We see that the integral (9) for $W(s)$ converges absolutely regardless of $s \in \mathbb{C}$. In the general theory, this will not happen too often. Even for the simple example $W_m(s)$, it is necessary for $\Re s > 0$ for the integral representation of $W_m(s)$ in (1) to converge absolutely. Further, the factor $e^{-2\pi i \cot\theta}$ in the definition of f_s is designed to cancel the additive character in the integral defining the attached Whittaker function. This greatly simplifies many asymptotic matters.

The motivation for the construction of f_s in Theorem 4.1 came from more complicated examples, and rather indirectly. It is perhaps of independent interest to see what is happening within the proof of holomorphy of smooth Whittaker functions, if such a function is of infinite order. Ultimately, this specifically, as well as related ideas, leads to some motivation for the rather easy construction of f_s in Theorem 4.1. Let us briefly explain.

In the case of $SL_2(\mathbb{R})$, let $f_s \in I_\infty(s,\eta)$ be *any* flat section with trivial η, and suppose the attached Whittaker function, $W(s)$, is of infinite order. Suppose $s \in \mathbb{C}$ with $\Re s > 0$. At the point $-s \in \mathbb{C}$, in showing $W(-s)$ is holomorphic, Shahidi's proof

[15] needs derivatives (from $U(\mathfrak{sl}_{2\mathbb{C}})$) of f_s where *these derivatives depend* on $-s$. Using the specific computation of Bernstein polynomials in Cohn [3, Appendix I], this proof can be made *effective*. Using this and other computations of Shahidi, such as the local coefficient $C(s)$ [14–17], it can be shown that the relevant derivatives of f_s that are needed lead naturally to a *discrete* Mellin transform (as a function in s), involving the Fourier coefficients of $f_s|_K$. The fact that this function is of infinite order is forced because $W(s)$ is of infinite order, by assumption. Trying to construct an infinite order discrete Mellin transform with specific estimates (which is connected to reversing this process) was part of the basis for the construction of f_s in Theorem 4.1.

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