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# Convergence for the fractional *p*-Laplacian and its application to the extended Nirenberg problem

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The main objective of this paper is to establish the convergence for the fractional $p$-Laplacian of sequences of nonnegative functions with $p > 2$. Furthermore, we show the blow-up phenomena for solutions to the extended Nirenberg problem modelled by fractional $p$-Laplacian with the prescribed negative functions.

*Keywords:* fractional $p$-Laplacian; convergence; extended Nirenberg problem; blow-up

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## 1. Introduction and main results

The fractional Laplacian has nowadays become a focus of research due to its extensive applications in describing anomalous diffusions in plasmas, flames propagation and chemical reactions in liquids, population dynamics, geophysical fluid dynamics, see [4, 7, 8] and the references therein. Moreover, it also has important applications in the fields of probability and finance, for example, see [1–3]. In particular, it can be regarded as the infinitesimal generator of an isotropic stable Lévy diffusion process. To better apply theories of the fractional Laplacian to practice, it is significantly important to make clear its own properties, especially those different from the classical Laplacian operator.

Before listing our main results, we first fix some notations. Let $n \geqslant 1$, $p \geqslant 2$ and $0 < \sigma < 1$. Define the fractional $p$-Laplacian $(-\Delta)_p^\sigma$ as follows:

$$(-\Delta)_p^\sigma u(x) = c_{n,\sigma p} P.V. \int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+\sigma p}} \, \mathrm{d}y,$$

where $c_{n,\sigma p}$ is a positive constant and $P.V.$ represents the Cauchy principal value. It is worth pointing out that $(-\Delta)_p^\sigma$ becomes the linear fractional Laplacian operator $(-\Delta)^\sigma$ if $p = 2$, while it is a nonlinear nonlocal operator if $p > 2$. The definition of $(-\Delta)_p^\sigma u$ is valid under the condition that $u \in C_{loc}^{\sigma p+\alpha}(\mathbb{R}^n) \cap \mathcal{L}_{\sigma p}(\mathbb{R}^n)$ for some $\alpha > 0$, where $C_{loc}^{\sigma p+\alpha} := C_{loc}^{[\sigma p+\alpha],\sigma p+\alpha-[\sigma p+\alpha]}$ with $[\sigma p + \alpha]$ denoting the integer part of $\sigma p+\alpha$,

$$\mathcal{L}_{\sigma p}(\mathbb{R}^n):=\left\{u\in L^{p-1}_{loc}(\mathbb{R}^n)\,\Big|\,\int_{\mathbb{R}^n}\frac{|u(x)|^{p-1}}{1+|x|^{n+\sigma p}}\,\mathrm{d}x<\infty\right\}.$$

Recently, Du *et al.* [11] derived the following fact:

> 'If $u_i\to u$ in $C^{2\sigma+\alpha}_{loc}$ as $i\to\infty$, and $\{(-\Delta)^\sigma u_i\}$ converges pointwisely, then $(-\Delta)^\sigma u_i\to(-\Delta)^\sigma u-\theta$ for some $\theta\geqslant 0$.'

In particular, they constructed an example showing that the nonnegative constant $\theta$ can be strictly positive, which is different from the classical Laplacian operator. This discrepancy essentially stems from the nonlocal behaviour of the fractional Laplacian operator. Inspired by their proof for the linear fractional Laplacian, in this paper we further overcome the nonlinear difficulty for the fractional $p$-Laplacian operator and prove that the above fact also holds for the nonlinear nonlocal operator $(-\Delta)^\sigma_p$ with $p>2$. Moreover, our result can be extended to more general nonlinear nonlocal operators. The principal result of this paper is stated as follows.

THEOREM 1.1. *Let $n\geqslant 1$, $p>2$, $0<\sigma<1$ and $\alpha>0$. Assume that a sequence of nonnegative functions $\{u_i\}\subset\mathcal{L}_{\sigma p}(\mathbb{R}^n)\cap C^{\sigma p+\alpha}_{loc}(\mathbb{R}^n)$ converges in $C^{\sigma p+\alpha}_{loc}(\mathbb{R}^n)$ to a function $u\in\mathcal{L}_{\sigma p}(\mathbb{R}^n)$, and $\{(-\Delta)^\sigma_p u_i\}$ converges pointwisely in $\mathbb{R}^n$. Then for any $x\in\mathbb{R}^n$,*

$$\lim_{i\to\infty}(-\Delta)^\sigma_p u_i(x)=(-\Delta)^\sigma_p u(x)-\theta,$$

*where $\theta$ is a nonnegative constant given by*

$$\theta=c_{n,\sigma p}\lim_{R\to\infty}\lim_{i\to\infty}\int_{B^c_R}\frac{u_i^{p-1}(x)}{|x|^{n+\sigma p}}\,\mathrm{d}x.$$

*Proof.* For any fixed $x\in\mathbb{R}^n$ and $R>>|x|+1$, let

$$\begin{aligned}
&(-\Delta)^\sigma_p u(x)-(-\Delta)^\sigma_p u_i(x)\\
&=c_{n,\sigma p}\int_{B_R(0)}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))-|u_i(x)-u_i(y)|^{p-2}(u_i(x)-u_i(y))}{|x-y|^{n+\sigma p}}\,\mathrm{d}y\\
&\quad+c_{n,\sigma p}\int_{B^c_R(0)}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+\sigma p}}\,\mathrm{d}y\\
&\quad+c_{n,\sigma p}\int_{B^c_R(0)}\frac{-|u_i(x)-u_i(y)|^{p-2}(u_i(x)-u_i(y))}{|x-y|^{n+\sigma p}}\,\mathrm{d}y\\
&=:\Phi_i(x,R)+\mathfrak{G}(x,R)+\Psi_i(x,R).
\end{aligned}\tag{1.1}$$

In light of the fact that $u_i\to u$ in $C^{\sigma p+\alpha}(B_{2R}(0))$, we obtain that for each $0<\varepsilon<1$, there exists an integer $N>0$ such that for every $i>N$,

$$\|u_i-u\|_{C^{\sigma p+\alpha}(B_{2R}(0))}\leqslant\varepsilon^{\frac{p}{\min\{1,p-2\}}},\quad\|u_i\|_{C^{\sigma p+\alpha}(B_{2R}(0))}\leqslant\|u\|_{C^{\sigma p+\alpha}(B_{2R}(0))}+1.\tag{1.2}$$

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Define

$$\Phi_i(x, R\setminus\varepsilon) := \Phi_i(x,R) - \Phi_i(x,\varepsilon), \quad \mathcal{M} := \|u\|_{C^{\sigma p+\alpha}(B_{2R}(0))} + 1,$$

where $\Phi_i(x,\varepsilon)$ denotes the integral in $\Phi_i(x,R)$ with the domain $B_R(0)$ replaced by $B_\varepsilon(x)$. Using (1.2), we deduce that for $x, y \in B_{2R}(0)$, $i > N$,

$$\begin{aligned}
&\big||u(x)-u(y)|^{p-2}(u(x)-u(y)) - |u_i(x)-u_i(y)|^{p-2}(u_i(x)-u_i(y))\big| \\
&\leqslant |u(x)-u(y)|^{p-2}|(u-u_i)(x)-(u-u_i)(y)| \\
&\quad + \big||u(x)-u(y)|^{p-2} - |u_i(x)-u_i(y)|^{p-2}\big|\,|u_i(x)-u_i(y)| \\
&\leqslant C(p,\mathcal{M})\|u_i-u\|^{\min\{1,p-2\}}_{L^\infty(B_{2R}(0))} \leqslant C(p,\mathcal{M})\varepsilon^p,
\end{aligned}$$

which yields that

$$|\Phi_i(x,R\setminus\varepsilon)| \leqslant C(p,\mathcal{M})\varepsilon^p \int_{B_{2R}(x)\setminus B_\varepsilon(x)} \frac{\mathrm{d}y}{|x-y|^{n+\sigma p}} \leqslant C(p,n,\sigma,\mathcal{M})\varepsilon^{(1-\sigma)p}. \tag{1.3}$$

On the other hand, if $\sigma p+\alpha \in (0,1]$, then it follows from (1.2) that

$$\begin{aligned}
|\Phi_i(x,\varepsilon)| &\leqslant C(p,\sigma,\alpha,\mathcal{M}) \int_{B_\varepsilon(x)} \frac{|x-y|^{(\sigma p+\alpha)(p-1)}}{|x-y|^{n+\sigma p}} \\
&\leqslant C(p,n,\sigma,\alpha,\mathcal{M})\varepsilon^{(\sigma p+\alpha)(p-2)+\alpha}.
\end{aligned} \tag{1.4}$$

When $\sigma p+\alpha \in (1,\infty)$, utilizing (1.2) again, it follows from Taylor expansion that

$$\begin{aligned}
&\big||u_i(x)-u_i(y)|^{p-2}(u_i(x)-u_i(y)) - |\nabla u_i(x)(x-y)|^{p-2}\nabla u_i(x)(x-y)\big| \\
&\leqslant C(p,\sigma,\alpha,\mathcal{M})\left(|\nabla u_i(x)(x-y)|^{p-2} + |x-y|^{\min\{2,\sigma p+\alpha\}(p-2)}\right)|x-y|^{\min\{2,\sigma p+\alpha\}} \\
&\leqslant C(p,\sigma,\alpha,\mathcal{M})|x-y|^{\min\{p,(\sigma+1)p+\alpha-2\}},
\end{aligned}$$

where we utilized the following element inequality:

$$\big||a|^{p-2}a - |b|^{p-2}b\big| \leqslant C(p)|a-b|\left(|a-b|^{p-2} + |b|^{p-2}\right), \quad \text{for } a,b\in\mathbb{R}^n.$$

By the same argument, we have

$$\begin{aligned}
&\big||u(x)-u(y)|^{p-2}(u(x)-u(y)) - |\nabla u(x)(x-y)|^{p-2}\nabla u(x)(x-y)\big| \\
&\leqslant C(p,\sigma,\alpha,\mathcal{M})|x-y|^{\min\{p,(\sigma+1)p+\alpha-2\}}.
\end{aligned}$$

Therefore, we obtain that if $\sigma p+\alpha \in (1,\infty)$,

$$\begin{aligned}
|\Phi_i(x,\varepsilon)| &\leqslant C(p,\sigma,\alpha,\mathcal{M}) \int_{B_\varepsilon(x)} \frac{|x-y|^{\min\{p,(\sigma+1)p+\alpha-2\}}}{|x-y|^{n+\sigma p}}\,\mathrm{d}y \\
&\leqslant C(p,n,\sigma,\alpha,\mathcal{M})\varepsilon^{\min\{(1-\sigma)p,p+\alpha-2\}},
\end{aligned} \tag{1.5}$$

where we utilized the anti-symmetry of $\nabla u(x)(x-y)$ and $\nabla u_i(x)(x-y)$ with regard to the centre $x$. Consequently, combining (1.3)–(1.5), we deduce that for

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every $i > N$,

$$|\Phi_i(x,R)| \leqslant C(p,n,\sigma,\alpha,\mathcal{M}) \begin{cases} \varepsilon^{\min\{(1-\sigma)p,(\sigma p+\alpha)(p-2)+\alpha\}}, & \text{if } \sigma p+\alpha \in (0,1], \\ \varepsilon^{\min\{(1-\sigma)p,p+\alpha-2\}}, & \text{if } \sigma p+\alpha \in (1,\infty), \end{cases}$$

which implies that

$$\lim_{i\to\infty} \Phi_i(x,R) = 0. \tag{1.6}$$

Note that $\{(-\Delta)_p^\sigma u_i\}$ is a pointwise convergent sequence, we then deduce from (1.1) and (1.6) that

$$\lim_{i\to\infty} \Psi_i(x,R) \text{ exists and is finite.} \tag{1.7}$$

Since $u \in \mathcal{L}_{\sigma p}(\mathbb{R}^n)$ and $R >> |x|+1$, then

$$\begin{aligned} &\limsup_{R\to\infty} \int_{B_R^c(0)} \frac{|u(x)-u(y)|^{p-1}}{|x-y|^{n+\sigma p}}\,\mathrm{d}y \\ &\leqslant \limsup_{R\to\infty} \left(\frac{R}{R-|x|}\right)^{n+\sigma p} \int_{B_R^c(0)} \frac{C(p)(u^{p-1}(x)+u^{p-1}(y))}{|y|^{n+\sigma p}}\,\mathrm{d}y = 0, \end{aligned}$$

which yields that

$$\lim_{R\to\infty} \mathfrak{G}(x,R) = 0.$$

This, in combination with (1.1) and (1.6)–(1.7), leads to that $\lim\limits_{R\to\infty}\lim\limits_{i\to\infty} \Psi_i(x,R)$ exists and is finite,

$$(-\Delta)_p^\sigma u(x) - \lim_{i\to\infty} (-\Delta)_p^\sigma u_i(x) = \lim_{R\to\infty}\lim_{i\to\infty} \Psi_i(x,R). \tag{1.8}$$

Denote

$$\begin{aligned} \mathcal{K}_1 &:= -u_i^{p-2}(y)u_i(x), \\ \mathcal{K}_2 &:= \left(u_i^{p-2}(y) - |u_i(x)-u_i(y)|^{p-2}\right) u_i(x), \\ \mathcal{K}_3 &:= -\left(u_i^{p-2}(y) - |u_i(x)-u_i(y)|^{p-2}\right) u_i(y), \\ \Theta &:= -|u_i(x)-u_i(y)|^{p-2}(u_i(x)-u_i(y)). \end{aligned}$$

Then we have

$$u_i^{p-1}(y) - \sum_{j=1}^3 |\mathcal{K}_j| \leqslant \Theta = u_i^{p-1}(y) + \sum_{j=1}^3 \mathcal{K}_j \leqslant u_i^{p-1}(y) + \sum_{j=2}^3 |\mathcal{K}_j|. \tag{1.9}$$

For any given $\varepsilon > 0$, it follows from Young's inequality that

$$|\mathcal{K}_1| \leqslant \varepsilon u_i^{p-1}(y) + \frac{C(p)}{\varepsilon^{p-2}} u_i^{p-1}(x). \tag{1.10}$$

We now divide into three cases to estimate $\mathcal{K}_2$ and $\mathcal{K}_3$ in the following.

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**Case 1.** Consider $2 < p \leqslant 3$. Since

$$u_i^{p-2}(y) \leqslant (|u_i(y) - u_i(x)| + u_i(x))^{p-2} \leqslant |u_i(y) - u_i(x)|^{p-2} + u_i^{p-2}(x),$$
$$|u_i(y) - u_i(x)|^{p-2} \leqslant u_i^{p-2}(y) + u_i^{p-2}(x),$$

then

$$\left| u_i^{p-2}(y) - |u_i(x) - u_i(y)|^{p-2} \right| \leqslant u_i^{p-2}(x).$$

Hence it follows from Young's inequality that

$$|\mathcal{K}_2| \leqslant u_i^{p-1}(x), \quad |\mathcal{K}_3| \leqslant \varepsilon u_i^{p-1}(y) + \frac{C(p)}{\varepsilon^{\frac{1}{p-2}}} u_i^{p-1}(x). \tag{1.11}$$

Substituting (1.10)–(1.11) into (1.9), we derive

$$(1 - 2\varepsilon) u_i^{p-1}(y) - \frac{C(p)}{\varepsilon^{p-2}} u_i^{p-1}(x) \leqslant \Theta \leqslant (1 + \varepsilon) u_i^{p-1}(y) + \frac{C(p)}{\varepsilon^{\frac{1}{p-2}}} u_i^{p-1}(x). \tag{1.12}$$

**Case 2.** Consider the case when $p > 3$ is an integer. From the binomial theorem and Young's inequality, we have

$$\begin{aligned}(a+b)^{p-2} &= a^{p-2} + \sum_{j=1}^{p-2} C_{p-2}^{j} a^{p-2-j} b^j \leqslant (1+\varepsilon) a^{p-2} + C(p) b^{p-2} \sum_{j=1}^{p-2} \varepsilon^{-\frac{p-2-k}{k}} \\ &\leqslant (1+\varepsilon) a^{p-2} + \frac{C(p)}{\varepsilon^{p-3}} b^{p-2}, \quad \text{for any } a, b \geqslant 0. \end{aligned} \tag{1.13}$$

Using (1.13), we deduce

$$u_i^{p-2}(y) \leqslant (|u_i(y) - u_i(x)| + u_i(x))^{p-2} \leqslant (1+\varepsilon)|u_i(x) - u_i(y)|^{p-2} + \frac{C(p)}{\varepsilon^{p-3}} u_i^{p-2}(x),$$

which implies that

$$\begin{aligned} u_i^{p-2}(y) - |u_i(y) - u_i(x)|^{p-2} &\leqslant \varepsilon |u_i(x) - u_i(y)|^{p-2} + \frac{C(p)}{\varepsilon^{p-3}} u_i^{p-2}(x) \\ &\leqslant \varepsilon(1+\varepsilon) u_i^{p-2}(y) + \frac{C(p)}{\varepsilon^{p-3}} u_i^{p-2}(x). \end{aligned}$$

Analogously,

$$|u_i(y) - u_i(x)|^{p-2} - u_i^{p-2}(y) \leqslant \varepsilon u_i^{p-2}(y) + \frac{C(p)}{\varepsilon^{p-3}} u_i^{p-2}(x).$$

Hence, we have

$$\left| u_i^{p-2}(y) - |u_i(y) - u_i(x)|^{p-2} \right| \leqslant \varepsilon(1+\varepsilon) u_i^{p-2}(y) + \frac{C(p)}{\varepsilon^{p-3}} u_i^{p-2}(x). \tag{1.14}$$

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Utilizing (1.14) and Young's inequality, we obtain

$$|\mathcal{K}_2| \leqslant \varepsilon(1+\varepsilon)u_i^{p-2}(y)u_i(x) + \frac{C(p)}{\varepsilon^{p-3}}u_i^{p-1}(x) \leqslant \varepsilon u_i^{p-1}(y) + \frac{C(p)}{\varepsilon^{p-3}}u_i^{p-1}(x),$$

$$|\mathcal{K}_3| \leqslant \varepsilon(1+\varepsilon)u_i^{p-1}(y) + \frac{C(p)}{\varepsilon^{p-3}}u_i^{p-2}(x)u_i(y) \leqslant \varepsilon(2+\varepsilon)u_i^{p-1}(y) + \frac{C(p)}{\varepsilon^{p-2}}u_i^{p-1}(x),$$

which, in combination with (1.9)–(1.10), gives that

$$\Theta \leqslant (1+3\varepsilon+\varepsilon^2)u_i^{p-1}(y) + \frac{C(p)}{\varepsilon^{p-2}}u_i^{p-1}(x), \tag{1.15}$$

$$\Theta \geqslant (1-4\varepsilon-\varepsilon^2)u_i^{p-1}(y) - \frac{C(p)}{\varepsilon^{p-2}}u_i^{p-1}(x). \tag{1.16}$$

**Case 3.** Consider the case when $p > 3$ is not an integer. On one hand, making use of (1.13), we obtain

$$\begin{aligned} |u_i(x)-u_i(y)|^{p-2} &\leqslant (u_i(x)+u_i(y))^{[p-2]+(p-[p])} \\ &\leqslant \left((1+\varepsilon)u_i^{[p-2]}(y) + \frac{C(p)}{\varepsilon^{[p-3]}}u_i^{[p-2]}(x)\right)\left(u_i^{p-[p]}(y)+u_i^{p-[p]}(x)\right) \\ &= (1+\varepsilon)u_i^{p-2}(y) + \frac{C(p)}{\varepsilon^{[p-3]}}u_i^{p-[p]}(y)u_i^{[p-2]}(x) \\ &\quad + (1+\varepsilon)u_i^{[p-2]}(y)u_i^{p-[p]}(x) + \frac{C(p)}{\varepsilon^{[p-3]}}u_i^{p-2}(x). \end{aligned} \tag{1.17}$$

From Young's inequality, we deduce

$$\frac{C(p)}{\varepsilon^{[p-3]}}u_i^{p-[p]}(y)u_i^{[p-2]}(x) \leqslant \varepsilon u_i^{p-2}(y) + \frac{C(p)}{\varepsilon^{p-3}}u_i^{p-2}(x), \tag{1.18}$$

$$(1+\varepsilon)u_i^{[p-2]}(y)u_i^{p-[p]}(x) \leqslant \varepsilon u_i^{p-2}(y) + \frac{C(p)}{\varepsilon^{\frac{[p-2]}{p-[p]}}}u_i^{p-2}(x). \tag{1.19}$$

Substituting (1.18)–(1.19) into (1.17), it follows that

$$|u_i(x)-u_i(y)|^{p-2} - u_i^{p-2}(y) \leqslant 3\varepsilon u_i^{p-2}(y) + \frac{C(p)}{\varepsilon^{\frac{[p-2]}{p-[p]}}}u_i^{p-2}(x). \tag{1.20}$$

On the other hand, using (1.13) again, we have

$$\begin{aligned} u_i^{p-2}(y) &\leqslant (|u_i(y)-u_i(x)|+u_i(x))^{[p-2]+(p-[p])} \\ &\leqslant \left((1+\varepsilon)|u_i(x)-u_i(y)|^{[p-2]} + \frac{C(p)}{\varepsilon^{[p-3]}}u_i^{[p-2]}(x)\right) \\ &\qquad \cdot \left(|u_i(x)-u_i(y)|^{p-[p]} + u_i^{p-[p]}(x)\right) \\ &= (1+\varepsilon)|u_i(x)-u_i(y)|^{p-2} + \frac{C(p)}{\varepsilon^{[p-3]}}u_i^{[p-2]}(x)|u_i(x)-u_i(y)|^{p-[p]} \\ &\quad + (1+\varepsilon)u_i^{p-[p]}(x)|u_i(x)-u_i(y)|^{[p-2]} + \frac{C(p)}{\varepsilon^{[p-3]}}u_i^{p-2}(x). \end{aligned} \tag{1.21}$$

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It follows from Young's inequality that

$$\frac{C(p)}{\varepsilon^{[p-3]}}u_i^{[p-2]}(x)|u_i(x)-u_i(y)|^{p-[p]} \leqslant \varepsilon|u_i(x)-u_i(y)|^{p-2}+\frac{C(p)}{\varepsilon^{p-3}}u_i^{p-2}(x), \tag{1.22}$$

$$(1+\varepsilon)u_i^{p-[p]}(x)|u_i(x)-u_i(y)|^{[p-2]} \leqslant \varepsilon|u_i(x)-u_i(y)|^{p-2}+\frac{C(p)}{\varepsilon^{\frac{[p-2]}{p-[p]}}}u_i^{p-2}(x). \tag{1.23}$$

Combining (1.20)–(1.23), we deduce

$$\begin{aligned} u_i^{p-2}(y)-|u_i(x)-u_i(y)|^{p-2} &\leqslant 3\varepsilon|u_i(x)-u_i(y)|^{p-2}+\frac{C(p)}{\varepsilon^{\frac{[p-2]}{p-[p]}}}u_i^{p-2}(x) \\ &\leqslant 3\varepsilon(1+3\varepsilon)u_i^{p-2}(y)+\frac{C(p)}{\varepsilon^{\frac{[p-2]}{p-[p]}}}u_i^{p-2}(x). \end{aligned}$$

This, together with (1.20) again, gives that

$$\left|u_i^{p-2}(y)-|u_i(x)-u_i(y)|^{p-2}\right| \leqslant 3\varepsilon(1+3\varepsilon)u_i^{p-2}(y)+\frac{C(p)}{\varepsilon^{\frac{[p-2]}{p-[p]}}}u_i^{p-2}(x). \tag{1.24}$$

In light of (1.24), it follows from Young's inequality that

$$\begin{aligned} |\mathcal{K}_2| &\leqslant 3\varepsilon(1+3\varepsilon)u_i^{p-2}(y)u_i(x)+\frac{C(p)}{\varepsilon^{\frac{[p-2]}{p-[p]}}}u_i^{p-1}(x) \\ &\leqslant \varepsilon u_i^{p-1}(y)+\frac{C(p)}{\varepsilon^{\frac{[p-2]}{p-[p]}}}u_i^{p-1}(x), \end{aligned} \tag{1.25}$$

$$\begin{aligned} |\mathcal{K}_3| &\leqslant 3\varepsilon(1+3\varepsilon)u_i^{p-1}(y)+\frac{C(p)}{\varepsilon^{\frac{[p-2]}{p-[p]}}}u_i^{p-2}(x)u_i(y) \\ &\leqslant \varepsilon(4+9\varepsilon)u_i^{p-1}(y)+\frac{C(p)}{\varepsilon^{\frac{[p-1]}{p-[p]}}}u_i^{p-1}(x). \end{aligned} \tag{1.26}$$

Therefore, substituting (1.10) and (1.25)–(1.26) into (1.9), we derive

$$\Theta \leqslant (1+5\varepsilon+9\varepsilon^2)u_i^{p-1}(y)+\frac{C(p)}{\varepsilon^{\frac{[p-1]}{p-[p]}}}u_i^{p-1}(x), \tag{1.27}$$

$$\Theta \geqslant (1-6\varepsilon-9\varepsilon^2)u_i^{p-1}(y)-\frac{C(p)}{\varepsilon^{\frac{[p-1]}{p-[p]}}}u_i^{p-1}(x). \tag{1.28}$$

Observe that

$$\begin{aligned} \lim_{R\to\infty}\lim_{i\to\infty}\int_{B_R^c(0)}\frac{u_i^{p-1}(x)}{|x-y|^{n+\sigma p}}\,\mathrm{d}y &= u^{p-1}(x)\lim_{R\to\infty}\int_{B_R^c(0)}\frac{\mathrm{d}y}{|x-y|^{n+\sigma p}} \\ &\leqslant u^{p-1}(x)\lim_{R\to\infty}\int_{B_{R-|x|}^c(x)}\frac{\mathrm{d}y}{|x-y|^{n+\sigma p}}=0. \end{aligned} \tag{1.29}$$

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Since $\lim\limits_{R\to\infty}\lim\limits_{i\to\infty}\Psi_i(x,R)$ exists and is finite, it follows from (1.12), (1.15)–(1.16) and (1.27)–(1.29) that

$$\lim_{R\to\infty}\lim_{i\to\infty}\Psi_i(x,R)\leqslant c_{n,\sigma p}(1+\varepsilon_p^{(1)})\liminf_{R\to\infty}\liminf_{i\to\infty}\int_{B_R^c(0)}\frac{u_i^{p-1}(y)}{|x-y|^{n+\sigma p}}\,\mathrm{d}y,$$

$$\lim_{R\to\infty}\lim_{i\to\infty}\Psi_i(x,R)\geqslant c_{n,\sigma p}(1-\varepsilon_p^{(2)})\limsup_{R\to\infty}\limsup_{i\to\infty}\int_{B_R^c(0)}\frac{u_i^{p-1}(y)}{|x-y|^{n+\sigma p}}\,\mathrm{d}y,$$

where

$$\varepsilon_p^{(1)}=\begin{cases}\varepsilon, & \text{if}\, 2<p\leqslant 3,\\ \varepsilon(3+\varepsilon), & \text{if}\, p>3 \text{ is an integer},\\ \varepsilon(5+9\varepsilon), & \text{if}\, p>3 \text{ is not an integer},\end{cases}$$

$$\varepsilon_p^{(2)}=\begin{cases}2\varepsilon, & \text{if }\ 2<p\leqslant 3,\\ \varepsilon(4+\varepsilon), & \text{if }\ p>3 \text{ is an integer},\\ 3\varepsilon(2+3\varepsilon), & \text{if }\ p>3 \text{ is not an integer}.\end{cases}$$

Due to the fact that $R>>|x|+1$, we have

$$\frac{(R-|x|)|y|}{R}\leqslant|y-x|\leqslant\frac{(R+|x|)|y|}{R},\quad \text{for } y\in B_R^c(0).$$

Hence, we deduce

$$\lim_{R\to\infty}\lim_{i\to\infty}\Psi_i(x,R)\leqslant c_{n,\sigma p}(1+\varepsilon_p^{(1)})\liminf_{R\to\infty}\liminf_{i\to\infty}\int_{B_R^c(0)}\frac{u_i^{p-1}(y)}{|y|^{n+\sigma p}}\,\mathrm{d}y,$$

$$\lim_{R\to\infty}\lim_{i\to\infty}\Psi_i(x,R)\geqslant c_{n,\sigma p}(1-\varepsilon_p^{(2)})\limsup_{R\to\infty}\limsup_{i\to\infty}\int_{B_R^c(0)}\frac{u_i^{p-1}(y)}{|y|^{n+\sigma p}}\,\mathrm{d}y.$$

By virtue of the arbitrariness of $\varepsilon$ and $\{u_i\}$ is nonnegative, we obtain

$$\lim_{R\to\infty}\lim_{i\to\infty}\Psi_i(x,R)=c_{n,\sigma p}\lim_{R\to\infty}\lim_{i\to\infty}\int_{B_R^c(0)}\frac{u_i^{p-1}(y)}{|y|^{n+\sigma p}}\,\mathrm{d}y\geqslant 0.$$

This, together with (1.8), yields that theorem 1.1 holds. □

In order to show that the limit constant $\theta$ captured in theorem 1.1 may be positive, we consider a sequence of nonnegative functions in the following. Choose a smooth cut-off function $\eta$ satisfying that

$$\eta(t)\equiv 0 \text{ in } (-\infty,0],\eta(t)\equiv 1 \text{ in } [1,\infty)\text{and } 0\leqslant\eta(t)\leqslant 1 \text{ in } [0,1]. \tag{1.30}$$

Then for any $0<s<t$ and $j\geqslant 1$, define

$$v_j(x):=j^{-s}w_j(R_j^{-1}x),\quad w_j(x):=\begin{cases}j^s+j^t\phi(x), \text{ in } B_6,\\ (1-\psi(x))(j^s+j^t), \text{ in } B_6^c,\end{cases} \tag{1.31}$$

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where $\phi(x) = \eta(|x| - 3)$, and $\psi(x) = \eta(|x| - 6)$, $R_j = j^{\frac{(t-s)(p-1)}{\sigma p}} \beta^{\frac{1}{\sigma p}}$ with

$$\beta := c_{n,\sigma p} \left( \int_{B_4 \setminus B_3} \frac{\phi^{p-1}(y)}{|y|^{n+\sigma p}} \, \mathrm{d}y + \int_{B_6 \setminus B_4} \frac{\mathrm{d}y}{|y|^{n+\sigma p}} + \int_{B_6^c} \frac{(1-\psi(y))^{p-1}}{|y|^{n+\sigma p}} \, \mathrm{d}y \right). \tag{1.32}$$

Example 1.2. Let $n \geqslant 1$, $p > 2$ and $0 < \sigma < 1$. If condition (1.31) holds, then we obtain that $v_j$ converges to 1 in $C^2_{loc}(\mathbb{R}^n)$, and

$$\lim_{i \to \infty} (-\Delta)^\sigma_p v_j(x) = -1.$$

Remark 1.3. We here would like to point out that the examples constructed in example 1.2 and theorem 2.1 were first given in [**11**].

*Proof.* It is easily seen from (1.31) that $v_j \in C^\infty_c(\mathbb{R}^n)$, $v_j \geqslant 0$ in $\mathbb{R}^n$, $v_j = 1$ in $B_{R_j}$, and $\|v_j - 1\|_{C^2_{loc}} \to 0$, as $i \to \infty$. A direct computation gives that

$$(-\Delta)^\sigma_p v_j(x) = j^{-s(p-1)} R_j^{-\sigma p} (-\Delta)^\sigma_p w_j(R_j^{-1} x), \quad \text{for } x \in B_{R_j}. \tag{1.33}$$

For any fixed $x \in \mathbb{R}^n$, we have

$$\begin{aligned}
& j^{-t(p-1)} (-\Delta)^\sigma_p w_j(R_j^{-1} x) \\
&= -c_{n,\sigma p} \int_{B_4 \setminus B_3} \frac{\phi^{p-1}(y)}{|R_j^{-1} x - y|^{n+\sigma p}} \, \mathrm{d}y - c_{n,\sigma p} \int_{B_6 \setminus B_4} \frac{\mathrm{d}y}{|R_j^{-1} x - y|^{n+\sigma p}} \\
&\quad + \frac{c_{n,\sigma p}}{j^{t-s}} \int_{B_6^c} \frac{\psi(y) |\psi(y) - 1 + \psi(y) j^{-(t-s)}|^{p-2}}{|R_j^{-1} x - y|^{n+\sigma p}} \, \mathrm{d}y \\
&\quad - c_{n,\sigma p} \int_{B_6^c} \frac{(1-\psi(y)) |\psi(y) - 1 + \psi(y) j^{-(t-s)}|^{p-2}}{|R_j^{-1} x - y|^{n+\sigma p}} \, \mathrm{d}y \\
&\quad \to -\beta, \quad \text{as } j \text{ goes to } \infty,
\end{aligned}$$

where $\beta$ is defined by (1.32). This, together with (1.33), gives that

$$\lim_{j \to \infty} (-\Delta)^\sigma_p v_j(x) = \beta^{-1} \lim_{j \to \infty} j^{-t(p-1)} (-\Delta)^\sigma_p w_j(R_j^{-1} x) = -1, \quad \text{in } \mathbb{R}^n.$$

The proof is complete. □

## 2. Blow-up analysis for the extended fractional Nirenberg problem

The extended fractional Nirenberg problem is equivalent to investigating the following equation:

$$(-\Delta)^\sigma_p u(x) = K(x) u^{q(p-1)}(x), \quad \text{for } x \in \mathbb{R}^n, \tag{2.1}$$

where $p \geqslant 2$ and $q \in \mathbb{R}$. It has been shown in [**11**] that there arises blow-up phenomena for the linear fractional Laplacian due to the nonzero constant $\theta$ captured in theorem 1.1. Specially, for $p = 2$, the compactness of solutions to (2.1) will fail in the region where $K$ is negative. In the following, we follow the proof of theorem 1.3 in [**11**] and extend the result to the nonlinear case of $p > 2$. On the other hand, when $K$ is positive, Jin *et al.* [**12**–**14**] derived a priori estimates for the fractional equation (2.1) with $p = 2$.

While these above-mentioned works are related to the fractional Nirenberg problem, there is another direction of research to study the classical elliptic equation $-\Delta u = K(x)u^p$. When $n = 1, 2$ and $1 < p < \infty$, or $n \geqslant 3$ and $1 < p < \frac{n+2}{n-2}$, $p$ is called a subcritical Sobolev exponent, while it is the critical Sobolev exponent if $n \geqslant 3$ and $p = \frac{n+2}{n-2}$. In particular, the elliptic equation in the case of critical Sobolev exponent corresponds to the Nirenberg problem, which is to seek a new metric conformal to the flat metric on $\mathbb{R}^n$ so that its scalar curvature is $K(x)$. Generally, it needs to establish priori estimates of the solutions for the purpose of obtaining the existence of solutions. We refer to [**9**, **10**] for the subcritical case. With regard to the critical case, see [**5**, **15**, **17**] for positive functions $K$ and [**6**, **16**, **18**] for $K$ changing signs, respectively.

THEOREM 2.1. *Assume that $n \geqslant 1$, $p > 2$, $0 < \sigma < 1$, $q \in \mathbb{R}$ and $s > -\frac{\sigma p}{p-1}$. Then there exist two positive constants $c_0 = c_0(n, \sigma, p, q, s)$ and $C_0 = C_0(n, \sigma, p, q, s)$, a sequence of functions $\{K_j\} \subset C^\infty(\mathbb{R}^n)$ satisfying*

$$-C_0 \leqslant K_j(x) \leqslant -c_0, \;\; c_0 \leqslant |\nabla K_j(x)| \leqslant C_0, \;\; \text{and} \; |\nabla^2 K_j(x)| \leqslant C_0, \quad \text{in } B_2,$$

*and a sequence of positive functions $\{u_j\} \subset C^\infty(\mathbb{R}^n)$ such that*

$$(-\Delta)_p^\sigma u_j(x) = K_j(x) u_j^{q(p-1)}(x), \;\; \text{for } x \in \mathbb{R}^n, \quad |x|^s u_j(x) \to 1, \quad \text{as } |x| \to \infty,$$

*and*

$$\min_{\overline{B}_1} u_j \to \infty, \quad \text{as } j \to \infty.$$

*Proof.* Let $\eta$ and $\phi$ be defined in (1.30) and (1.31). For $q \in \mathbb{R}$ and $s > -\frac{\sigma p}{p-1}$, let

$$u_j(x) = \begin{cases} j + j^q \phi(x), & \text{in } B_R, \\ (1 - \varphi(x))(j + j^q) + \varphi(x)|x|^{-s}, & \text{in } B_R^c, \end{cases}$$

where $\varphi(x) = \eta(|x| - R)$ and $R = R(n, p, q, \sigma, s, j) > 9$ is a sufficiently large constant to be determined later. Then $u_j \in C^\infty(\mathbb{R}^n) \cap \mathcal{L}_{\sigma p}(\mathbb{R}^n)$ and $u_j > 0$ in $\mathbb{R}^n$. Denote

$$K_j(x) := \frac{(-\Delta)_p^\sigma u_j(x)}{u_j^{q(p-1)}(x)}, \quad \text{in } \mathbb{R}^n.$$

Then $K_j \in C^\infty(\mathbb{R}^n)$. Moreover, $\{K_j\}$ satisfies the following properties: there exists four positive constants $C_i := C_i(n, \sigma, p)$, $i = 1, 2, 3, 4$, such that for every $j \geqslant 1$,

(**K1**) $-C_1 \leqslant K_j(x) \leqslant -C_2$, and $\sum_{i=1}^{3} |\nabla^i K_j(x)| \leqslant C_3$ in $B_2$;

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(**K2**) $\nabla^2 K_j(0) \leqslant -C_4\mathbf{I}_n$, where $\mathbf{I}_n$ denotes $n\times n$ identity matrix.

We first prove (**K1**). Observe that

$$c_{n,\sigma p}^{-1}K_j(x) = -\int_{B_4\setminus B_3}\frac{\phi^{p-1}(y)}{|x-y|^{n+\sigma p}}\,\mathrm{d}y - \int_{B_R\setminus B_4}\frac{\mathrm{d}y}{|x-y|^{n+\sigma p}} + \int_{B_R^c}\frac{|\mathcal{A}_\varphi(y)|^{p-2}\mathcal{A}_\varphi(y)}{|x-y|^{n+\sigma p}}\,\mathrm{d}y := \sum_{i=1}^{3} J_i, \tag{2.2}$$

where $\mathcal{A}_\varphi(y) := \varphi(y) - 1 + j^{1-q}\varphi(y) - j^{-q}|y|^{-s}\varphi(y)$. For simplicity, let

$$\gamma := \gamma(n,\sigma,p) = \int_{B_1^c}\frac{\mathrm{d}y}{|y|^{n+\sigma p}} = \frac{|\mathbb{S}^{n-1}|}{\sigma p},$$

$$\tau := \tau(n,\sigma,p,s) = \int_{B_1^c}\frac{\mathrm{d}y}{|y|^{n+\sigma p+s(p-1)}} = \frac{|\mathbb{S}^{n-1}|}{\sigma p + s(p-1)}.$$

A straightforward computation yields that

$$0 \geqslant J_1 \geqslant -\int_{B_1(x)^c}\frac{\mathrm{d}y}{|x-y|^{n+\sigma p}} = -\gamma,$$

and

$$-\gamma \leqslant J_2 \leqslant -\int_{B_{R-2}\setminus B_6}\frac{\mathrm{d}y}{|y|^{n+\sigma p}} = -(6^{-\sigma p} - (R-2)^{-\sigma p})\gamma.$$

For $x\in B_2$, $y\in B_R^c$, we have $|x-y|\geqslant |y|/2$ in virtue of $R>9$. Then

$$|J_3| \leqslant 2^{(\sigma+1)p+n-2}\int_{B_R^c}\frac{(1+j^{1-q})^{p-1} + j^{-q(p-1)}|y|^{-s(p-1)}}{|y|^{n+\sigma p}}\,\mathrm{d}y = 2^{(\sigma+1)p+n-2}\gamma(1+j^{1-q})^{p-1}R^{-\sigma p} + 2^{(\sigma+1)p+n-2}\tau j^{-q(p-1)}R^{-\sigma p - s(p-1)}.$$

For a sufficiently large $R>9$, we have

$$(R-2)^{-\sigma p}\gamma + 2^{(\sigma+1)p+n-2}R^{-\sigma p}\left(\gamma(1+j^{1-q})^{p-1} + \tau j^{-q(p-1)}R^{-s(p-1)}\right) \leqslant \frac{\gamma 6^{-\sigma p}}{2},$$

which implies that

$$-3c_{n,\sigma p}\gamma \leqslant K_j(x) \leqslant -\frac{c_{n,\sigma p}\gamma 6^{-\sigma p}}{2}, \quad \forall\, x\in B_2,\ j\geqslant 1.$$

Furthermore, after differentiating (2.2), it follows from a similar calculation that

$$\sum_{i=1}^{3}|\nabla^i K_j(x)| \leqslant C(n,\sigma,p), \quad \text{for } x\in B_2,\ j\geqslant 1.$$

We proceed to verify property (**K2**). A simple calculation shows that for $y\in B_3^c$,

$$\partial^2_{x_k x_l}\left(\frac{1}{|x-y|^{n+\sigma p}}\right)(0) = \frac{(n+\sigma p)[(n+\sigma p+2)y_k y_l - \delta_{kl}|y|^2]}{|y|^{n+\sigma p+4}}. \tag{2.3}$$

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Since the integral domain is symmetric, then we see from (2.2) to (2.3) that

$$\partial^2_{x_k x_l} K_j(0) = 0, \quad \text{for } k \neq l.$$

If $k = l$, it follows from the radial symmetry of $\phi$ and $\varphi$ that

$$\begin{aligned}
&[(n+\sigma p)c_{n,\sigma p}]^{-1}\partial^2_{x_k x_k} K_j(0) \\
&= -\frac{\sigma p+2}{n}\left(\int_{B_4\setminus B_3} \frac{\phi^{p-1}(y)}{|y|^{n+\sigma p+2}}\,\mathrm{d}y + \int_{B_R^c} \frac{|\mathcal{A}_\varphi(y)|^{p-2}(j^{1-q}\varphi(y)-\mathcal{A}_\varphi(y))}{|y|^{n+\sigma p+2}}\,\mathrm{d}y\right) \\
&\quad -\frac{\sigma p+2}{n}\left(\int_{B_R\setminus B_4} \frac{\mathrm{d}y}{|y|^{n+\sigma p+2}} - j^{1-q}\int_{B_R^c} \frac{|\mathcal{A}_\varphi(y)|^{p-2}\varphi(y)}{|y|^{n+\sigma p+2}}\,\mathrm{d}y\right) \\
&\leqslant -|B_1|\left(4^{-(\sigma p+2)} - R^{-(\sigma p+2)} - 3j^{1-q}R^{-(\sigma p+2)}\right) \\
&\leqslant -|B_1|4^{-(\sigma p+3)}, \quad \text{for a sufficiently large } R > 9,
\end{aligned}$$

where we used the fact that $|\mathcal{A}_\varphi(y)|^{p-2}\varphi(y) \leqslant 3$ in $B_R^c$. That is, property (**K2**) holds.

From the radial symmetry of $u_j$ with respect to the origin, we know that $K_j$ is also radially symmetric. Then we have

$$\nabla K_j(0) = 0,$$

which, together with (**K1**)–(**K2**), leads to that for $j \geqslant 1$,

$$|\nabla K_j(x)| \geqslant c_1, \quad \text{in } B_{2\varepsilon_0}(4\varepsilon_0 e_1), \tag{2.4}$$

where $e_1 = (1, 0, ..., 0) \in \mathbb{R}^n$, $\varepsilon_0 := \varepsilon_0(n, p, \sigma) \in (0, 1/4)$ is a small constant and $c_1 := c_1(n, p, \sigma)$ is a positive constant.

Define

$$\bar{u}_j(x) := \varepsilon_0^s u_j(\varepsilon_0(x + 4e_1)), \quad \text{and } \bar{K}_j(x) := \varepsilon_0^{\sigma p - s(p-1)(q-1)} K_j(\varepsilon_0(x + 4e_1)).$$

Therefore,

$$(-\Delta)_p^\sigma \bar{u}_j = \bar{K}_j(x)\bar{u}_j^{q(p-1)}, \quad \text{for } x \in \mathbb{R}^n.$$

Then combining (**K1**) and (2.4), we obtain

$$-\bar{C} \leqslant \bar{K}_j(x) \leqslant -\bar{c}, \;\; \bar{c} \leqslant |\nabla \bar{K}_j(x)| \leqslant \bar{C}, \text{ and } |\nabla^2 \bar{K}_j(x)| \leqslant \bar{C}, \quad \text{in } B_2,$$

where $\bar{c} = \bar{c}(n, \sigma, p, q, s)$ and $\bar{C} = \bar{C}(n, \sigma, p, q, s)$. Moreover, recalling the definition of $u_j$, we have

$$\lim_{|x|\to\infty} |x|^s \bar{u}_j = 1, \quad \text{and } \min_{\overline{B}_1} \bar{u}_j = \varepsilon_0^s j \to \infty, \quad \text{as } j \to \infty.$$

The proof is finished. □

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## Data availability statement

The data used to support the findings of this study are available from the corresponding author upon request.

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## Conflict of interest

None.

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