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HOMOMORPHISMS BETWEEN LATTICES OF ZERO-SETS

BY

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ABSTRACT. For a completely regular Hausdorff topological space X, let $Z(X)$ denote the lattice of zero-sets of X. If τ is a continuous map from X to Y, then there is a lattice homomorphism τ' from $Z(Y)$ to $Z(X)$ induced by τ which is defined by $\tau'(A)=\tau^{\leftarrow}(A)$. A characterization is given of those lattice homomorphisms from $Z(Y)$ to $Z(X)$ which are induced in the above way by a continuous function from X to Y.

1. Introduction. The theory of duality linking topology and algebra has been studied in depth in the past. Most notably in this area, is the work of M. H. Stone in [3], which describes the duality between compact, Hausdorff, 0-dimensional spaces (i.e. spaces with a base of open-and-closed sets) and Boolean algebras. In particular, it is shown that if X and Y are compact, Hausdorff, 0-dimensional spaces and $B(X)$ and $B(Y)$ are their Boolean algebras of clopen sets then if $t:B(Y)\to B(X)$ is a homomorphism such that $t(Y)=X$, then there is a continuous map $f:X\to Y$ such that $t(A)=f^{\leftarrow}(A)$ for all $A\in B(Y)$. In chapter 10 of [1] this aspect of duality is studied where the algebraic object is $C(X)$, the ring of continuous functions of a completely regular, Hausdorff space. It is shown that if X and Y are realcompact spaces and if $t:C(Y)\to C(X)$ is a homomorphism such that $t((1_Y)=1_X$ (where 1 denotes the constant function whose range is $\{1\}$), then there is a continuous function $f:X\to Y$ such that $t(g)=g\circ f$ for all $g\in C(Y)$.

In this paper we consider the lattice $Z(X)$, of zero-sets of a completely regular, Hausdorff space and characterize these lattice homomorphisms between zero-set lattices that arise, in the natural way described above, from continuous functions.

1.1. DEFINITION. (a) Let X and Y be spaces and let $Z(X)$ and $Z(Y)$ denote their respective zero-set lattices. By a *σ-homomorphism* t from $Z(Y)$ to $Z(X)$

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we will mean a lattice homomorphism $t: Z(Y) \to Z(X)$ such that t preserves countable meets (i.e. if $\{A_i\}_{i=1}^{\infty} \subseteq Z(Y)$, then $t(\bigcap_{i=1}^{\infty} A_i) = \bigcap_{i=1}^{\infty} t(A_i)$), and such that $t(\phi) = \phi$, $t(Y) = X$.

(b) Let X and Y be spaces and let $\tau: X \to Y$ be a continuous map. Then the "*homomorphism induced by* τ" is the map $\tau': Z(Y) \to Z(X)$ defined by $\tau'(A) = \tau^{\leftarrow}(A)$ for $A \in Z(Y)$.

Note that in 1.1(b) τ' is actually a *σ-homomorphism* in the sense of 1.1(a). In section 2 it is shown that these are precisely the homomorphisms induced by continuous maps.

The notation will be that of [1]. The set of integers is denoted by N. All spaces discussed are assumed to be completely regular and Hausdorff.

2. The homomorphism induced by a continuous map. Before getting to the main characterization theorem, some results of general interest on the induced homomorphism are obtained.

2.1 DEFINITION. (a) A subspace W of X is *G_δ-dense* in X if every G_δ in X meets W in a non-empty set.

(b) A subspace W of X is *z-embedded* in X if every zero-set in W is the restriction to W of a zero-set in X.

2.2. THEOREM. *Let τ be a continuous map from X to Y and let τ' be the induced homomorphism from $Z(Y)$ to $Z(X)$. Then*

(*a*) *τ' is one-to-one iff $\tau(X)$ is G_δ-dense in Y, and*

(b) *τ' is onto iff τ is a homeomorphism onto a z-embedded subset of Y.*

Proof. (a) Necessity. Suppose $\tau(X)$ is not G_δ-*dense* in Y. Then there is a non-empty G_δ set $H = \bigcap_{i \in N} U_i$ in Y (where U_i is open in Y for all $i \in N$) such that $H \cap \tau(X) = \phi$. Let $y \in H$. Then there is a zero-set Z in Y such that $y \in Z \subseteq H$ (for each $i \in N$ there is a zero-set Z_i in Y such that $y \in Z_i \subseteq U_i$ since Y is completely regular, thus $y \in Z = \bigcap_{i \in N} \subseteq Z_i \ \bigcap_{i \in N} U_i = H$). Hence $\tau'(Z) = \tau^{\leftarrow}(Z) = \phi = \tau^{\leftarrow}(\phi) = \tau'(\phi)$ and thus τ' is not one-to-one.

Sufficiency. Suppose τ' is not one-to-one. Then there exist $Z_1, Z_2 \in Z(Y)$ such that $\tau^{\leftarrow}(Z_1) = \tau^{\leftarrow}(Z_2)$ but $Z_1 \neq Z_2$. Let $p \in Z_1 - Z_2$. Then there is a $Z_3 \in Z(Y)$ such that $p \in Z_3$ and $Z_3 \cap Z_2 = \phi$ (as $Y - Z_2$ is a neighborhood of p). Let $Z = Z_1 \cap Z_3$. Then $\tau'(Z) = \tau^{\leftarrow}(Z_1 \cap Z_3) = \tau'(Z_1) \cap \tau^{\leftarrow}(Z_3) = \tau^{\leftarrow}(Z_2) \cap \tau^{\leftarrow}(Z_3) = \tau^{\leftarrow}(Z_2 \cap Z_3) = \tau^{\leftarrow}(\phi) = \phi$. Thus $Z \cap \tau(X) = \phi$ and hence $\tau(X)$ is not G_δ dense in Y.

(b) This result has been obtained independently by Mandelker in [2, p. 619], to which the reader is referred for a proof.

2.3. DEFINITION. Let X be a space. A *z-filter* $\mathcal{F}$ on X is a filter on the lattice $Z(X)$. That is $\mathcal{F} \subseteq Z(X)$ such that i) $\phi \notin \mathcal{F}$, ii) if $Z_1, Z_2 \in \mathcal{F}$ then $Z_1 \cap Z_2 \in \mathcal{F}$, and iii) if $Z_1 \in \mathcal{F}$ and $Z \in Z(X)$ such that $Z_1 \subseteq Z$, then $Z \in \mathcal{F}$. A z-filter $\mathcal{F}$ on X

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is *prime* if $Z_1 \cup Z_2 \in \mathcal{F}$ implies that either $Z_1 \in \mathcal{F}$ or $Z_2 \in \mathcal{F}$. A *z-ultrafilter* is a maximal z-filter. A *real z-ultrafilter* is a z-ultrafilter closed under countable intersection (i.e. if $\{Z_i\}_{i\in N} \subseteq \mathcal{F}$, then $\bigcap_{i\in N} Z_i \in \mathcal{F}$).

2.4. REMARK. If $\{p\}$ is the one point space, then $Z(\{p\}) = \{\phi, \{p\}\}$ is the two-point lattice which will be referred to here as $\{0, 1\}$. There is a one-to-one correspondence between σ-homomorphisms from $Z(Y)$ onto $\{0, 1\}$ and real z-ultrafilters on Y. Clearly if $\mathcal{F}$ is a real z-ultrafilter on Y, then $t: Z(Y) \to \{0, 1\}$ defined by $t(Z) = 1$ if $Z \in \mathcal{F}$, $t(Z) = 0$ otherwise, is a σ-homomorphism. On the other hand, if $t: Z(Y) \to \{0, 1\}$ is a σ-homomorphism, then clearly $\mathcal{F} = t^{\leftarrow}(\{1\})$ is a prime z-filter on Y. Since t is a σ-homomorphism, $\mathcal{F}$ is closed under countable intersection. Thus, by [1, 7H4], $\mathcal{F}$ is a real z-ultrafilter. It is also apparent that there is a one-to-one correspondence between all homomorphisms from $Z(Y)$ to $\{0, 1\}$ and all prime z-filters on Y.

2.5. THEOREM. *Let Y be a realcompact space and let t be a homomorphism from $Z(Y)$ to $Z(X)$. The following are equivalent.*

(i) *t is a σ-homomorphism.*
(ii) *$t = \tau'$ for a unique continuous map $\tau: X \to Y$.*

Proof. (ii) implies (i). We have already noted in the introduction that this implication is true.

(i) implies (ii). Let $x \in X$. Let $A_x = \{Z \in Z(X) \mid x \in Z\}$. Then A_x is a real z-ultrafilter on X. Let $t^{\leftarrow}(A_x) = \{Z \in Z(Y) \mid t(Z) \in A_x\}$. Then $t^{\leftarrow}(A_x)$ is clearly a prime z-filter on Y. Furthermore, $t^{\leftarrow}(A_x)$ is closed under countable intersection (since t is a σ-homomorphism). Thus, by [1, 7H4], $t^{\leftarrow}(A_x)$ is a real z-ultrafilter on Y. Since Y is realcompact, there is a unique $y \in Y$ such that $\bigcap t^{\leftarrow}(A_x) = \{y\}$. Define $\tau(x) = y$. Then τ is a well-defined map from X to Y.

Let $Z \in Z(Y)$. Then $t(Z) = \tau^{\leftarrow}(Z)$, for $x \in t(Z)$ iff $t(Z) \in A_x$ iff $Z \in t^{\leftarrow}(A_x)$ iff $x \in \tau^{\leftarrow}(Z)$. This shows that $t = \tau'$ and τ is continuous, as every closed subset of Y is an intersection of zero-sets.

If $\sigma: X \to Y$ such that $\sigma' = \tau' = t$, then $\sigma^{\to}(Z) = \tau^{\leftarrow}(Z)$ for every $Z \in Z(Y)$. Thus $\sigma^{\leftarrow}(\{y\}) = \sigma^{\leftarrow}(\bigcap A_y) = \tau^{\leftarrow}(\bigcap A_y) = \tau^{\leftarrow}(\{y\})$. Hence $\sigma = \tau$, and τ is unique.

The condition "Y is realcompact" in 2.5 cannot be dropped because, as is shown in [1, 8D2], $Z(X) \cong Z(\upsilon X)$ (lattice isomorphic by the isomorphism $Z \to cl_{\upsilon X} Z$, where υX denotes the Hewitt realcompactification of X) for any space X, thus $Z(X)$ does not distinguish between X and υX. This immediately gives the following corollary to 2.5.

2.6. COROLLARY. *If t is a σ-homomorphism from $Z(Y)$ to $Z(X)$, then there is a unique continuous map $\tau: X \to \upsilon Y$ such that $t(Z) = \tau^{\leftarrow}(cl_{\upsilon Y} Z)$ for all $Z \in Z(Y)$.*

Theorem 2.5 also yields the following results.

2.7. COROLLARY. *If X and Y are two spaces then $Z(X)\cong Z(Y)$ (lattice isomorphic) iff $\upsilon X\cong \upsilon Y$.*

Proof. Sufficiency. This follows from the fact, noted above, that $Z(X)\cong Z(\upsilon X)$ and $Z(Y)\cong Z(\upsilon Y)$.

Necessity. Let $t:Z(\upsilon Y)\to Z(\upsilon X)$ be an isomorphism. Then $s:Z(\upsilon X)\to Z(\upsilon Y)$ defined by $s(Z)=t^{\leftarrow}(Z)$ is also an isomorphism. By 2.5 there are continuous maps $\tau:\upsilon X\to \upsilon Y$ and $\sigma:\upsilon Y\to \upsilon X$ such that $t=\tau'$ and $s=\sigma'$ (as an isomorphism is, in particular, a σ-homomorphism). Then $\sigma'\circ\tau'=s\circ t=1_{Z(\upsilon Y)}$ (the identity map) and $\tau'\circ\sigma'=t\circ s=1_{Z(\upsilon X)}$, i.e. $\sigma'\circ\tau'=(1_{\upsilon Y})'$ and $\tau'\circ\sigma'=(1_{\upsilon X})'$ But $\sigma'\circ\tau'=(\tau\circ\sigma)'$ (as $\sigma'\circ\tau'(Z)=\sigma^{\leftarrow}(\tau^{\leftarrow}(Z))=(\tau\circ\sigma)^{\leftarrow}(Z)=(\tau\circ\sigma)'(Z)$), and $\tau'\circ\sigma'=(\sigma\circ\tau)'$. Thus $(\tau\circ\sigma)'=(1_{\upsilon Y})'$ and $(\sigma\circ\tau)'=(1_{\upsilon X})'$. By uniqueness $\tau\circ\sigma=1_{\upsilon Y}$, $\sigma\circ\tau=1_{\upsilon X}$. Thus σ and τ are homeomorphisms.

2.8. COROLLARY. *υY contains a continuous image of X iff $Z(X)$ contains a σ-homomorphic image of $Z(Y)$.*

2.2 together with 2.5 yield the following results.

2.9. COROLLARY. *$Z(X)$ contains a σ-isomorphic copy of $Z(Y)$ iff υY contains a G_δ-dense continuous image of X.*

2.10. COROLLARY. *υY contains a Z-embedded copy of X iff $Z(X)$ is a σ-homomorphic image of $Z(Y)$.*

Results 2.5 to 2.10 are the $Z(X)$-analogues of 10.6 and 10.9 in [1].

3. The continuous map induced by a homomorphism. In section 2 we were concerned only with σ-homomorphisms. We now show that any lattice homomorphism $t:Z(Y)\to Z(X)$ (such that $t(Y)=X$ and $t(\phi)=\phi$) induces a continuous map $\tau:\beta X\to\beta Y$.

3.1. DEFINITION. Let X be a space. Let

$$M^p=\{Z\in Z(X)\mid p\in cl_{\beta X}Z\}\quad\text{and}$$

$$0^p=\{Z\in Z(X)\mid cl_{\beta X}z\quad\text{is a neighborhood in } \beta X \text{ of } p\}.$$

3.2. PROPOSITION. *Let $t:Z(Y)\to Z(X)$ be a lattice homomorphism. Then there exists a continuous map $\tau:\beta X\to\beta Y$ such that $\tau^{\leftarrow}(cl_{\beta Y}Z)\supseteq cl_{\beta X}t(Z)$. For a given $Z\in Z(Y)$, the above containment is equality if $t^{\leftarrow}(M^p)=M^{\tau(p)}$ for every $p\in\tau(cl_{\beta Y}Z)$* (i.e. *is $t^{\leftarrow}(M^p)$ is an ultrafilter for every $p\in\tau^{\leftarrow}(cl_{\beta Y}Z)$*).

Proof. Let $t:Z(Y)\to Z(X)$ be as hypothesized and let $x\in\beta X$. Then $t^{\leftarrow}(M^x)$ is a prime z-filter on Y. Thus, by [1.2.11], $t^{\leftarrow}(M^x)$ is contained in a unique z-ultrafilter M^p, on Y. Let $\tau:\beta X\to\beta Y$ be defined by $\tau(x)=p$. Let $Z\in Z(Y)$. Let $x\in cl_{\beta X}t(Z)$. Then $t(Z)\in M^x$, and thus $Z\in t^{\leftarrow}(M^x)\subseteq M^{\tau(x)}$. So $\tau(x)\in cl_{\beta Y}Z$, and hence $x \in \tau^{\leftarrow}(cl_{\beta Y}Z)$, i.e. $cl_{\beta X}t(Z) \subseteq \tau^{\leftarrow}(cl_{\beta Y}Z)$. Now we show that τ is continuous.

First we show that $t(0^{\tau(x)}) \subseteq 0^x$ for every $x \in X$. Let $x \in X$. Consider $A = \{P \subseteq Z(X) \mid 0^x \subseteq P$, and P is a prime z-filter$\}$. Then $t^{\leftarrow}(P)$ is prime for every $P \in A$, and $t^{\leftarrow}(P) \subseteq t^{\leftarrow}(M^x) \subseteq M^{\tau(x)}$ (as 0^x is contained in only one z-ultrafilter, namely M^x). Thus, by [1, 7.15], $t^{\leftarrow}(P) \supseteq 0^{\tau(x)}$ for every $P \in A$. But by [1, 2.8] $0^x = \bigcap_{P \in A} P$. Therefore $t(0^{\tau(x)}) \subseteq \bigcap_{P \in A} P = 0^x$.

Let $x \in \beta X$, and let $W \in 0^{\tau(x)}$ (i.e. $cl_{\beta Y}W$ is a neighborhood of $\tau(x)$). Then $t(W) \in 0^x$. However, $cl_{\beta X}t(W) \subseteq \tau^{\leftarrow}(cl_{\beta Y}W)$, and the former set is a neighborhood of x in βX (as $t(W) \in 0^x$). Thus $\tau^{\leftarrow}(cl_{\beta Y}W)$ is a neighborhood of x. Since the closures in βY of the sets in $0^{\tau(x)}$ form a neighborhood base $\tau(x)$, this shows that τ is a continuous map.

Suppose $Z \in Z(Y)$ and $t^{\leftarrow}(M^p)$ is maximal for every $p \in \tau^{\leftarrow}(cl_{\beta Y}Z)$. If $p \in \tau^{\leftarrow}(cl_{\beta Y}Z)$, then $\tau(p) \in cl_{\beta Y}Z$ and hence $Z \in M^{\tau(p)}$. Since $t^{\leftarrow}(M^p)$ is maximal and is contained in $M^{\tau(p)}$ we must have $t^{\leftarrow}(M^p) = M^{\tau(p)}$. Thus $Z \in t^{\leftarrow}(M^p)$ and so $t(Z) \in M^p$ and $p \in cl_{\beta X}t(Z)$. Hence $cl_{\beta X}t(Z) = \tau^{\leftarrow}(cl_{\beta X}Z)$.

It is evident that distinct homomorphisms from $Z(Y)$ to $Z(X)$ may induce the same map from βX to βY. All that is required for two homomorphisms t_α and t_β from $Z(Y)$ to $Z(X)$ to induce the same map from βX to βY is that given $x \in \beta X$, there is a point $p \in \beta Y$ such that $t_\alpha^{\leftarrow}(M^x)$, $t_\beta^{\leftarrow}(M^x) \subseteq M^p$.

The continuous map τ induced by t in 3.2 in turn induces a homomorphism $t': Z(\beta Y) \to Z(\beta X)$ defined by $t'(Z) = \tau^{\leftarrow}(Z)$. Thus if $Z \in Z(\beta Y)$, then $t'(Z) \cap X = \tau^{\leftarrow}(Z) \cap X \supseteq \tau^{\leftarrow}(cl_{\beta Y}(Z \cap Y)) \cap X \supseteq t(Z \cap Y)$. So if $Z \in Z(Y)$ and $W \in Z(\beta Y)$ such that $Z = W \cap Y$ then $t'(W) \cap X \supseteq t(W \cap Y) = t(Z)$.

Thus it can be seen that if $f: X \to Y$ is continuous, then $t: Z(Y) \to Z(X)$ defined by $t(Z) = f^{\leftarrow}(Z)$ is a (σ-) homomorphism. Then by Proposition 3.2 there is a map $\tau: \beta X \to \beta Y$ such that $\tau^{\leftarrow}(cl_{\beta Y}Z) \supseteq cl_{\beta X}t(Z) = cl_{\beta X}(f^{\leftarrow}(Z))$ for any $Z \in Z(Y)$. Thus τ must agree with f on X. For if $\tau(x) \neq f(x)$, let $Z \in Z(Y)$ be such that $f(x) \in Z$ and $\tau(x) \notin cl_{\beta Y}Z$. Clearly $x \in cl_{\beta X}(f^{\leftarrow}(Z))$, hence, $x \in \tau^{\leftarrow}(cl_{\beta Y}Z)$. But then $\tau(x) \in cl_{\beta Y}Z$ contrary to assumption. Therefore τ is precisely the Stone extension of f. Since t is also a σ-homomorphism it follows that $\tau(\upsilon X) \subseteq \upsilon Y$.

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