Packing Graphs of Bounded Codegree

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Two graphs G_1 and G_2 on n vertices are said to *pack* if there exist injective mappings of their vertex sets into $[n]$ such that the images of their edge sets are disjoint. A longstanding conjecture due to Bollobás and Eldridge and, independently, Catlin, asserts that if $(\Delta(G_1)+1)(\Delta(G_2)+1) \leqslant n+1$, then G_1 and G_2 pack. We consider the validity of this assertion under the additional assumption that G_1 or G_2 has bounded codegree. In particular, we prove for all $t \geqslant 2$ that if G_1 contains no copy of the complete bipartite graph $K_{2,t}$ and $\Delta(G_1) > 17t \cdot \Delta(G_2)$, then $(\Delta(G_1)+1)(\Delta(G_2)+1) \leqslant n+1$ implies that G_1 and G_2 pack. We also provide a mild improvement if moreover G_2 contains no copy of the complete tripartite graph $K_{1,1,s}$, $s \geqslant 1$.

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1. Introduction

Let G_1 and G_2 be graphs on n vertices. (All graphs are assumed to have neither loops nor multiple edges.) We say that G_1 and G_2 *pack* if there exist injective mappings of their vertex sets into $[n] = \{1, \dots, n\}$ so that their edge sets have disjoint images. Equivalently, G_1 and G_2 pack if G_1 is a subgraph of the complement of G_2. The *maximum codegree* $\Delta^\wedge(G)$ of a graph G is the maximum over all vertex pairs of their common degree, *i.e.* $\Delta^\wedge(G) < t$ if and only if G contains no copy of the complete bipartite graph $K_{2,t}$. The *maximum adjacent codegree* $\Delta^\triangle(G)$ of G is the maximum over all pairs of *adjacent* vertices of their common degree, *i.e.* $\Delta^\triangle(G) < s$ if and only if G contains no copy of the complete tripartite graph $K_{1,1,s}$. Clearly, $\Delta^\triangle(G) \leqslant \Delta^\wedge(G)$ always. We let Δ_1 and Δ_2 denote the maximum degrees of G_1 and G_2, respectively, and $\Delta_1^\wedge$ and $\Delta_2^\triangle$ the corresponding maximum (adjacent) codegrees. We provide sufficient conditions for G_1 and G_2 to pack in terms of Δ_1, Δ_2, $\Delta_1^\wedge$, $\Delta_2^\triangle$.

For integers $t \geqslant 2$ and $\Delta_2 \geqslant 1$, we define

$$\alpha^*(t, \Delta_2) := \frac{1}{2}(2 + \gamma + \sqrt{4\gamma + \gamma^2}), \quad \text{where} \quad \gamma = \frac{\Delta_2}{\Delta_2 + 1} \cdot \frac{t-1}{t}.$$

Note $\alpha^* = \alpha^*(t, \Delta_2)$ is the larger solution to the equation $(\alpha - 1)^2 - \gamma\alpha = 0$ and

$$\frac{1}{8}(9 + \sqrt{17}) \leqslant \alpha \leqslant \frac{1}{2}(3 + \sqrt{5}).$$

Theorem 1.1. *Let G_1 and G_2 be graphs on n vertices with respective maximum degrees Δ_1 and Δ_2. Let $\Delta_1^\wedge$ be the maximum codegree of G_1. Let $t \geqslant 2$ be an integer and let $\alpha > \alpha^* = \alpha^*(t, \Delta_2)$ and $0 < \varepsilon < 1/2$ be reals. Then G_1 and G_2 pack if $\Delta_1^\wedge < t$ and n is larger than each of the following quantities:*

$$\left(t + \frac{\alpha(\alpha - 1)}{(\alpha - 1)^2 - \alpha}\right) \cdot \Delta_2 + \Delta_1\Delta_2, \tag{1.1}$$

$$(2\alpha t + 2) \cdot \Delta_2 + ((2\alpha + 1)t - 1) \cdot \Delta_2^2 + (1 - \varepsilon) \cdot \Delta_1\Delta_2, \tag{1.2}$$

$$1 + \left(2 + \frac{\varepsilon}{1 - 2\varepsilon}\right) \cdot \Delta_2 + \Delta_1\Delta_2, \quad \text{and} \tag{1.3}$$

$$\left(t + \frac{3 - \varepsilon}{2}\right) \cdot \Delta_2 + \frac{3 - \varepsilon}{2}(t - 1) \cdot \Delta_2^2 + \frac{1 + \varepsilon}{2} \cdot \Delta_1\Delta_2. \tag{1.4}$$

Theorem 1.2. *Let G_1 and G_2 be graphs on n vertices with respective maximum degrees Δ_1 and Δ_2. Let $\Delta_1^\wedge$ be the maximum codegree of G_1 and $\Delta_2^\triangle$ the maximum adjacent codegree of G_2. Let $s \geqslant 1$ and $t \geqslant 2$ be integers and let $\alpha > \alpha^* = \alpha^*(t, \Delta_2)$ be real. Then G_1 and G_2 pack if $\Delta_1^\wedge < t$, $\Delta_2^\triangle < s$, and n is larger than both of the following quantities:*

$$\left(t + \frac{\alpha(\alpha - 1)}{(\alpha - 1)^2 - \alpha}\right) \cdot \Delta_2 + \Delta_1\Delta_2 \quad \text{and} \tag{1.5}$$

$$(2 + 2\alpha t) \cdot \Delta_2 + (s - 1) \cdot \Delta_1 + ((2\alpha + 1)t - 1) \cdot \Delta_2^2. \tag{1.6}$$

For better context, we compare Theorems 1.1 and 1.2 to a line of work on graph packing that was initiated in the 1970s [2, 6, 7, 17]. The following is a central problem in the area.

Conjecture 1.3 (Bollobás and Eldridge [2] and Catlin [7]). *Let G_1 and G_2 be graphs on n vertices with respective maximum degrees Δ_1 and Δ_2. Then G_1 and G_2 pack if $(\Delta_1 + 1)(\Delta_2 + 1) \leqslant n + 1$.*

If true, the statement would be sharp and would significantly generalize a celebrated result of Hajnal and Szemerédi [12] on equitable colourings. Sauer and Spencer [17] showed that $2\Delta_1\Delta_2 < n$ is a sufficient condition for G_1 and G_2 to pack, which is seen to be sharp when one of the graphs is a perfect matching. Thus far the Bollobás–Eldridge–Catlin (BEC) conjecture has been confirmed in the following special cases: $\Delta_1 = 2$ [1]; $\Delta_1 = 3$ and n sufficiently large [10]; G_1 bipartite and n sufficiently large [9]; and G_1 d-degenerate, $\Delta_1 \geqslant 40d$ and $\Delta_2 \geqslant 215$ [5]. We would

also like to highlight the following three results that can be considered approximate forms of the BEC conjecture. (a) The condition $(\Delta_1 + 1)(\Delta_2 + 1) \leqslant 3n/5 + 1$ is sufficient for G_1 and G_2 to pack, provided that $\Delta_1, \Delta_2 \geqslant 300$ [15]. (b) The BEC condition is sufficient for G_1 and G_2 to admit a 'near packing' in that the subgraph induced by the intersection of their images has maximum degree at most 1 [11]. (c) If G_2 is chosen as a binomial random graph of parameters n and p such that np in place of Δ_2 satisfies the BEC condition, then G_1 and G_2 pack with probability tending to 1 as $n \to \infty$ [3].

Corollary 1.4. *Let $G_1, G_2, \Delta_1, \Delta_2$ and $\Delta_1^{\wedge}$ be as before. Let $t \geqslant 2$ be an integer. Then G_1 and G_2 pack if $\Delta_1 \Delta_2 + \Delta_1 \leqslant n + 1$ and $\Delta_1^{\wedge} < t$ and $\Delta_1 > 17t \cdot \Delta_2$.*

Proof. Choose $\varepsilon = (2t - 2)/(4t - 3)$ and $\alpha = 3$ in Theorem 1.1. Using that

$$\Delta_1 > 17t\Delta_2 > \frac{(4t - 3)(7t - 1)}{2t - 2} \cdot \Delta_2,$$

it follows that $\max((1.1),(1.2),(1.3),(1.4)) \leqslant (\Delta_1 + 1)(\Delta_2 + 1) - 1 \leqslant n$. So G_1 and G_2 pack. □

We have the following results concerning the BEC conjecture.

Corollary 1.5. *Given an integer $t \geqslant 2$, the BEC conjecture holds under the additional condition that the maximum codegree $\Delta_1^{\wedge}$ of G_1 is less than t and $\Delta_1 > 17t \cdot \Delta_2$.*

We were unable to avoid the linear dependence on Δ_2 in the lower bound condition on Δ_1. Although we have not seriously attempted to optimize the factor $17t$ above, Theorem 1.2 improves on this factor under the additional assumption that $\Delta_2^{\wedge}$ is bounded, as exemplified by the following corollary.

Corollary 1.6. *Given an integer $t \geqslant 2$, the BEC conjecture holds under the additional condition that the maximum codegree $\Delta_1^{\wedge}$ of G_1 is less than t, G_2 is triangle-free, and $\Delta_1 > (4 + \sqrt{5})t \cdot \Delta_2$.*

Proof. Choose

$$\alpha = \frac{1}{4t}(6t + 1 + \sqrt{20t^2 + 4t + 1})$$

and $s = 1$ in Theorem 1.2. Using that

$$t + \frac{\alpha(\alpha - 1)}{(\alpha - 1)^2 - \alpha} - 1 = (2\alpha + 1)t - 1$$

and that $\Delta_1 > (4 + \sqrt{5})t \cdot \Delta_2 > ((2\alpha + 1)t - 1) \cdot \Delta_2$, it follows that $\max((1.5),(1.6)) \leqslant (\Delta_1 + 1)(\Delta_2 + 1) - 1 \leqslant n$. So G_1 and G_2 pack. □

Structure of the paper
In the next section, we provide some notation and preliminary observations. In Section 3, we discuss the common features of a hypothetical critical counterexample to one of our theorems. In Section 4, we prove Theorems 1.1 and 1.2. We conclude the paper with some remarks about the results, proofs and further possibilities.

2. Notation and preliminaries

Here we introduce some terminology which we use throughout. We often call G_1 the *blue* graph and G_2 the *red* graph. We treat the injective vertex mappings as labellings of the vertices from 1 to n. However, rather than saying, 'the vertex in G_1 (or G_2) corresponding to the label i', we often only say, 'vertex i', since this should never cause any confusion. Our proofs rely on accurately specifying the neighbourhood structure as viewed from a particular vertex. Let $i \in [n]$. The *blue neighbourhood $N_1(i)$ of i* is the set $\{j \mid ij \in E(G_1)\}$ and the *blue degree* $\deg_1(i)$ *of i* is $|N_1(i)|$. The *red neighbourhood $N_2(i)$* and *red degree $N_2(i)$* are defined analogously. For $j \in [n]$, a *red–blue link (or 2–1 link) from i to j* is a vertex i' such that $ii' \in E(G_2)$ and $i'j \in E(G_1)$. The *red–blue neighbourhood $N_1(N_2(i))$ of i* is the set $\{j \mid \exists \text{ red–blue link from } i \text{ to } j\}$. A *blue–red link (or 1–2 link)* and the *blue–red neighbourhood $N_2(N_1(i))$* are defined analogously.

In search of a certificate that G_1 and G_2 pack, without loss of generality, we keep the vertex labelling of the blue graph G_1 fixed, and permute only the labels in the red graph G_2. This can be thought of as 'moving' the red graph above a fixed ground set $[n]$. In particular, we seek to avoid the situation that there are $i, j \in [n]$ for which ij is an edge in both G_1 and G_2 – in this situation, we call ij a *purple* edge induced by the labellings of G_1 and G_2. So G_1 and G_2 pack if and only if they admit a pair of vertex labellings that induces no purple edge. In our search, we make small cyclic sub-permutations of the labels (of G_2), which are referred to as follows. For $i_0, \dots, i_{\ell-1} \in [n]$, a $(i_0, \dots, i_{\ell-1})$-*swap* is a relabelling of G_2 so that for each $k \in \{0, \dots, \ell-1\}$ the vertex labelled i_k is re-assigned the label $i_{k+1 \bmod \ell}$. In fact, we shall only require swaps having $\ell \in \{1, 2\}$. The following observation describes when a swap could be helpful in the search for a packing certificate. This is identical to Lemma 1 in [15].

Lemma 2.1. *Let $u_0, \dots, u_{\ell-1} \in [n]$. For every $k, k' \in \{0, \dots, \ell-1\}$, suppose that there is no red–blue link from u_k to $u_{k+1 \bmod \ell}$ and that if $u_k u_{k'} \in E(G_2)$, then $u_{(k+1 \bmod \ell)} u_{(k'+1 \bmod \ell)} \notin E(G_1)$. Then there is no purple edge incident to any of $u_0, \dots, u_{\ell-1}$ after a $(u_0, \dots, u_{\ell-1})$-swap.*

We will use a classic extremal set theoretic result to upper-bound the size of certain vertex subsets.

Lemma 2.2 (Corrádi [8]). *Let $A_1, \dots, A_N$ be k-element sets and X be their union. If $|A_i \cap A_j| \leqslant t-1$ for all $i \neq j$, then $|X| \geqslant k^2 N / (k + (N-1)(t-1))$.*

In particular, this implies the following.

Corollary 2.3. *Let $A_1, \dots, A_N$ be size $\geqslant k$ subsets of a set X. If $k^2 > (t-1) \cdot |X|$ and $|A_i \cap A_j| \leqslant t-1$ for all $i \neq j$, then*

$$N \leqslant |X| \cdot \frac{k - (t-1)}{k^2 - (t-1) \cdot |X|}.$$

Proof. Consider arbitrary subsets $A_1^* \subset A_1, \dots, A_N^* \subset A_N$ of size k. An application of Corrádi's lemma to $A_1^*, \dots, A_N^*$ yields that $|X| \geqslant k^2 \cdot N / (k + (N-1)(t-1))$, which is easily seen to be equivalent to $(k^2 - (t-1) \cdot |X|) \cdot N \leqslant (k-t+1) \cdot |X|$. The corollary follows after dividing both

sides of the inequality by $k^2 - (t-1) \cdot |X|$. Note that this division does not cause a sign change because of the assumption that $k^2 > (t-1) \cdot |X|$. $\qquad\square$

3. Hypothetical critical counterexamples

The overall proof structure we use for both theorems is the same, and in this section we describe common features and some further notation. Suppose the theorem (one of Theorem 1.1 or 1.2) is false. Then there must exist a counterexample, that is, a pair (G_1, G_2) of non-packable graphs on n vertices that satisfy the conditions of the theorem.

Moreover, we may assume that (G_1, G_2) is a *critical* pair in the sense that G_2 is edge-minimal among all counterexamples. In other words, G_1 and $G_2 - e$ pack for *any* $e \in E(G_2)$. There is no loss of generality, since the removal of an edge from G_2 increases neither Δ_2 nor $\Delta_2^\triangle$ and obviously affects none of Δ_1, $\Delta_1^\wedge$ and n, thus maintaining the required conditions.

Now choose *any* edge $e = uv \in E(G_2)$. Criticality implies that there is a pair of labellings of G_1 and G_2 such that e is the *unique* purple edge, for otherwise G_1 and $G_2 - e$ do not pack. Let us fix such a pair of labellings so that we can further describe the neighbourhood structure as viewed from u (or v). Estimation of the sizes of subsets in this neighbourhood structure is our main method for deriving upper bounds on n that in turn yield the desired contradiction from which the theorem follows.

We need the definition of the following vertex subsets (which are analogously defined for v also):

$$
\begin{aligned}
A(u) &:= N_2(N_1(u)) \setminus (N_1(u) \cup N_2(u) \cup N_1(N_2(u))), \\
B(u) &:= N_1(N_2(u)) \setminus (N_1(u) \cup N_2(u) \cup N_2(N_1(u))), \\
A^*(u) &:= N_2(N_1(u)) \setminus (N_2(u) \cup N_1(N_2(u))), \quad \text{and} \\
N_1^*(u) &:= N_1(u) \cap (N_1(N_2(u)) \setminus (N_2(u) \cup N_2(N_1(u)))).
\end{aligned}
$$

One justification for specifying the above subsets is that the following two claims (which are essentially Claims 1 and 2 in [15]) hold.

Claim 3.1. *For all $w \in [n] \setminus \{v\}$, there is a red–blue link or a blue–red link from u to w.*

Proof. If not, then by Lemma 2.1, a (u,w)-swap yields a new labelling such that uv is no longer purple and no new purple edges are created. Thus G_1 and G_2 pack, a contradiction. See Figure 1. $\qquad\square$

Claim 3.2. *For all $a \in A^*(u)$ and $b \in B(u)$, there is a red–blue link from a to b.*

Proof. Since $B(u) \cap N_1(u) = B(u) \cap N_2(u) = \emptyset$ and $A^*(u) \cap N_2(u) = \emptyset$, we have that $bu \notin E(G_1) \cup E(G_2)$ and $ua \notin E(G_2)$. Furthermore, since $A^*(u) \cap N_1(N_2(u)) = B(u) \cap N_2(N_1(u)) = \emptyset$, there is no red–blue link from u to a or from b to u. Now suppose that there is also no red–blue link from a to b. Then it follows from Lemma 2.1 that after a (u,a,b)-swap there is no purple edge incident to any of u, a, b, which implies that there is no purple edge at all. So we have obtained a packing of G_1 and G_2, a contradiction. $\qquad\square$
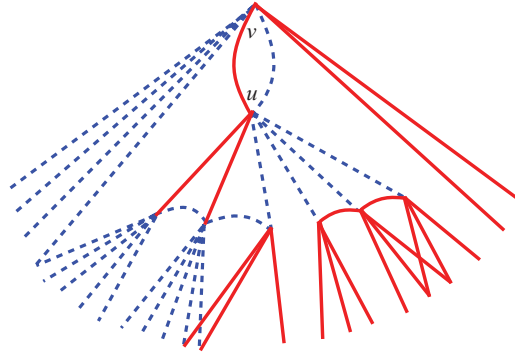
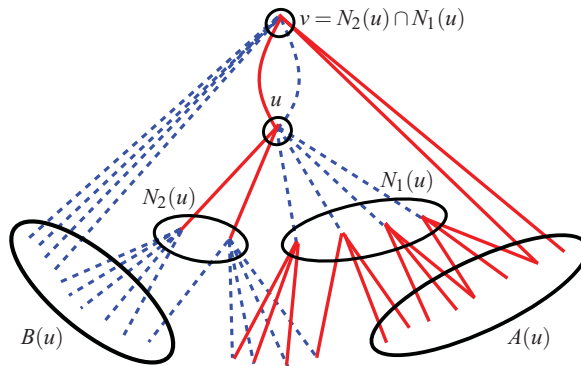*Figure 1.* All vertices (except possibly v) are reachable by a link from u (Claim 3.1).

Figure 2. The neighbourhood structure of a hypothetical critical counterexample, as seen from u.

See Figure 2

In the next claim, we list three upper bounds on the total number n of vertices in terms of the sizes of the vertex subsets defined above. In the proofs of Theorems 1.1 and 1.2, we consider several cases for which we prove at least one of these upper bounds to be small enough for a contradiction with the assumed lower bounds on n.

Claim 3.3. *The total number n of vertices is at most each of the following quantities:*

(i) $|N_2(u)| + |A^*(u)| + |N_1(N_2(u))|$,

(ii) $|N_1^*(u)| + |N_2(u)| + |B(u)| + |N_2(N_1(u))|$,

(iii) $|A^*(v)| + |A^*(u)| + |(N_2(u) \cup N_1(N_2(u))) \cap (N_2(v) \cup N_1(N_2(v)))|$.

Proof. In all cases, $[n]$ equals the union of the neighbourhood sets that occur in the upper bound.

(i) The union of $N_2(u)$, $A^*(u)$ and $N_1(N_2(u))$ covers $\{v\} \cup N_2(N_1(u)) \cup N_1(N_2(u))$, which by Claim 3.1 equals $[n]$.

(ii) The union of $N_1^*(u)$, $N_2(u)$, $B(u)$ and $N_2(N_1(u))$ covers $\{v\} \cup N_2(N_1(u)) \cup N_1(N_2(u))$, which equals $[n]$.

(iii) By the proof of (i), $[n]$ is the union of $A^*(u)$ and $N_2(u) \cup N_1(N_2(u))$ as well as the union of $A^*(v)$ and $N_2(v) \cup N_1(N_2(v))$. It follows that $[n]$ also is the union of $A^*(u)$, $A^*(v)$ and $(N_2(u) \cup N_1(N_2(u))) \cap (N_2(v) \cup N_1(N_2(v)))$. $\square$

The reason for working with $N_1^*(u)$ and $A^*(u)$ rather than the simpler sets $N_1(u)$ and $A(u)$ is the following. Under the requirement that the codegree $\Delta_1^\wedge$ of G_1 is less than t, we can upper-bound $|N_1^*(u)|$ entirely in terms of Δ_2. This is sharper than the trivial bound $|N_1(u)| \leqslant \Delta_1$ because we work under conditions with Δ_1 rather larger than Δ_2. Similarly, since $N_1^*(u) \subset N_1(u)$, we need to compensate for the loss of covered vertices by working with the slightly enlarged set $A^*(u)$, rather than $A(u)$. The following claims use the condition $\Delta_1^\wedge < t$ (which is assumed by both theorems).

Claim 3.4. $|N_1^*(u)| \leqslant (t-1) \cdot \Delta_2$.

Proof. Suppose $|N_1(u) \cap N_1(N_2(u))| \geqslant (t-1) \cdot \Delta_2 + 1$; then there is at least one $x \in N_2(u)$ such that

$$|N_1(u) \cap N_1(x)| \geqslant \frac{1}{|N_2(u)|} \cdot ((t-1) \cdot \Delta_2 + 1) > t-1,$$

which contradicts $\Delta_1^\wedge < t$. $\square$

The following claim (in combination with Corrádi's lemma) is useful for an upper bound on $|B(u)|$ that is only linear in Δ_2, provided that $|A^*(u)|$ is at least quadratic in Δ_2. See case (i) in the proof of Theorem 1.1.

Claim 3.5. *For any $b \in B(u)$, $|N_1(b) \cap A^*(u)| \geqslant |A^*(u)|/\Delta_2 - t(\Delta_2 + 1)$.*

Proof. For all $b \in N_1(N_2(u))$ it holds that

$$|N_1(b) \cap N_1(N_2(u))| \leqslant (t-1) \cdot |N_2(u)| \leqslant (t-1) \cdot \Delta_2.$$

Indeed, otherwise there would exist a blue copy of $K_{2,t}$ in the graph induced by $N_1(N_2(u)) \cup N_2(u)$. Similarly, $|N_1(b) \cap N_1(u)| \leqslant t$ and $|N_1(b) \cap N_2(u)| \leqslant \Delta_2$. So for every $b \in N_1(N_2(u))$, at most $t \cdot (\Delta_2 + 1)$ blue neighbours of b are in $[n] \setminus A(u)$. So in particular, for every $b \in B(u)$, at most $t \cdot (\Delta_2 + 1)$ blue neighbours of b are in $[n] \setminus A^*(u)$.

Using Claim 3.2 and the fact that each blue neighbour of a fixed $b \in B(u)$ has at most Δ_2 red neighbours in $A^*(u)$, we see that every $b \in B(u)$ has at least $\lceil |A^*(u)|/\Delta_2 \rceil$ blue neighbours, and thus at least $|A^*(u)|/\Delta_2 - t(\Delta_2 + 1)$ blue neighbours in $A^*(u)$. $\square$

4. Proofs

4.1. Proof of Theorem 1.1

Suppose the theorem is false. Consider a critical counterexample, a pair of non-packable graphs (G_1, G_2), with G_2 edge-minimal, satisfying the constraints of the theorem. We distinguish three cases, for each of which we derive an upper bound on n, given by one of the inequalities (4.2), (4.4) and (4.10). At least one of these three inequalities should hold, so together they contradict
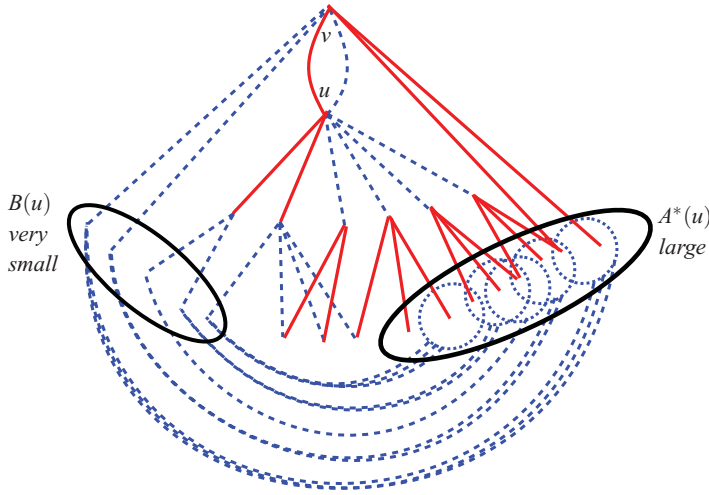
Figure 3. A depiction of case (i) of Theorem 1.1, that $|A^*(u)| = \Omega(\Delta_2^2)$ implies $|B(u)| = O(\Delta_2)$.

the condition that

$$\max((4.2),(4.4),(4.10)) = \max((1.1),(1.2),(1.3),(1.4)) < n,$$

thus proving the theorem.

(i) There exists a vertex $u \in [n]$ and there are labellings of G_1 and G_2 such that u is incident to the unique purple edge and $|A^*(u)| \geqslant \alpha t \cdot \Delta_2(\Delta_2 + 1)$.

(ii) Case (i) does not hold and furthermore $|N_2(u) \cap N_2(v)| < (1 - \varepsilon) \cdot \Delta_2$ for some edge $uv \in E(G_2)$.

(iii) Case (i) does not hold and $|N_2(u) \cap N_2(v)| \geqslant (1 - \varepsilon) \cdot \Delta_2$ for every $uv \in E(G_2)$.

We now proceed with deriving upper bounds on n for each of these three cases.

Bound for case (i). Choose a vertex $u \in [n]$ and labellings of G_1 and G_2 such that u is incident to the unique purple edge and $|A^*(u)| \geqslant \alpha t \cdot \Delta_2(\Delta_2 + 1)$. See Figure 3 for a depiction of the argumentation in this case. From now on, we write $k := |A^*(u)|/\Delta_2 - t(\Delta_2 + 1)$. Our first tool is Claim 3.5, which yields that all $b \in B(u)$ satisfy $|N_1(b) \cap A^*(u)| \geqslant k$. Note that $k \geqslant 1$, since $\alpha > 1$. Our second tool is Corrádi's lemma, or rather Corollary 2.3, which we apply with $X = A^*(u)$ and $N = |B(u)|$ and with size $\geqslant k$ subsets $A_1, \ldots, A_N \subset X$ given by $N_1(b) \cap A^*(u)$, for all $b \in B(u)$. Note that $|A_i \cap A_j| \leqslant t - 1$ for all $i \neq j$, or else there would be a blue copy of $K_{2,t}$.

In order to apply Corollary 2.3, we need to check that its condition $k^2 > (t - 1) \cdot |A^*(u)|$ holds. For that, we write $\beta := |A^*(u)|/(t\Delta_2(\Delta_2 + 1))$, so that $k = (\beta - 1)t(\Delta_2 + 1)$. Now

$$k^2 - (t-1) \cdot |A^*(u)| = ((\beta - 1)t(\Delta_2 + 1))^2 - \beta t \Delta_2(\Delta_2 + 1)(t - 1)$$
$$= ((\beta - 1)^2 - \gamma \cdot \beta) \cdot (t(\Delta_2 + 1))^2,$$

which is positive if and only if $(\beta - 1)^2 - \gamma \beta > 0$, which holds true because $\beta \geqslant \alpha > \alpha^*$. Thus, by Corollary 2.3, we obtain

$$|B(u)| \leqslant |A^*(u)| \cdot \frac{k - (t-1)}{k^2 - (t-1) \cdot |A^*(u)|} = \frac{1 - (t-1)/k}{k/|A^*(u)| - (t-1)/k}.$$

The numerator and denominator of the right-hand side are both positive, so we can bound and rearrange as follows:

$$|B(u)| \leqslant \left(\frac{k}{|A^*(u)|} - \frac{t-1}{k} \right)^{-1} = \left(\frac{(\beta-1)t(\Delta_2+1)}{\beta t \Delta_2(\Delta_2+1)} - \frac{t-1}{(\beta-1)t(\Delta_2+1)} \right)^{-1}$$

$$= \Delta_2 \cdot \left(\frac{\beta-1}{\beta} - \frac{1}{\beta-1} \cdot \frac{\Delta_2}{\Delta_2+1} \cdot \frac{t-1}{t} \right)^{-1} = \Delta_2 \cdot \left(\frac{\beta-1}{\beta} - \frac{\gamma}{\beta-1} \right)^{-1}$$

$$\leqslant \Delta_2 \cdot \frac{\alpha(\alpha-1)}{(\alpha-1)^2 - \gamma\alpha}, \tag{4.1}$$

where the last step holds because $\beta \geqslant \alpha > \alpha^*$ and α^* is the larger singular point of

$$\frac{\beta(\beta-1)}{(\beta-1)^2 - \gamma\beta},$$

which is a decreasing function of β for all $\beta > \alpha^*$.

Evaluating (4.1) and Claim 3.4 in the upper bound of Claim 3.3(ii) yields

$$n \leqslant |N_1^*(u)| + |N_2(u)| + |B(u)| + |N_2(N_1(u))|$$

$$\leqslant (t-1) \cdot \Delta_2 + \Delta_2 + \frac{\alpha(\alpha-1)}{(\alpha-1)^2 - \alpha} \cdot \Delta_2 + \Delta_1 \Delta_2$$

$$= \left(t + \frac{\alpha(\alpha-1)}{(\alpha-1)^2 - \alpha} \right) \cdot \Delta_2 + \Delta_1 \Delta_2. \tag{4.2}$$

Bound for case (ii). Choose labellings of G_1 and G_2 such that there is a unique purple edge uv that satisfies $|N_2(u) \cap N_2(v)| < (1-\varepsilon) \cdot \Delta_2$. Note that the inequalities $|A^*(u)| < \alpha t \cdot \Delta_2(\Delta_2+1)$ and $|A^*(v)| < \alpha t \cdot \Delta_2(\Delta_2+1)$ are satisfied as well, as a direct consequence of the assumptions of case (ii).

We proceed with deriving a technical estimate on an intersection of neighbourhood sets. For each $x \in N_2(u) \setminus N_2(v)$ and $y \in N_2(v) \setminus N_2(u)$ we have $x \neq y$ and therefore absence of blue copies of $K_{2,t}$ implies the inequality $|N_1(x) \cap N_1(y)| \leqslant t-1$. So

$$|N_1(N_2(u) \setminus N_2(v)) \cap N_1(N_2(v) \setminus N_2(u))| \leqslant \sum_{x \in N_2(u) \setminus N_2(v)} \sum_{y \in N_2(v) \setminus N_2(u)} |N_1(x) \cap N_1(y)|$$

$$\leqslant |N_2(u) \setminus N_2(v)| \cdot |N_2(v) \setminus N_2(u)| \cdot (t-1)$$

$$\leqslant (\Delta_2 - |N_2(u) \cap N_2(v)|)^2 \cdot (t-1).$$

Furthermore, since $|N_2(u) \cap N_2(v)| < (1-\varepsilon) \cdot \Delta_2$,

$$|N_1(N_2(u)) \cap N_1(N_2(v))| \leqslant |N_1(N_2(u) \cap N_2(v))| + |N_1(N_2(u) \setminus N_2(v)) \cap N_1(N_2(v) \setminus N_2(u))|$$

$$< \Delta_1 \cdot |N_2(u) \cap N_2(v)| + (\Delta_2 - |N_2(u) \cap N_2(v)|)^2 \cdot (t-1)$$

$$\leqslant \max_{p \in \{0,1,2,\dots,\lfloor(1-\varepsilon)\cdot\Delta_2\rfloor\}} (\Delta_1 \cdot p + (\Delta_2 - p)^2 \cdot (t-1)).$$
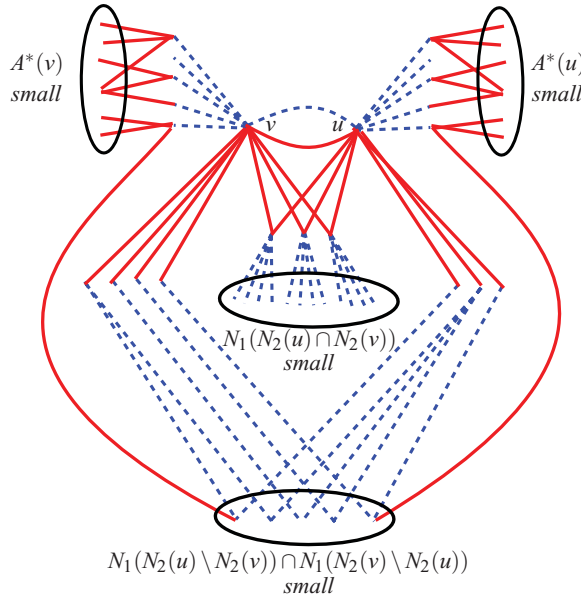
Figure 4. A depiction of case (ii) of Theorem 1.1, that $|N_1(N_2(u)) \cap N_1(N_2(v))|$ is small.

See Figure 4. Finally, we evaluate this in Claim 3.3(iii) to find the following bound on n:

$$
\begin{aligned}
n &\leqslant |A^*(v)| + |A^*(u)| + |(N_2(u) \cup N_1(N_2(u))) \cap (N_2(v) \cup N_1(N_2(v)))| \\
&\leqslant |A^*(v)| + |A^*(u)| + |N_2(u)| + |N_2(v)| + |N_1(N_2(u)) \cap N_1(N_2(v))| \\
&\leqslant 2\alpha t \cdot \Delta_2(\Delta_2 + 1) + 2\Delta_2 + \max_{p \in \{0,1,2,\dots,\lfloor (1-\varepsilon) \cdot \Delta_2 \rfloor\}} (\Delta_1 \cdot p + (\Delta_2 - p)^2 \cdot (t-1)).
\end{aligned}
\tag{4.3}
$$

In particular, this implies the slightly rougher bound

$$
n \leqslant 2\alpha t \cdot \Delta_2(\Delta_2 + 1) + 2\Delta_2 + (1-\varepsilon) \cdot \Delta_1 \Delta_2 + \Delta_2^2 \cdot (t-1).
\tag{4.4}
$$

Bound for case (iii). Choose a pair of labellings of G_1 and G_2 that induces a unique purple edge uv. The assumptions of this case imply, in particular, that in the red graph the neighbourhoods of each pair of adjacent vertices overlap significantly: $|N_2(x) \cap N_2(y)| \geqslant (1-\varepsilon) \cdot \Delta_2$ for each $xy \in E(G_2)$.

We will derive two consequences, namely the implication

$$
\left(|A^*(u)| \geqslant 1 + \Delta_2 + \frac{\varepsilon \cdot \Delta_2}{1 - 2\varepsilon} \right) \implies (|B(u)| \leqslant (t-1) \cdot \Delta_2^2)
\tag{4.5}
$$

and the inequality

$$
|N_2(N_1(u))| \leqslant \frac{1+\varepsilon}{2} \Delta_1 \Delta_2 + \frac{1-\varepsilon}{2}(t-1) \cdot \Delta_2^2 + \frac{3}{2}\Delta_2.
\tag{4.6}
$$

We start by proving the statement (4.5), the first consequence. See Figure 5. Suppose $a \in A^*(u) \setminus N_2(u)$ has a red neighbour $x \in N_2(u)$. Then ux and ax are edges of G_2, so $|N_2(a) \cap N_2(x)| \geqslant (1-\varepsilon)\Delta_2$ and $|N_2(u) \cap N_2(x)| \geqslant (1-\varepsilon)\Delta_2$. Combining this with the obvious fact that $|N_2(x)| \leqslant \Delta_2$
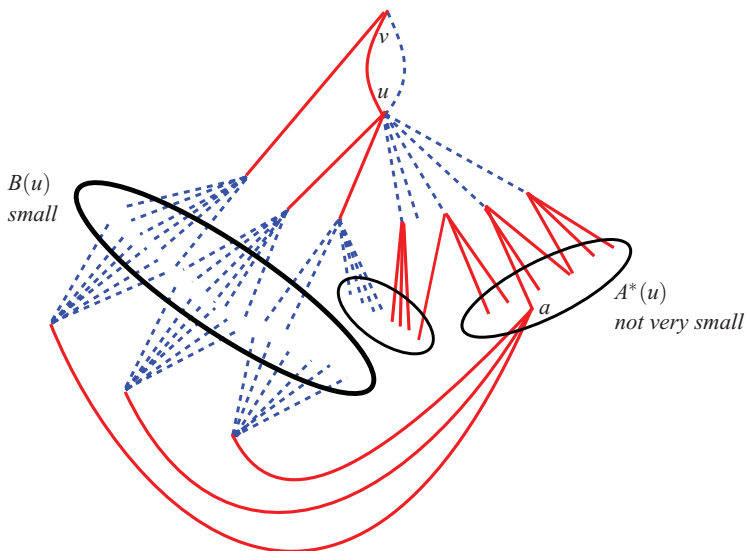
Figure 5. A depiction of (4.5) in case (iii) of Theorem 1.1.

yields that

$$|N_2(a) \cap N_2(u)| \geqslant (1 - 2\varepsilon) \cdot \Delta_2. \tag{4.7}$$

Let us define

$$A^{**}(u) := \{a \in A^*(u) \mid a \text{ has a red neighbour in } N_2(u)\}.$$

It follows from (4.7) that

$$\sum_{a \in A^{**}(u)} |N_2(a) \cap N_2(u)| \geqslant |A^{**}(u)| \cdot (1 - 2\varepsilon) \cdot \Delta_2,$$

so

$$\sum_{x \in N_2(u)} |N_2(x)| \geqslant \sum_{x \in N_2(u)} |N_2(x) \cap N_2(u)| + \sum_{a \in A^{**}(u)} |N_2(a) \cap N_2(u)|$$

$$\geqslant (1 - \varepsilon)\Delta_2 \cdot |N_2(u)| + |A^{**}(u)| \cdot (1 - 2\varepsilon) \cdot \Delta_2,$$

and (crucially) since $\sum_{x \in N_2(u)} |N_2(x)| \leqslant \Delta_2 \cdot |N_2(u)|$, it follows that

$$|A^{**}(u)| \leqslant \frac{|N_2(u)| \cdot \Delta_2 - (1 - \varepsilon) \cdot \Delta_2 |N_2(u)|}{(1 - 2\varepsilon) \cdot \Delta_2} = \frac{\varepsilon \cdot |N_2(u)|}{1 - 2\varepsilon}. \tag{4.8}$$

Next, suppose we would have that $|A^*(u)| \geqslant 1 + |N_2(u)| + |A^{**}(u)|$. Then there exists a vertex $a \in A^*(u) \setminus A^{**}(u)$. By the definition of $A^{**}(u)$, this vertex satisfies $N_2(a) \cap N_2(u) = \emptyset$. Furthermore, since $a \in A^*(u)$, we have that for all $b \in B(u)$ there is a red–blue link from a to b. In other words, $B(u) = N_1(N_2(a)) \cap B(u)$. This implies that

$$|B(u)| = |N_1(N_2(a)) \cap B(u)| \leqslant |N_1(N_2(a)) \cap N_1(N_2(u))| \leqslant (t - 1) \cdot \Delta_2^2,$$

where the last inequality is a consequence of the facts that $N_2(a) \cap N_2(u) = \emptyset$ and G_1 does not contain a copy of $K_{2,t}$. In summary, we have shown the implication

$$|A^*(u)| \geqslant 1 + |N_2(u)| + |A^{**}(u)| \implies |B(u)| \leqslant (t-1) \cdot \Delta_2^2. \tag{4.9}$$

Combining (4.8) and (4.9) yields our first desired main consequence (4.5).

We now prove inequality (4.6), the second consequence. See Figure 6. First, the absence of blue copies of $K_{2,t}$ implies that for every $x \in N_2(u)$ we have $|N_1(x) \cap N_1(u)| \leqslant t - 1$. Therefore

$$|N_1(u) \cap N_1(N_2(u))| \leqslant |N_2(u)| \cdot \max_{x \in N_2(u)} (|N_1(x) \cap N_1(u)|) \leqslant \Delta_2 \cdot (t-1).$$

In other words, there is a red–blue link from u to y for at most $\Delta_2 \cdot (t-1)$ vertices $y \in N_1(u)$. Recalling that there is a link from u to every vertex (possibly with the exception of v), it follows that there are at least $h := |N_1(u)| - (t-1)\Delta_2 - 1$ vertices $y \in N_1(u)$ for which there is a blue–red link (and no red–blue link) from u to y. In other words, $m := |N_1(u) \cap N_2(N_1(u))| \geqslant h$. It follows from the definition of blue–red link that any $y_1 \in N_1(u) \cap N_2(N_1(u))$ is connected to at least one other vertex $y_2 \in N_1(u) \cap N_2(N_1(u))$ by a *red edge*.

This means that $N_1(u) \cap N_2(N_1(u))$ can be covered by a collection of vertex-disjoint red stars $S_1, S_2, \ldots$ that each have at least two vertices (unless $m \in \{0, 1\}$, in which case inequality (4.6) is clearly satisfied). Let S be one such star, with central vertex y^* and leaves $y_1, y_2 \ldots, y_{|S|-1}$. Each of its edges has a large common red neighbourhood: for all $j \in \{1, 2, \ldots, |S| - 1\}$ it holds that $|N_2(y^*) \cap N_2(y_j)| \geqslant (1 - \varepsilon) \cdot \Delta_2$. Therefore

$$\left| \bigcup_{y \in S} N_2(y) \right| \leqslant |N_2(y^*)| + \sum_{y \in S \setminus \{y^*\}} |N_2(y) \setminus N_2(y^*)| \leqslant (1 + \varepsilon \cdot (|S| - 1)) \cdot \Delta_2,$$

which is at most $(1 + \varepsilon)/2 \cdot |S| \cdot \Delta_2$. So

$$|N_2(N_1(u) \cap N_2(N_1(u)))| = \left| \bigcup_i \bigcup_{y \in S_i} N_2(y) \right| \leqslant \sum_i \left| \bigcup_{y \in S_i} N_2(y) \right|$$

$$\leqslant \sum_i \frac{1 + \varepsilon}{2} \cdot |S_i| \cdot \Delta_2 = \frac{m}{2} \cdot (1 + \varepsilon) \cdot \Delta_2.$$

Last, note that

$$|N_1(u) \cap (N_1(N_2(u)) \setminus N_2(N_1(u)))| = |N_1(u)| - m - \mathbb{1}_{\{\nexists \text{ link from } u \text{ to } v\}} \leqslant |N_1(u)| - m.$$

We are now ready to derive (4.6):

$$|N_2(N_1(u))| \leqslant |N_2(N_1(u) \cap N_2(N_1(u)))| + |N_2(N_1(u) \cap (N_1(N_2(u)) \setminus N_2(N_1(u))))| + |N_2(v)|$$

$$\leqslant \frac{m}{2} \cdot (1 + \varepsilon) \cdot \Delta_2 + (|N_1(u)| - m) \cdot \Delta_2 + \Delta_2 =: g(m).$$
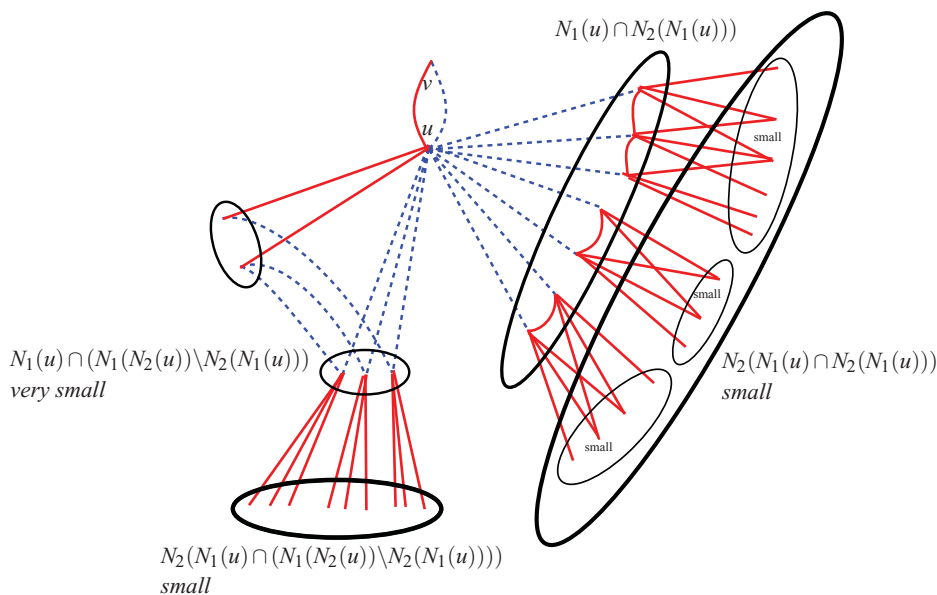
Figure 6. A depiction of (4.6) in case (iii) of Theorem 1.1.

Since $\Delta_2 \geqslant 0$ and $\varepsilon < 1/2$, the function $g(x)$ is non-increasing on the whole of $\mathbb{R}$. Since $h \leqslant m$, it follows that $g(m) \leqslant g(h)$. So

$$
\begin{aligned}
|N_2(N_1(u))| &\leqslant g(|N_1(u)| - (t-1)\Delta_2 - 1) \\
&= \frac{1+\varepsilon}{2} \cdot (|N_1(u)| - (t-1)\cdot\Delta_2 - 1) \cdot \Delta_2 + (t-1)\cdot\Delta_2^2 + 2\Delta_2 \\
&\leqslant \frac{1+\varepsilon}{2}\cdot\Delta_1\Delta_2 + \frac{1-\varepsilon}{2}\cdot(t-1)\cdot\Delta_2^2 + \frac{3-\varepsilon}{2}\cdot\Delta_2,
\end{aligned}
$$

as desired.

Finally, we evaluate (4.5) and (4.6) in the bounds on n given by Claim 3.3, parts (i) and (ii), to obtain

$$
\begin{aligned}
n &\leqslant \min(|N_1(N_2(u))| + |A^*(u)| + |N_2(u)|, \; |N_2(N_1(u))| + |N_2(u)| + |N_1^*(u)| + |B(u)|) \\
&\leqslant \min\left(\Delta_1\Delta_2 + \Delta_2 + |A^*(u)|, \; \frac{1+\varepsilon}{2}\Delta_1\Delta_2 + \frac{1-\varepsilon}{2}(t-1)\Delta_2^2 + \left(t + \frac{3-\varepsilon}{2}\right)\cdot\Delta_2 + |B(u)|\right) \\
&= \Delta_1\Delta_2 + \Delta_2 + \min\left(|A^*(u)|, \; |B(u)| + \left(t + \frac{1-\varepsilon}{2}\right)\cdot\Delta_2 - \frac{1-\varepsilon}{2}(\Delta_1\Delta_2 - (t-1)\Delta_2^2)\right) \\
&\leqslant \Delta_1\Delta_2 + \Delta_2 + \max\left(1 + \Delta_2 + \frac{\varepsilon\Delta_2}{1-2\varepsilon}, \; \frac{3-\varepsilon}{2}(t-1)\cdot\Delta_2^2 - \frac{1-\varepsilon}{2}\Delta_1\Delta_2 + \left(t + \frac{1-\varepsilon}{2}\right)\cdot\Delta_2\right),
\end{aligned}
$$
(4.10)

where we employed Claim 3.3 in the first line, Claim 3.4 and inequality (4.6) in the second line and implication (4.5) in the last line.

4.2. Proof of Theorem 1.2

Suppose the theorem is false. Consider a critical counterexample, a pair of non-packable graphs (G_1, G_2) satisfying the constraints of the theorem, such that there is a near-packing with a unique purple edge uv. We distinguish two cases, (i) and (ii). From the first we derive the inequality (4.11) and from the second we obtain the inequality (4.12). Together they contradict the condition that $\max((1.5), (1.6)) < n$, thus proving the theorem.

(i) $|A^*(u)| \geqslant \alpha t \cdot \Delta_2(\Delta_2 + 1)$ or $|A^*(v)| \geqslant \alpha t \cdot \Delta_2(\Delta_2 + 1)$.

Without loss of generality, we assume $|A^*(u)| \geqslant \alpha t \cdot \Delta_2(\Delta_2 + 1)$. From here on the proof is the same as for case (i) in the proof of Theorem 1.1, leading to the same bound,

$$n \leqslant \left(t + \frac{\alpha(\alpha - 1)}{(\alpha - 1)^2 - \alpha} \right) \cdot \Delta_2 + \Delta_1 \Delta_2. \tag{4.11}$$

(ii) Case (i) does not hold.

From here on we proceed almost exactly as for case (ii) in the proof of Theorem 1.1, the difference being that instead of the upper bound $|N_2(u) \cap N_2(v)| < (1 - \varepsilon) \cdot \Delta_2$ we use $|N_2(u) \cap N_2(v)| < s$, which holds due to the additional condition $\Delta_2^{\triangle} < s$. (Compare with (4.4).) It follows that

$$n \leqslant 2\alpha t \cdot \Delta_2(\Delta_2 + 1) + 2\Delta_2 + \Delta_1 \cdot (s - 1) + \Delta_2^2 \cdot (t - 1). \tag{4.12}$$

4.3. Concluding remarks

We wish to make the following remarks about Theorems 1.1 and 1.2.

- In Theorem 1.1, the bottleneck is the quantity (1.2), which corresponds to the bound (4.4) of case (ii). So improving in this case would improve the overall bound on n, albeit not by much.
- The condition in Theorem 1.2 that $\Delta_2^{\triangle} < s$ is equivalent to '$|N_2(x) \cap N_2(y)| < s$ for all $xy \in E(G_2)$'. With a little adaptation, we can replace this with the weaker but perhaps obscure condition that G_2 has *no* subgraph $G_2^!$ such that $|N_2(x) \cap N_2(y)| \geqslant s$ for *all* $xy \in E(G_2^!)$. Indeed, this property is invariant under edge removal, and so holds for an edge-minimal critical counterexample, which therefore has an edge uv with $|N(u) \cap N(v)| < s$, for which we can choose labellings such that uv is the unique purple edge. From here on, one again proceeds exactly as in case (ii) of the proof of Theorem 1.1.
- Theorem 1.2 yields a better bound than Theorem 1.1 only if Δ_1 is much larger than Δ_2 and s, t are both small.
- By taking G_2 to be a collection of (nearly) equal-sized cliques, Corollary 1.4 implies that if G is a $K_{2,t}$-free graph of maximum degree Δ with $\Delta \geqslant \sqrt{17t} \cdot \sqrt{n}$, then the equitable chromatic number of G is at most Δ. Note that this result cannot be obtained by the result of Hajnal and Szemerédi on equitable colourings [12].

The BEC conjecture notwithstanding, naturally one might wonder whether Theorem 1.1, or rather Corollary 1.5, could be improved according to a weaker form of the BEC condition, as was the case for d-degenerate G_1 [5]. In other words, it would be interesting to improve upon the $\Omega(\Delta_1 \Delta_2)$ terms appearing in each of (1.1)–(1.4). We leave this to further study, but point out the

following constructions where G_1 has low maximum codegree, which mark boundaries for this problem.

- When n is even, there are non-packable pairs (G_1, G_2) of graphs where G_1 is a perfect matching (so $\Delta_1^{\wedge} = 0$) and $2\Delta_1\Delta_2 = n$: see [14].
- Bollobás, Kostochka and Nakprasit [4] exhibited a family of non-packable pairs (G_1, G_2) of graphs where G_1 is a forest (so $\Delta_1^{\wedge} = 1$) and $\Delta_1 \ln \Delta_2 \geqslant cn$ for some $c > 0$.
- If $\Delta^{\wedge}(G) = 1$, then the chromatic number of G satisfies $\chi(G) = O(\Delta(G)/\ln \Delta(G))$ as $\Delta(G) \to \infty$, and there are standard examples having arbitrarily large girth that show this bound to be sharp up to a constant factor: see [16, Ex. 12.7]. Since the equitable chromatic number is at least the chromatic number, these examples moreover yield non-packable pairs (G_1, G_2) of graphs having $(\Delta_1/\ln \Delta_1)(\Delta_2 + 1) \geqslant cn$ for some $c > 0$ and $\Delta_1^{\wedge} = 1$.

Since the examples can also have the maximum adjacent codegree $\Delta_1^{\triangle}$ being zero, this last remark hints at another natural line to pursue, which could significantly extend both the result of Csaba [9] and a result of Johansson [13]. If Δ_1 is large enough and G_1 is triangle-free, is some condition of the form $(\Delta_1/\ln \Delta_1)(\Delta_2 + 1) \leqslant cn$ for some constant $c > 0$ sufficient for G_1 and G_2 to pack?

Acknowledgement

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