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ON THE SET OF BETTI ELEMENTS OF A PUISEUX MONOID

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Abstract

Let M be a Puiseux monoid, that is, a monoid consisting of nonnegative rationals (under standard addition). In this paper, we study factorisations in atomic Puiseux monoids through the lens of their associated Betti graphs. The Betti graph of $b \in M$ is the graph whose vertices are the factorisations of b with edges between factorisations that share at least one atom. If the Betti graph associated to b is disconnected, then we call b a Betti element of M. We explicitly compute the set of Betti elements for a large class of Puiseux monoids (the atomisations of certain infinite sequences of rationals). The process of atomisation is quite useful in studying the arithmetic of Puiseux monoids, and it has been actively considered in recent literature. This leads to an argument that for every positive integer n, there exists an atomic Puiseux monoid with exactly n Betti elements.

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1. Introduction

Let M be an (additive) monoid that is cancellative and commutative. We say that a noninvertible element of M is an atom if it cannot be written in M as a sum of two noninvertible elements, and we say that M is atomic if every noninvertible element of M can be written as a sum of finitely many atoms (allowing repetitions). A formal sum of atoms which add up to $b \in M$ is called a factorisation of b, while the number of atoms in a factorisation z (counting repetitions) is called the length of z. Assume now that M is an atomic monoid. If b is a noninvertible element of M, then the Betti graph of b is the graph whose elements are the factorisations of b and whose set of edges consists of all pairs of factorisations having at least one atom in common. A noninvertible element of M is called a Betti element if its Betti graph is disconnected. For a more general notion of a Betti element, namely, the *syzygies* of an $\mathbb{N}^k$-graded module, see [29]. Following [19], we say that an additive submonoid of $\mathbb{Q}$ is a Puiseux monoid if it consists of nonnegative rationals. Factorisations in the setting of Puiseux monoids have been actively investigated in the past few years (see [8, 15]). The primary purpose of this paper is to further understand factorisations in Puiseux monoids, now through the lens of Betti graphs. Using Theorem 4.2, we construct a large class of atomic Puiseux monoids for which we can explicitly describe the sets of Betti elements. In fact, we show in Proposition 4.5 that any positive integer b can serve as the cardinality of the set of Betti elements of a Puiseux monoid.

Betti graphs are relevant in the theory of nonunique factorisation because several of the most relevant factorisation and length-factorisation (global) invariants are either attained at Betti elements or can be computed using Betti elements. For instance, Chapman *et al.* [6] proved that the catenary degree of every finitely generated reduced monoid is attained at a Betti element. In addition, Chapman *et al.* [7] used Betti elements to describe the delta set of atomic monoids satisfying the bounded factorisation property (the catenary degree and the delta set are two of the most relevant factorisation invariants). Betti elements have been studied by García-Sánchez and Ojeda [13] in connection with uniquely presented numerical semigroups. In addition, García-Sánchez *et al.* [12] characterised affine semigroups having exactly one Betti element and, for those semigroups, they explicitly found various factorisation invariants, including the catenary degree and the delta set. In the same direction, Chapman *et al.* [5] recently proved that every length-factorial monoid that is not a unique factorisation monoid has a unique Betti element. Even more recently, the sets of Betti elements of additive monoids of the form $(\mathbb{N}_0[\alpha], +)$ for certain positive algebraic numbers α have been explicitly computed by Ajran *et al.* [2].

This paper is organised as follows. In Section 2, we discuss most of the terminology and nonstandard results needed to follow the subsequent sections. In Section 3, we provide some motivating examples and perform explicit computations of the sets of Betti elements of some Puiseux monoids. These examples should provide certain intuition to better understand our main results. In Section 4, which is the section containing our main results, we discuss the notion of atomisation, which is a method introduced by Gotti and Li in [26] that one can use to construct atomic Puiseux monoids with certain desired factorisation properties. Indeed, most of the Puiseux monoids with applications in commutative ring theory can be constructed using atomisation (see [26, 27]). However, techniques involving atomisation reach far more deeply. For instance, these ideas play a key role in obtaining realisation theorems for full systems of sets of lengths in the setting of both numerical monoids and Puiseux monoids (see [17, 21]). As the main result of this paper, we describe the set of Betti elements of Puiseux monoids constructed by atomisation, and we completely determine the sets of Betti elements for certain special types of atomised Puiseux monoids. Finally, we provide the following application of our main result: for any possible size b, there exists an atomic Puiseux monoid having precisely b Betti elements.

2. Background

2.1. General notation and terminology. We use terminology standard in the general area of nonunique factorisation theory. We briefly review some of these definitions and direct the reader to [16] for any undefined notation. We let $\mathbb{N}$ denote the set of positive integers and we set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. In addition, we let $\mathbb{P}$ stand for the set of primes. As it is customary, we let $\mathbb{Z}$ and $\mathbb{Q}$ denote the set of integers and the set of rationals, respectively. If $b, c \in \mathbb{Z}$, then we let $[\![b, c]\!]$ denote the discrete closed interval from b to c; that is, $[\![b, c]\!] := \{n \in \mathbb{Z} \mid b \le n \le c\}$ (observe that $[\![b, c]\!]$ is empty if $b > c$). For a subset X consisting of rationals and $q \in \mathbb{Q}$, we set

$$X_{\ge q} := \{x \in X \mid x \ge q\},$$

and we define $X_{>q}$ in a similar manner. For $q \in \mathbb{Q}_{>0}$, we let $\mathsf{n}(q)$ and $\mathsf{d}(q)$ denote the unique elements of $\mathbb{N}$ satisfying $\gcd(\mathsf{n}(q), \mathsf{d}(q)) = 1$ and $q = \mathsf{n}(q)/\mathsf{d}(q)$. For $p \in \mathbb{P}$ and $n \in \mathbb{N}$, the value $v_p(n)$ is the exponent of the largest power of p dividing n. Moreover, the *p-adic valuation* is the map $v_p \colon \mathbb{Q}_{\ge 0} \to \mathbb{Z}$ defined by $v_p(q) = v_p(\mathsf{n}(q)) - v_p(\mathsf{d}(q))$ for $q \in \mathbb{Q}_{>0}$ and $v_p(0) = \infty$. One can verify that the p-adic valuation satisfies the inequality $v_p(q_1 + \cdots + q_n) \ge \min\{v_p(q_1), \ldots, v_p(q_n)\}$ for every $n \in \mathbb{N}$ and $q_1, \ldots, q_n \in \mathbb{Q}_{>0}$.

2.2. Monoids. Throughout this paper, we tacitly assume that the term *monoid* refers to a cancellative and commutative semigroup with an identity element. Unless we specify otherwise, monoids in this paper will be additively written. Let M be a monoid. We let $M^\bullet$ denote the set $M \setminus \{0\}$. The group of invertible elements of M is denoted by $\mathcal{U}(M)$. A subset S of M is called a *generating set* if the only submonoid of M containing S is M itself, in which case we write $M = \langle S \rangle$. For $b, c \in M$, we say that b *divides* c in M and write $b \mid_M c$ if there exists $b' \in M$ such that $c = b + b'$. The monoid M is called a *valuation monoid* if for any pair of elements $b, c \in M$, either $b \mid_M c$ or $c \mid_M b$.

A noninvertible element $a \in M$ is called an *atom* provided that for all $u, v \in M$, the fact that $a = u + v$ implies that $u \in \mathcal{U}(M)$ or $v \in \mathcal{U}(M)$. The set consisting of all the atoms of M is denoted by $\mathcal{A}(M)$. Following Coykendall *et al.* [10], we say that M is antimatter if $\mathcal{A}(M)$ is empty. An element $b \in M$ is called *atomic* if either b is invertible or b can be written as a sum of atoms (with repetitions allowed), while the whole monoid M is called *atomic* if every element of M is atomic. It is well known that every monoid satisfying the ascending chain condition on principal ideals (ACCP) is atomic [16, Proposition 1.1.4]. The converse does not hold and we will discuss such examples in the following sections.

2.3. Factorisations. Let M be a monoid. The set $M_{\mathrm{red}} := \{b + \mathcal{U}(M) \mid b \in M\}$ is also a monoid under the natural addition induced by that of M. (One can verify that M_{red} is atomic if and only if M is atomic.) We let $\mathsf{Z}(M)$ denote the free commutative monoid on the set $\mathcal{A}(M_{\mathrm{red}})$, that is, the monoid consisting of all formal sums of atoms in M_{red}. The monoid $\mathsf{Z}(M)$ plays an important role in this paper, and the formal sums in $\mathsf{Z}(M)$ are called *factorisations*. The *greatest common divisor* of two factorisations z and z' in $\mathsf{Z}(M)$, denoted by $\gcd(z, z')$, is the factorisation consisting of all the atoms z and z' have in common (counting repetitions). If a factorisation $z \in \mathsf{Z}(M)$ consists of ℓ atoms of M_{red} (counting repetitions), then we call ℓ the *length* of z, in which case we

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often write $|z|$ as an alternative for ℓ. We say that $a \in \mathcal{A}(M)$ *appears* in z provided that $a + \mathcal{U}(M)$ is one of the formal summands of z.

There is a unique monoid homomorphism $\pi_M \colon \mathsf{Z}(M) \to M_{\mathrm{red}}$ satisfying $\pi(a) = a$ for all $a \in \mathcal{A}(M_{\mathrm{red}})$, which is called the *factorisation homomorphism* of M. When there seems to be no risk of ambiguity, we write π instead of π_M. The set

$$\ker \pi := \{(z, z') \in \mathsf{Z}(M)^2 \mid \pi(z) = \pi(z')\}$$

is called the *kernel* of π, and it is a congruence in the sense that it is an equivalence relation on $\mathsf{Z}(M)$ such that if $(z, z') \in \ker \pi$, then $(z + w, z' + w) \in \ker \pi$ for all $w \in \mathsf{Z}(M)$. An element $(z, z') \in \ker \pi$ is called a *factorisation relation*. For each $b \in M$, we set

$$\mathsf{Z}(b) := \mathsf{Z}_M(b) := \pi^{-1}(b + \mathcal{U}(M)) \subseteq \mathsf{Z}(M)$$

and we call $\mathsf{Z}(b)$ the *set of factorisations* of b. Observe that $\mathsf{Z}(u) = \{0\}$ if and only if $u \in \mathcal{U}(M)$. If $|\mathsf{Z}(b)| = 1$ for every $b \in M$, then M is called a *unique factorisation monoid* (UFM). For each $b \in M$, we set

$$\mathsf{L}(b) := \mathsf{L}_M(b) := \{|z| : z \in \mathsf{Z}(b)\} \subset \mathbb{N}_0$$

and we call $\mathsf{L}(b)$ the *set of lengths* of b. If $|\mathsf{L}(b)| = 1$ for every $b \in M$, then M is called a *half-factorial monoid* (HFM). Note that every UFM is an HFM (see [4] for examples of HFMs that are not UFMs). Moreover, if $1 \le |\mathsf{L}(b)| < \infty$ for every $b \in M$, then M is called a *bounded factorisation monoid* (BFM). Finally, if for each $b \in M$ and each distinct pair of factorisations z_1 and z_2 taken from $\mathsf{Z}(b)$ we have $|z_1| \neq |z_2|$, then M is called a *length-factorial monoid* (LFM).

It follows directly from the definitions that every HFM is a BFM. Cofinite submonoids of $(\mathbb{N}_0, +)$ are called *numerical monoids*, and every numerical monoid different from $\mathbb{N}_0$ is a BFM that is not an HFM. In addition, it is well known that every BFM satisfies the ACCP [28, Corollary 1]. The converse does not hold as we will see in Example 3.4. For a recent survey on factorisations in commutative monoids, see [18].

2.4. Betti elements and Betti graphs. A finite sequence $z_0, \ldots, z_k$ of factorisations in $\mathsf{Z}(M)$ is called a *chain of factorisations* from z_0 to z_k provided that $\pi(z_0) = \pi(z_1) = \cdots = \pi(z_k)$. Let $\mathcal{R}$ be the subset of $\mathsf{Z}(M)^2$ consisting of all pairs (z, z') such that there exists a chain of factorisations $z_0, \ldots, z_k$ from z to z' with $\gcd(z_{i-1}, z_i) \neq 0$ for every $i \in [\![1, k]\!]$. It follows immediately that $\mathcal{R}$ is an equivalence relation on $\mathsf{Z}(M)$ that refines $\ker \pi$. Fix $b \in M$. We let $\mathcal{R}_b$ denote the set of equivalence classes of $\mathcal{R}$ inside $\mathsf{Z}(b)$, and the element b is called a *Betti element* provided that $|\mathcal{R}_b| \ge 2$. The *Betti graph* ∇_b of b is the graph whose set of vertices is $\mathsf{Z}(b)$ having an edge between factorisations $z, z' \in \mathsf{Z}(x)$ precisely when $\gcd(z, z') \neq 0$. Observe that an element of M is a Betti element if and only if its Betti graph is disconnected. We let $\mathrm{Betti}(M)$ denote the set of Betti elements of M.

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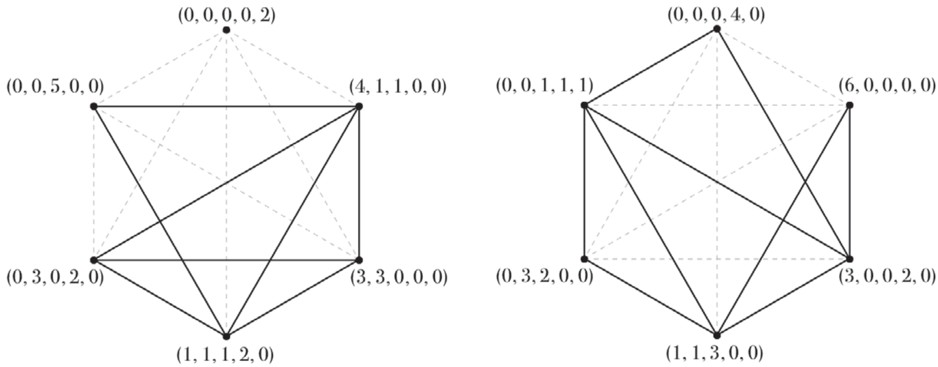

FIGURE 1. For $N = \langle 14, 16, 18, 21, 45\rangle$, the figure shows the Betti graph of $90 \in \text{Betti}(N)$ on the left and that of $84 \notin \text{Betti}(N)$ on the right.

EXAMPLE 2.1. Consider the numerical monoid $N := \langle 14, 16, 18, 21, 45\rangle$. Using the SAGE package called `numericalsgps GAP`, we obtain $|\text{Betti}(N)| = 9$. Also, $90 \in \text{Betti}(N)$, while $84 \notin \text{Betti}(N)$. Figure 1 (taken from [25]) shows the Betti graphs of both 84 and 90 in N.

3. Basic observations and motivating examples

It is clear that if a monoid is a UFM, then its set of Betti elements is empty. Following Coykendall and Zafrullah [11], we say that a monoid M is an *unrestricted unique factorisation monoid* (U-UFM) if its atomic elements have at most one factorisation. It follows directly from the definitions that every UFM is a U-UFM and that an atomic U-UFM is a UFM. We conclude this subsection by characterising U-UFMs in terms of the existence of Betti elements.

PROPOSITION 3.1. *A monoid is a U-UFM if and only if its set of Betti elements is empty.*

PROOF. The direct implication follows immediately because if a monoid is a U-UFM, then the Betti graph of each element has at most one vertex and is, therefore, connected.

For the reverse implication, assume that M is a monoid containing no Betti elements. Now suppose, by way of contradiction, that M is not a U-UFM. This means that there exists an element $x_0 \in M$ such that $|\mathsf{Z}(x_0)| \geq 2$. Let z_0 and z_0' be two distinct factorisations of x_0. After dropping the common atoms of z_0 and z_0', we can assume that $\gcd(z_0, z_0')$ is the empty factorisation. Since x_0 is not a Betti element, z_0 and z_0' must be connected in ∇_{x_0}, and so there exists a factorisation w_0 of x_0 with $w_0 \neq z_0$ such that $\gcd(z_0, w_0)$ is nonempty. Now set $z_1 := z_0 - \gcd(z_0, w_0)$. Note that z_1 is a sub-factorisation of z_0 satisfying $|z_0| > |z_1|$ (because $\gcd(z_0, w_0)$ is nonempty). Take $x_1 \in M$ such that z_1 is a factorisation of x_1, and observe that x_1 has at least two factorisations, namely, z_1 and $w_0 - \gcd(z_0, w_0)$. Because x_1 is not a Betti element, there must be a factorisation w_1 of x_1 with $w_1 \neq z_1$ such that $\gcd(z_1, w_1)$ is nonempty. Now

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set $z_2 := z_1 - \gcd(z_1, w_1)$. Note that z_2 is a sub-factorisation of z_1 satisfying $|z_1| > |z_2|$ (because $\gcd(z_1, w_1)$ is nonempty). Proceeding in this fashion, we can find a sequence $(z_n)_{n\geq 0}$ of factorisations in M such that $|z_n| > |z_{n+1}|$ for every $n \in \mathbb{N}_0$. However, this contradicts the well-ordering principle. Hence, M must be a U-UFM. □

We state the following simple corollary to Proposition 3.1 which may be of special interest.

COROLLARY 3.2. *If M is an atomic monoid which is not a UFM, then* $\mathrm{Betti}(M) \neq \emptyset$.

It is well known that a Puiseux monoid is a UFM if and only if it is an HFM. This occurs if and only if it can be generated by one element, in which case it is isomorphic to $\mathbb{N}_0$ (see [22, Proposition 4.3]). There are, however, non-HFM atomic Puiseux monoids that contain finitely many Betti elements. The next two examples illustrate this observation.

EXAMPLE 3.3. Let M be a finitely generated Puiseux monoid. If $M \neq \langle q \rangle$ for any element $q \in \mathbb{Q}_{>0}$, then it follows from [13, Remark 2] that M contains at least one Betti element. Since M is finitely generated, it must be isomorphic to a numerical monoid and, therefore, M has finitely many Betti elements (see [14, Section 9.3]). Thus, every finitely generated Puiseux monoid that is not generated by a single rational has a nonempty finite set of Betti elements.

It was proved in [5, Proposition 3.5] that if a monoid is an LFM that is not a UFM, then it contains exactly one Betti element, and it follows directly from [5, Proposition 5.7] that a Puiseux monoid is an LFM if and only if it can be generated by two elements. However, there are nonfinitely generated atomic Puiseux monoids with exactly one Betti element. This is illustrated in the following example.

EXAMPLE 3.4. Consider the Puiseux monoid $M := \langle 1/p \mid p \in \mathbb{P} \rangle$. It is well known that M is atomic with $\mathcal{A}(M) = \{1/p \mid p \in \mathbb{P}\}$. It follows from [3, Example 3.3] (see [23, Proposition 4.2(2)] for more details) that every element $q \in M$ can be written uniquely as

$$q = c + \sum_{p\in\mathbb{P}} c_p \frac{1}{p},$$

where $c \in \mathbb{N}_0$ and $c_p \in [\![0, p-1]\!]$ for every $p \in \mathbb{P}$ (here, all but finitely many of the coefficients c_p are zero). From this, we can infer that for any element $q \in M$, the conditions $|\mathsf{Z}(q)| = 1$ and $1 \nmid_M q$ are equivalent. We claim that $\mathrm{Betti}(M) = \{1\}$. To argue this equality, fix $q \in M^\bullet$. If $1 \nmid_M q$, then $|\mathsf{Z}(q)| = 1$ and so ∇_q is trivially connected, whence q is not a Betti element. Assume, however, that $1 \mid_M q$ and, therefore, that $|\mathsf{Z}(q)| \geq 2$. Suppose first that $q \neq 1$. Because M is atomic, we can write $q = 1 + \sum_{i=1}^k a_i$ for some $k \in \mathbb{N}$ and $a_1, \ldots, a_k \in \mathcal{A}(M)$. Observe that any two factorisations in $\mathsf{Z}(q)$ of the form $p(1/p) + a_1 + \cdots + a_k$ with $p \in \mathbb{P}$ are connected in the graph ∇_q. In addition, any other factorisation in $\mathsf{Z}(q)$ contains an atom $1/p_0$ for some $p_0 \in \mathbb{P}$, so this factorisation must be connected in ∇_q to the factorisation $p_0(1/p_0) + a_1 + \cdots + a_k$. Hence, ∇_q is connected when $1 \mid_M q$ and $q \neq 1$, and so q is not a Betti element. Finally,

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we see that $q = 1$ is a Betti element: indeed, in this case, $\mathsf{Z}(1) = \{p(1/p) \mid p \in \mathbb{P}\}$, so the Betti graph of 1 contains no edges. Hence, $\text{Betti}(M) = \{1\}$.

Additionally, M offers us an example of a Puiseux monoid which satisfies the ACCP, but is not a BFM. To see this, let $c(q)$ denote the c in the canonical representation of q established above, and let $s(q) := \sum_{p \in \mathbb{P}} c_p$ be the sum of the coefficients in the same canonical representation of q. To argue that M satisfies the ACCP, let $(q_n + M)_{n \geq 1}$ be an ascending chain of principal ideals. Observe that $q_n + M \subseteq q_{n+1} + M$ implies that $c(q_n) \geq c(q_{n+1})$, and so the sequence $(c(q_n))_{n \geq 1}$ must become stationary from some point on. Then, after dropping finitely many terms from the initial ascending chain of ideals, we can assume that the sequence $(c(q_n))_{n \geq 1}$ is constant, with all its terms being c. After replacing each term $q_n + M$ by $q_n - c + M$, we can further assume that in the initial chain of principal ideals, $c(q_n) = 0$ for every $n \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$, the uniqueness of the canonical representation of q_n guarantees that q_n has a unique factorisation in M. Thus, the ascending chain of principal ideals $(q_n + M)_{n \geq 1}$ must stabilise, and so M satisfies the ACCP. To see that M is not a BFM, it suffices to observe that $\mathbb{P} \subseteq \mathsf{L}(1)$.

The Puiseux monoids in the examples we have discussed so far have finitely many Betti elements. However, there exist atomic Puiseux monoids having infinitely many Betti elements. We provide an example showing this in the next section (Example 4.4).

4. Atomisation and Betti elements

It turns out that we can construct Puiseux monoids with any prescribed number of Betti elements. Before doing so, we need to introduce the notion of atomisation, which is a useful technique to construct Puiseux monoids satisfying certain desired properties. Let $(q_n)_{n \geq 1}$ be a sequence consisting of positive rationals and let $(p_n)_{n \geq 1}$ be a sequence of pairwise distinct primes such that $\gcd(p_i, \mathsf{n}(q_i)) = \gcd(p_i, \mathsf{d}(q_j)) = 1$ for all $i, j \in \mathbb{N}$. Following Gotti and Li [26], we say that

$$M := \left\langle \frac{q_n}{p_n} \;\middle|\; n \in \mathbb{N} \right\rangle$$

is the *Puiseux monoid* of $(q_n)_{n \geq 1}$ *atomised* at $(p_n)_{n \geq 1}$. It is not hard to argue that M is atomic with $\mathscr{A}(M) = \{q_n/p_n \mid n \in \mathbb{N}\}$ (see [26, Proposition 3.1] for the details). It turns out that we can determine the Betti elements of certain Puiseux monoids obtained by atomisation. We will pursue this further in Theorem 4.2. First, we need the following technical lemma.

LEMMA 4.1. *Let $(q_n)_{n \geq 1}$ be a sequence consisting of positive rational numbers and let $(p_n)_{n \geq 1}$ be a sequence of prime numbers whose terms are pairwise distinct such that $\gcd(p_i, \mathsf{n}(q_i)) = \gcd(p_i, \mathsf{d}(q_j)) = 1$ for all $i, j \in \mathbb{N}$. Let M be the Puiseux monoid of $(q_n)_{n \geq 1}$ atomised at $(p_n)_{n \geq 1}$. Then every element $q \in M$ can be uniquely written as*

$$q = n_q + \sum_{n \in \mathbb{N}} c_n \frac{q_n}{p_n}, \tag{4.1}$$

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where $n_q \in \langle q_n \mid n \in \mathbb{N}\rangle$ *and* $c_n \in [\![0, p_n - 1]\!]$ *for every* $n \in \mathbb{N}$ *(here* $c_n = 0$ *for all but finitely many* $n \in \mathbb{N}$*).*

PROOF. It suffices to prove the existence and uniqueness of the decomposition in (4.1) for every nonzero element $q \in M$. Fix $q \in M^\bullet$. Let N be the submonoid of M generated by the sequence $(q_n)_{n\geq 1}$, that is,

$$N := \langle q_n \mid n \in \mathbb{N}\rangle.$$

It follows from [26, Proposition 3.1] that M is an atomic Puiseux monoid with

$$\mathcal{A}(M) = \left\{\frac{q_n}{p_n} \;\middle|\; n \in \mathbb{N}\right\}.$$

For the existence of the decomposition in (4.1), we first decompose q as in (4.1) without imposing the condition that $c_n < p_n$ for all $n \in \mathbb{N}$. Since M is atomic, there is at least one way to decompose q in the specified way (with $n_q = 0$). Among all such decompositions, choose $q = n_q + \sum_{n\in\mathbb{N}} c_n(q_n/p_n)$ to be one minimising the sum $\sum_{n\in\mathbb{N}} c_n$. We claim that in the chosen decomposition, $c_n < p_n$ for every $n \in \mathbb{N}$. Observe that if there existed $k \in \mathbb{N}$ such that $c_k \geq p_k$, then

$$q = n'_q + (c_k - p_k)\frac{q_k}{p_k} + \sum_{n\in\mathbb{N}\setminus\{k\}} c_n \frac{q_n}{p_n},$$

where $n'_q := n_q + q_k \in N$ would be another decomposition with smaller corresponding sum, which is not possible given the minimality of $\sum_{n\in\mathbb{N}} c_n$. Hence, every element $q \in M$ has a decomposition as in (4.1) satisfying $c_n \in [\![0, p_n - 1]\!]$ for every $n \in \mathbb{N}$.

For the uniqueness, suppose that q has a decomposition as in (4.1) and also a decomposition $q = n'_q + \sum_{n\in\mathbb{N}} c'_n(q_n/p_n)$ satisfying $n'_q \in N$ and $c'_n \in [\![0, p_n - 1]\!]$ for every $n \in \mathbb{N}$ (with $c'_n = 0$ for all but finitely many $n \in \mathbb{N}$). Observe that for each $n \in \mathbb{N}$, the p_n-adic valuation of each element of N is nonnegative and the p_n-adic valuation of q_k/p_k is also nonnegative when $k \neq n$. Thus, for each $n \in \mathbb{N}$, after applying the p_n-adic valuation to both sides of $n'_q - n_q = \sum_{n\in\mathbb{N}}(c_n - c'_n)(q_n/p_n)$, we find that $p_n \mid c_n - c'_n$, which implies that $c'_n = c_n$ (here we are using the fact that $c_n, c'_n \in [\![0, p_n - 1]\!]$). Therefore, $c'_n = c_n$ for every $n \in \mathbb{N}$ and so $n'_q = n_q$. As a consequence, we can conclude that the decomposition in (4.1) is unique. □

With notation as in the statement of Lemma 4.1, we call the equality in (4.1) the *canonical decomposition* of q. We are now in a position to argue the main result of this section. Our proof of the following theorem is motivated by the argument given in Example 3.4.

THEOREM 4.2. *Let* $(q_n)_{n\geq 1}$ *be a sequence consisting of positive rational numbers and let* $(p_n)_{n\geq 1}$ *be a sequence of prime numbers whose terms are pairwise distinct such that* $\mathsf{gcd}(p_i, \mathsf{n}(q_i)) = \mathsf{gcd}(p_i, \mathsf{d}(q_j)) = 1$ *for all* $i, j \in \mathbb{N}$. *Let* M *be the Puiseux monoid of* $(q_n)_{n\geq 1}$ *atomised at* $(p_n)_{n\geq 1}$. *Then the following statements hold.*

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(1) *For each $j \in \mathbb{N}$, the length-p_j factorisation $p_j(q_j/p_j)$ is an isolated vertex in ∇_{q_j}.*
(2) $\mathrm{Betti}(M) \subseteq \langle q_n \mid n \in \mathbb{N} \rangle$.
(3) $\{q_n \mid n \in \mathbb{N}\} \subseteq \mathrm{Betti}(M)$ *if $\langle q_n \mid n \in \mathbb{N} \rangle$ is antimatter.*
(4) $\mathrm{Betti}(M) \subseteq \{q_n \mid n \in \mathbb{N}\}$ *if $\langle q_n \mid n \in \mathbb{N} \rangle$ is a valuation monoid.*

PROOF. Set $N := \langle q_n \mid n \in \mathbb{N} \rangle$. As mentioned in Lemma 4.1, the Puiseux monoid M is atomic with

$$\mathcal{A}(M) = \left\{ \frac{q_n}{p_n} \,\middle|\, n \in \mathbb{N} \right\}.$$

(1) Fix $j \in \mathbb{N}$ and let us argue that $z := p_j(q_j/p_j)$ is an isolated factorisation in the Betti graph of q_j. If $|\mathsf{Z}(q_j)| = 1$, then we are done. Suppose, however, that $|\mathsf{Z}(q_j)| \geq 2$ and take $c_1, \ldots, c_k \in \mathbb{N}_0$ such that $z' := \sum_{i=1}^{k} c_i(q_i/p_i)$ is a factorisation of q_j in M with $z \neq z'$ (we can assume, without loss of generality, that $k \geq j$). Because $v_{p_j}(q_j) = 0$, we can apply the p_j-adic valuation to both sides of the equality $q_j = \sum_{i=1}^{k} c_i(q_i/p_i)$ to find that $p_j \mid c_j$. Thus, the fact that $z \neq z'$ ensures that $c_j = 0$. As a consequence, $\gcd(z, z') = 0$. We can conclude, therefore, that z is an isolated factorisation in the Betti graph ∇_{q_j}.

(2) Fix $q \in M$. It suffices to prove that if $q \notin N$, then q is not a Betti element. To do so, assume that $q \notin N$. In light of Lemma 4.1, we can write q uniquely as

$$q = n_q + \sum_{n \in \mathbb{N}} c_n \frac{q_n}{p_n},$$

where $n_q \in N$ and $c_n \in [\![0, p_n - 1]\!]$ for every $n \in \mathbb{N}$. Since $q \notin N$, there exist $k \in \mathbb{N}$ such that $c_k \neq 0$. In this case, the p_k-adic valuation of q is negative and, therefore, every factorisation of q must contain the atom q_k/p_k, whence ∇_q is connected. Hence, $\mathrm{Betti}(M) \subseteq N$.

(3) Assume that N is an antimatter monoid. For any $j \in \mathbb{N}$, recall from part (1) that $z := p_j(q_j/p_j)$ is an isolated factorisation in the Betti graph ∇_{q_j}. Also, since N is an antimatter monoid, there exist $k \in \mathbb{N}$ and $s \in N^\bullet$ such that $q_j = q_k + s$. Now set

$$z' := p_k \frac{q_k}{p_k} + z'',$$

where z'' is a factorisation of s in M. Since $k \neq j$, we see that z' is a factorisation of q_j in M that is different from z. Since z is isolated, $\gcd(z, z') = 0$ and so ∇_{q_j} is disconnected. Hence, q_j is a Betti element of M. As a result, the inclusion $\{q_n \mid n \in \mathbb{N}\} \subseteq \mathrm{Betti}(M)$ holds.

(4) Lastly, assume that N is a valuation monoid. Fix $q \in M^\bullet \setminus \{q_n \mid n \in \mathbb{N}\}$ and let us argue that q is not a Betti element of M. If $q \notin N$, then it follows from part (2) that $q \notin \mathrm{Betti}(M)$. Hence, we assume that $q \in N$. Fix two factorisations

$$z := \sum_{n \in \mathbb{N}} c_n \frac{q_n}{p_n} \quad \text{and} \quad z' := \sum_{n \in \mathbb{N}} c'_n \frac{q_n}{p_n}$$

of q (here, all but finitely many c_n and all but finitely many c'_n equal 0). For each $n \in \mathbb{N}$, the fact that $q \in N$ implies that q has nonnegative p_n-adic valuation, and

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so after applying the p_n-adic valuation to both equalities $q = \sum_{n\in\mathbb{N}} c_n(q_n/p_n)$ and $q = \sum_{n\in\mathbb{N}} c'_n(q_n/p_n)$, we find that $p_n \mid c_n$ and $p_n \mid c'_n$. Because q is nonzero, we can take $k, \ell \in \mathbb{N}$ such that $c_k \geq p_k$ and $c'_\ell \geq p_\ell$. Since N is a valuation monoid, either $q_k \mid_N q_\ell$ or $q_\ell \mid_N q_k$. Assume, without loss of generality, that $q_\ell \mid_N q_k$. Then there exists $s \in N$ such that $q_k = q_\ell + s$. Now take a factorisation z_s of s in M and set

$$z'' := z - p_k \frac{q_k}{p_k} + p_\ell \frac{q_\ell}{p_\ell} + z_s.$$

Notice that z'' is a factorisation of q in M. As $q \notin \{q_n \mid n \in \mathbb{N}\}$, it follows that $\gcd(z, z'') \neq 0$. Also, the atom q_ℓ/p_ℓ has nonzero coefficients in both z' and z'', which implies that $\gcd(z', z'') \neq 0$. As the factorisations z and z' are both adjacent to z'' in the Betti graph ∇_q, there is a length-2 path between them. Since z and z' were arbitrarily taken, the graph ∇_q is connected, which means that q is not a Betti element. Hence, $\mathrm{Betti}(M) \subseteq \{q_n \mid n \in \mathbb{N}\}$. □

As an immediate consequence of Theorem 4.2, we obtain the following corollary.

COROLLARY 4.3. *Let $(q_n)_{n\geq1}$ be a sequence consisting of positive rational numbers and let $(p_n)_{n\geq1}$ be a sequence of prime numbers whose terms are pairwise distinct such that $\gcd(p_i, \mathsf{n}(q_i)) = \gcd(p_i, \mathsf{d}(q_j)) = 1$ for all $i, j \in \mathbb{N}$. Let M be the Puiseux monoid of $(q_n)_{n\geq1}$ atomised at $(p_n)_{n\geq1}$. If $\langle q_n \mid n \in \mathbb{N}\rangle$ is an antimatter valuation monoid, then*

$$\mathrm{Betti}(M) = \{q_n \mid n \in \mathbb{N}\}.$$

As an application of Corollary 4.3, we can easily determine the set of Betti elements of Grams' monoid.

EXAMPLE 4.4. Let $(p_n)_{n\geq0}$ be the strictly increasing sequence whose underlying set consists of all odd primes, and consider the Puiseux monoid

$$M := \left\langle \frac{1}{2^n p_n} \;\middle|\; n \in \mathbb{N}_0 \right\rangle.$$

The monoid M is often referred to as *Grams' monoid* as it was the crucial ingredient in Grams' construction of the first atomic integral domain not satisfying the ACCP (see [27] for the details of the construction). Observe that M is the atomisation of the sequence $(1/2^n)_{n\geq0}$ at the sequence of primes $(p_n)_{n\geq0}$. As a consequence, it follows from [26, Proposition 3.1] that M is an atomic Puiseux monoid with

$$\mathcal{A}(M) = \left\{ \frac{1}{2^n p_n} \;\middle|\; n \in \mathbb{N}_0 \right\}.$$

However, M does not satisfy the ACCP because $(1/2^n + M)_{n\geq0}$ is an ascending chain of principal ideals of M that does not stabilise. Since $\langle 1/2^n \mid n \in \mathbb{N}_0\rangle$ is an antimatter valuation monoid, it follows from Corollary 4.3 that

$$\mathrm{Betti}(M) = \left\{ \frac{1}{2^n} \;\middle|\; n \in \mathbb{N}_0 \right\}.$$

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As a final application of Theorem 4.2, we construct atomic Puiseux monoids with any prescribed number of Betti elements.

PROPOSITION 4.5. *For each $b \in \mathbb{N} \cup \{\infty\}$, there exists an atomic Puiseux monoid M such that $|\mathrm{Betti}(M)| = b$.*

PROOF. We have seen in Example 4.4 that Grams' monoid is an atomic Puiseux monoid and we have also seen in the same example that the Grams' monoid has infinitely many Betti elements. Therefore, it suffices to assume that $b \in \mathbb{N}$.

Fix $b \in \mathbb{N}$. Now consider the sequence $(q_n)_{n\geq 1}$ whose terms are defined as $q_{kb+r} := r + 1$ for every $k \in \mathbb{N}_0$ and $r \in [\![0, b-1]\!]$. Now let $(p_n)_{n\geq 1}$ be a strictly increasing sequence of primes such that $p_n > b$ for every $n \in \mathbb{N}$. Then $\gcd(p_i, \mathsf{n}(q_i)) = \gcd(p_i, \mathsf{d}(q_j)) = 1$ for all $i, j \in \mathbb{N}$. Let M be the Puiseux monoid we obtain after atomising the sequence $(q_n)_{\geq 1}$ at the sequence $(p_n)_{n\geq 1}$. It follows from [26, Proposition 3.1] that M is an atomic Puiseux monoid with

$$\mathcal{A}(M) := \left\{ \frac{q_n}{p_n} \;\middle|\; n \in \mathbb{N} \right\}.$$

Observe that $\langle q_n \mid n \in \mathbb{N} \rangle = \langle 1, \ldots, b \rangle = \mathbb{N}_0$, which is a valuation monoid. As a consequence, it follows from Theorem 4.2(4) that $\mathrm{Betti}(M) \subseteq \{q_n \mid n \in \mathbb{N}\} = [\![1, b]\!]$. Now fix $m \in [\![1, b]\!]$, and let us check that m is a Betti element. To do this, first observe that the Betti graph ∇_m contains infinitely many vertices because

$$\left\{ p_{kb+(m-1)} \frac{m}{p_{kb+(m-1)}} \;\middle|\; k \in \mathbb{N} \right\} \subseteq \mathsf{Z}(m).$$

Therefore, ∇_m must be disconnected as it follows from Theorem 4.2(1) that $p_{m-1}(m/p_{m-1})$ is an isolated vertex in ∇_m. Hence, $\mathrm{Betti}(M) = [\![1, b]\!]$ and so $|\mathrm{Betti}(M)| = b$, as desired. □

We note that using [13, Example 14], given $b \neq \infty$ as in Proposition 4.5, one can construct a numerical monoid N with $|\mathrm{Betti}(N)| = b$. Obviously, these examples are finitely generated and differ greatly from those presented above.

Among the examples of atomic Puiseux monoids we have discussed so far, the only one having infinitely many Betti elements is Grams' monoid, which does not satisfy the ACCP. However, there are Puiseux monoids containing infinitely many Betti elements that are finite factorisation monoids (FFMs). The following example illustrates this observation.

EXAMPLE 4.6. Let q be a noninteger positive rational and consider the Puiseux monoid $M_q := \langle q^n \mid n \in \mathbb{N}_0 \rangle$. It is well known that M_q is atomic provided that $q^{-1} \notin \mathbb{N}$, in which case, $\mathcal{A}(M_q) = \{q^n \mid n \in \mathbb{N}_0\}$ (see [24, Theorem 6.2] and also [9, Theorem 4.2]). It follows from [2, Lemma 4.3] that $\mathrm{Betti}(M_q) = \{\mathsf{n}(q)q^n \mid n \in \mathbb{N}_0\}$. Thus, M_q is an atomic Puiseux monoid with infinitely many Betti elements. When $q > 1$, it follows from [20, Theorem 5.6] that M_q is an FFM (in particular, M_q satisfies the ACCP).

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As we have mentioned in Example 3.3, every finitely generated Puiseux monoid has finitely many Betti elements. Although the class of finitely generated monoids sits inside the class of FFMs (see [16, Proposition 2.7.8]), we have seen in Example 4.6 that inside the class of Puiseux monoids, the finite factorisation property is not enough to guarantee that the set of Betti elements is finite.

However, every atomic Puiseux monoid with finitely many Betti elements we have discussed so far satisfies the ACCP: these include the Puiseux monoids discussed in Examples 3.3 and 3.4 as well as the Puiseux monoids constructed in the proof of Proposition 4.5, which satisfy the ACCP in light of [1, Theorem 4.5]. We have not been able to construct an atomic Puiseux monoid with finitely many Betti elements that does not satisfy the ACCP. Thus, we conclude this paper with the following question.

QUESTION 4.7. Does every atomic Puiseux monoid with finitely many Betti elements satisfy the ACCP?

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References

[1] C. Aguilera, M. Gotti and A. F. Hamelberg, ‘Factorization in reciprocal Puiseux monoids’, Preprint, 2021, arXiv:2112.04048.

[2] K. Ajran, J. Bringas, B. Li, E. Singer and M. Tirador, ‘Factorization in additive monoids of evaluation polynomial semirings’, *Comm. Algebra* **51** (2023), 4347–4362.

[3] D. F. Anderson and F. Gotti, ‘Bounded and finite factorization domains’, in: *Rings, Monoids and Module Theory*, Springer Proceedings in Mathematics and Statistics, 382 (eds. A. Badawi and J. Coykendall) (Springer, Singapore, 2022), 7–57.

[4] S. T. Chapman, ‘A tale of two monoids: a friendly introduction to the theory of non-unique factorizations’, *Math. Mag.* **87** (2014), 163–173.

[5] S. T. Chapman, J. Coykendall, F. Gotti and W. W. Smith, ‘Length-factoriality in commutative monoids and integral domains’, *J. Algebra* **578** (2021), 186–212.

[6] S. T. Chapman, P. García-Sánchez, D. Llena, V. Ponomarenko and J. Rosales, ‘The catenary and tame degree in finitely generated commutative cancellative monoids’, *Manuscripta Math.* **120** (2006), 253–264.

[7] S. T. Chapman, P. A. García-Sánchez, D. Llena, A. Malyshev and D. Steinberg, ‘On the delta set and the Betti elements of a BF-monoid’, *Arab. J. Math.* **1** (2012), 53–61.

[8] S. T. Chapman, F. Gotti and M. Gotti, ‘When is a Puiseux monoid atomic?’, *Amer. Math. Monthly* **128** (2021), 302–321.

[9] J. Correa-Morris and F. Gotti, ‘On the additive structure of algebraic valuations of polynomial semirings’, *J. Pure Appl. Algebra* **226** (2022), Article no. 107104.

[10] J. Coykendall, D. E. Dobbs and B. Mullins, ‘On integral domains with no atoms’, *Comm. Algebra* **27** (1999), 5813–5831.

[11] J. Coykendall and M. Zafrullah, ‘AP-domains and unique factorization’, *J. Pure Appl. Algebra* **189** (2004), 27–35.

[12] P. García-Sánchez, I. Ojeda and J. Rosales, 'Affine semigroups having a unique Betti element', *J. Algebra Appl.* **12** (2013), Article no. 1250177.
[13] P. A. García-Sánchez and I. Ojeda, 'Uniquely presented finitely generated commutative monoids', *Pacific J. Math.* **248** (2010), 91–105.
[14] P. A. García-Sánchez and J. C. Rosales, *Numerical Semigroups*, Developments in Mathematics, 20 (Springer-Verlag, Dordrecht, 2009).
[15] A. Geroldinger, F. Gotti and S. Tringali, 'On strongly primary monoids, with a focus on Puiseux monoids', *J. Algebra* **567** (2021), 310–345.
[16] A. Geroldinger and F. Halter-Koch, *Non-unique Factorizations: Algebraic, Combinatorial and Analytic Theory*, Pure and Applied Mathematics, 278 (Chapman and Hall/CRC, Boca Raton, FL, 2006).
[17] A. Geroldinger and W. A. Schmid, 'A realization theorem for sets of lengths in numerical monoids', *Forum Math.* **30** (2018), 1111–1118.
[18] A. Geroldinger and Q. Zhong, 'Factorization theory in commutative monoids', *Semigroup Forum* **100** (2020), 22–51.
[19] F. Gotti, 'On the atomic structure of Puiseux monoids', *J. Algebra Appl.* **16** (2017), Article no. 1750126.
[20] F. Gotti, 'Increasing positive monoids of ordered fields are FF-monoids', *J. Algebra* **518** (2019). 40–56.
[21] F. Gotti, 'Systems of sets of lengths of Puiseux monoids', *J. Pure Appl. Algebra* **223** (2019), 1856–1868.
[22] F. Gotti, 'Irreducibility and factorizations in monoid rings', in: *Numerical Semigroups*, Springer INdAM Series, 40 (eds. V. Barucci, S. T. Chapman, M. D'Anna and R. Fröberg) (Springer, Cham, 2020), 129–139.
[23] F. Gotti, 'On semigroup algebras with rational exponents', *Comm. Algebra* **50** (2022), 3–18.
[24] F. Gotti and M. Gotti, 'Atomicity and boundedness of monotone Puiseux monoids', *Semigroup Forum* **96** (2018), 536–552.
[25] F. Gotti and M. Gotti, 'On the molecules of numerical semigroups, Puiseux monoids, and monoid algebras', in: *Numerical Semigroups*, Springer INdAM Series, 40 (eds. V. Barucci, S. T. Chapman, M. D'Anna and R. Fröberg) (Springer, Cham, 2020), 141–161.
[26] F. Gotti and B. Li, 'Atomic semigroup rings and the ascending chain condition on principal ideals', *Proc. Amer. Math. Soc.* **151** (2023), 2291–2302.
[27] A. Grams, 'Atomic rings and the ascending chain condition for principal ideals', *Proc. Cambridge Philos. Soc.* **75** (1974), 321–329.
[28] F. Halter-Koch, 'Finiteness theorems for factorizations', *Semigroup Forum* **44** (1992), 112–117.
[29] E. Miller and B. Sturmfels, *Combinatorial Commutative Algebra*, Graduate Texts in Mathematics, 227 (Springer-Verlag, New York, 2004).

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