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BANACH SPACES OF CONTINUOUS VECTOR-VALUED FUNCTIONS OF ORDINALS

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Abstract Let X be a Banach space and ξ an ordinal number. We study some isomorphic classifications of the Banach spaces X^ξ of the continuous X-valued functions defined in the interval of ordinals $[1,\xi]$ and equipped with the supremum norm. More precisely, first we use the continuum hypothesis to give an isomorphic classification of $C(I)^\xi$, $\xi \geqslant \omega_1$. Then we present a characterization of the separable Banach spaces X that are isomorphic to X^ξ, $\forall\xi$, $\omega \leqslant \xi < \omega_1$. Finally, we show that the isomorphic classifications of $(C(I) \oplus F^*)^\xi$ and $\ell_\infty(\mathbb{N})^\xi$, where F is the space of Figiel and $\omega \leqslant \xi < \omega_1$ are similar to that of $\mathbb{R}^\xi$ given by Bessaga and Pelczynski.

Keywords: continuous vector-valued functions; isomorphic classifications of Banach spaces

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1. Introduction

As in [**1**], X being a Banach space and ξ an ordinal number, X^ξ will indicate the Banach space of the continuous X-valued functions defined in the interval of ordinals $[1,\xi]$ and equipped with the supremum norm. $C(I)$ being the Banach space of continuous functions defined in the interval $I = [0,1]$ of the real line $\mathbb{R}$ with the supremum norm, it follows from Milutin's Theorem (see [**18**, p. 379]) that

$$C(I)^\xi \text{ is isomorphic to } C(I), \quad \forall\xi, \quad \omega \leqslant \xi < \omega_1. \tag{1.1}$$

Initially, using the continuum hypothesis we give an isomorphic classification of the Banach spaces $C(I)^\xi$, $\xi \geqslant \omega_1$ (Theorem 3.1). Afterwards, inspired by Bourgain [**15**] we exhibit a characterization of the separable Banach spaces X such that X^ξ, $\forall\xi$, $\omega \leqslant \xi < \omega_1$ is isomorphic to X (Theorem 4.1). Next, by Pisier's Theorem, we will generalize a result from Samuel [**16**], and we show that the presence of F^*, where F is the space of Figiel [**6**], together with $C(I)$ annihilates (1.1), since the isomorphic classification of $(C(I) \oplus F^*)^\xi$, $\omega \leqslant \xi < \omega_1$ is similar to that of $\mathbb{R}^\xi$ given in [**1**] (Corollary 5.6). Finally, we will prove that the same happens with the isomorphic classification of $\ell_\infty(\mathbb{N})^\xi$, $\omega \leqslant \xi < \omega_1$, where $\ell_\infty(\mathbb{N})$ is the Banach space of the bounded sequences with the supremum norm (Corollary 5.8). These results motivated the definition of ω_1 cancellable Banach space (Definition 5.9).

2. Preliminaries

To fix the notation, let us recall some definitions. If X and Y are Banach spaces, then Y is isomorphic to a closed subspace of X, $Y \hookrightarrow X$, if there is a one-to-one bounded linear operator from Y into X; Y is said to be isomorphic to X, $X \sim Y$, provided there is a one-to-one bounded linear operator from X onto Y, and Y is a quotient of X, $X \twoheadrightarrow Y$, if there is a surjective bounded linear operator from X onto Y. The notation $\hat{\hat{\otimes}}_n X$ will indicate the injective tensor product of n isomorphic copies of X, $n < \omega$.

Let Γ be a set. By $C_0(\Gamma, X)$ we denote the Banach space of X-valued functions defined on Γ such that for any positive ε the set $\{\gamma \in \Gamma : \|f(\gamma)\| \geqslant \varepsilon\}$ is finite, with the supremum norm, and by $\ell_1(\Gamma, X)$ we denote the Banach space of all absolutely summable X-functions defined on Γ.

If α is an ordinal number and X is a Banach space, we set $X_0^\alpha = \{f \in X^\alpha : f(\alpha) = 0\}$. The cardinality of the ordinal number ξ will be denoted by $\bar{\xi}$. The notation ω_1 will denote the first non-denumerable ordinal. If α is a non-denumerable regular ordinal and γ is any ordinal, we will denote by $\wedge_\gamma^\alpha$ the subset of $[1, \gamma]$ consisting of limit ordinals that are not limits of sets of cardinality strictly smaller than $\bar{\alpha}$.

The density character $\operatorname{dens} X$ of a Banach space X is the smallest cardinal number δ such that there exists a set of cardinality δ dense in X.

Let γ be an ordinal. A γ-sequence in a set A is the image of a function $f : [1, \gamma[\to A$ and will be denoted by $(x_\theta)_{\theta<\gamma}$. If A is a topological space and β is an ordinal, we will say that the γ-sequence is β-continuous if, for every β-sequence of ordinals $(\theta_\xi)_{\xi<\beta}$ on $[1, \gamma]$ that converges to θ_β when ξ converges to β, we have that x_{θ_ξ} converges to x_{θ_β}.

Let X be a Banach space. By X_s we will denote the set of $F \in X^{**}$ having the following property: for every ω-sequence $(x_n^*)_{n<\omega}$ in X^* such that $x_n^*(x) \xrightarrow{n\to\omega} 0$, for all $x \in X$, we have $F(x_n^*) \xrightarrow{n\to\omega} 0$. X is said to have Mazur's property (also, d-complete [**9**] or μB-spaces [**19**]) if $X_s = cX$, where cX is the canonical image of X in X^{**}. The class of Banach spaces with Mazur's property includes the weakly compactly generated (WCG) Banach spaces and, therefore, the separable Banach spaces. See [**11**] for more examples of Banach spaces having this property.

Let α be a non-denumerable regular ordinal, φ any ordinal, and X a Banach space. By X_α^φ we will denote the set of $F \in X^{**}$ having the following property: for every limit ordinal $\beta < \alpha$ and for every φ-sequence $x^\eta = (x_\xi^*(\eta))_{\xi<\beta}$ of β-sequences of X^* such that there exists $K \in \mathbb{R}$ with $\|x_\xi^*(\eta)\| \leqslant K$, $\forall \eta < \varphi$, $\forall \xi < \beta$ and such that $x_\xi^*(\eta)(x) \xrightarrow{\xi\to\beta} 0$, $\forall x \in X$, uniformly in η, we have $F(x_\xi^*(\eta)) \xrightarrow{\xi\to\beta} 0$ uniformly in η.

We say that the Banach spaces X and Y are totally incomparable if X and Y have no isomorphic closed subspaces of infinite dimension.

If $T : X \to Y$ is a surjective bounded linear operator and B_X and B_Y are the closed unit balls of X and Y respectively, we define $r_0(T) = \inf\{r > 0 : B_Y \subset rT(B_X)\}$.

Other notations are standard in conformity with [**18**].

3. Isomorphic classification of the Banach spaces $C(I)^\xi$, $\xi \geqslant \omega_1$

Our main aim here is to prove the following theorem, which provides the isomorphic classification of the Banach spaces $C(I)^\xi$, $\xi \geqslant \omega_1$, and for that we will suppose the continuum hypothesis, that is $2^{\aleph_0} = \aleph_1$.

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Theorem 3.1. *Let α be an initial non-denumerable ordinal and X a separable Banach space with $X^\omega \sim X$ and* $\operatorname{dens} X^* = 2^{\aleph_0}$.

(1) *If α is singular, then $X^\xi \sim X^\eta$ with $\xi \leqslant \eta$ and $\bar{\xi} = \bar{\eta} = \bar{\alpha}$ if and only if $\eta < \xi^\omega$.*

(2) *If α is regular, then*

(a) $X^\alpha \sim X^{\alpha\eta}$, *with $1 \leqslant \eta \leqslant \omega_1$ if and only if $\eta < \omega_1$;*

(b) $X^{\alpha\xi} \sim X^{\alpha\eta}$, *with $\omega_1 \leqslant \xi \leqslant \eta \leqslant \alpha$ if and only if $\bar{\xi} = \bar{\eta}$;*

(c) $X^\xi \sim X^\eta$, *with $\alpha^2 \leqslant \xi \leqslant \eta$ and $\bar{\xi} = \bar{\eta} = \bar{\alpha}$ if and only if $\eta < \xi^w$.*

We will need some auxiliary results.

Lemma 3.2. *Let X be a Banach space having Mazur's property and γ be any ordinal, then*

$$\frac{(X^\gamma)^\omega_{\omega_1}}{cX^\gamma} \sim C_0(\wedge^{\omega_1}_\gamma, X).$$

Proof. This is similar to the proof of Corollary 2.8 in [**7**], only noticing that the Statement (b) of the proof of Proposition 2b is also true in this case, since, if $F \in (X^\gamma)^\omega_{\omega_1}$ and H is the canonical isomorphism from $\ell_1([1,\gamma], X^*)$ onto $(X^\gamma)^*$, then $H^*(F) = (F_\theta)_{\theta<\gamma+1}$ is β-continuous, $\forall\beta$, $\beta < \omega_1$. Indeed, let $\beta < \omega_1$ and $(\theta_\xi)_{\xi<\beta}$ be a β-sequence of ordinals in $[1,\gamma]$ converging to θ_β when ξ converges to β.

Now suppose that $(F_{\theta_\xi})_{\xi<\beta}$ does not converge to F_{θ_ξ} when ξ converges to β. So there is $\varepsilon > 0$ and a strictly increasing ω-sequence of ordinals $(\xi_n)_{n<\omega}$ converging to β and a ω-sequence $(x^*_n)_{n<\omega}$ of elements of the unit ball of X^* such that $\|F_{\theta_{\xi_n}}(x^*_n) - F_{\theta_\beta}(x^*_n)\| \geqslant \varepsilon$. Let $P^n_{\xi_m}$ be in $\ell_1([1,\gamma], X^*)$ defined by $P^n_{\xi_m}(\theta) = x^*_n$ if $\theta = \theta_{\xi_m}$ and $P^n_{\xi_m}(\theta) = 0$ if $\theta = \theta_{\xi_m}$, $\forall n$, $m < \omega$, so $HP^n_{\xi_m}(f) = x^*_n f(\theta_{\xi_m})$ and, therefore, $|H(P^n_{\theta_{\xi_m}} - P^n_{\theta_\beta})(f)| \leqslant \|x^*_n\|\,\|f(\theta_{\xi_m}) - f(\theta_\beta)\| \xrightarrow{m\to\omega} 0$, $\forall f$, uniformly in n and $\|H(P^n_{\theta_{\xi_m}} - P^n_{\theta_\beta})\| \leqslant 2\|H\|$, $\forall n$, $m < \omega$. Thus, $FH(P^n_{\theta_{\xi_m}} - P^n_{\theta_\beta}) = F_{\theta_{\xi_m}}(x^*_n) - F_{\theta_\beta}(x^*_n) \xrightarrow{m\to\omega} 0$ uniformly in n, which is absurd. □

Now, we remark that the argument presented in the proof of Lemma 2 in [**1**] also proves the following result.

Lemma 3.3. *Let ξ be a limit ordinal and X a Banach space. If, for every $\beta < \xi$, $\mathbb{R}^\xi \not\hookrightarrow X^\beta$ holds, then $\mathbb{R}^{\xi^\omega} \not\hookrightarrow X^\xi$.*

Lemma 3.4. *Let ξ be a non-denumerable ordinal and X a separable Banach space, then $\mathbb{R}^{\xi^\omega} \not\hookrightarrow X^\xi$.*

Proof. Let us suppose that (a) $\mathbb{R}^{\xi^\omega} \hookrightarrow X^\xi$, thus $\mathbb{R}^\xi \hookrightarrow \mathbb{R}^{\xi^\omega} \hookrightarrow X^\xi$, so we can consider $\xi_0 = \min\{\theta : \exists\, m,\, m < \omega,\, \mathbb{R}^\xi \hookrightarrow (\hat{\otimes}_m X)^\theta\}$. Let m_0, $m_0 < \omega$, be such that (b) $\mathbb{R}^\xi \hookrightarrow (\hat{\otimes}_{m_0} X)^{\xi_0}$. It suffices to show that ξ_0 is finite to come to a contradiction, because, in this case, $(\hat{\otimes}_{m_0} X)^{\xi_0}$ is separable and $\mathbb{R}^\xi$ is not.

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We suppose that ξ_0 is infinite and we note that (c) $\mathbb{R}^{\xi_0} \not\hookrightarrow (\hat{\hat{\otimes}}_{m_0+1}X)^{\beta}$, $\forall\beta$, $\beta<\xi_0$. Indeed, otherwise there exists β_1, $\beta_1<\xi_0$ such that by using item (b), Theorem 20.5.6 in [**18**] and Proposition 7 in [**4**, p. 225], we have

$$\begin{aligned}\mathbb{R}^{\xi} \hookrightarrow (\hat{\hat{\otimes}}_{m_0}X)^{\xi_0} = \mathbb{R}^{\xi_0}\hat{\hat{\otimes}}(\hat{\hat{\otimes}}_{m_0}X) &\hookrightarrow (\hat{\hat{\otimes}}_{m_0+1}X)^{\beta_1}\hat{\hat{\otimes}}(\hat{\hat{\otimes}}_{m_0}X)\\ &= \mathbb{R}^{\beta_1}\hat{\hat{\otimes}}(\hat{\hat{\otimes}}_{m_0+1}X)\hat{\hat{\otimes}}_{m_0}X = (\hat{\hat{\otimes}}_{2m_0+1}X)^{\beta_1},\end{aligned}$$

which is absurd because of the choice of ξ_0. We state that ξ_0 is a limit ordinal. Indeed, if $\xi_0=\xi_1+n$, for some n, $0\leqslant n<\omega$ and ξ_1 infinite, then $n+\xi_1=\xi_1$ and, from Property II in [**1**, p. 54], it follows that $(\hat{\hat{\otimes}}_{m_0}X)^{\xi_0}\sim(\hat{\hat{\otimes}}_{m_0}X)^{\xi_1}$, so, by the minimality of ξ_0, we conclude that $n=0$.

We can use Lemma 3.3 and item (c) to conclude that (d) $\mathbb{R}^{\xi_0^w}\not\hookrightarrow(\hat{\hat{\otimes}}_{m_0+1}X)^{\xi_0}$. Since $\xi_0\leqslant\xi$, and bearing (a) and (b) in mind, we have

$$\mathbb{R}^{\xi_0^\omega}\hookrightarrow\mathbb{R}^{\xi^\omega}\hookrightarrow X^{\xi}=\mathbb{R}^{\xi}\hat{\hat{\otimes}}X\hookrightarrow(\hat{\hat{\otimes}}_{m_0}X)^{\xi_0}\hat{\hat{\otimes}}X=\mathbb{R}^{\xi_0}\hat{\hat{\otimes}}(\hat{\hat{\otimes}}_{m_0}X)\otimes X=(\otimes_{m_0+1}X)^{\xi_0},$$

which is absurd because of (d).

So ξ_0 must be finite. $\square$

Proof of Theorem 3.1.
(1) Let α be singular. If $X^{\xi}\sim X^{\eta}$ with $\xi\leqslant\eta$ and we also suppose $\xi^{\omega}\leqslant\eta$, then

$$\mathbb{R}^{\xi^\omega}\hookrightarrow\mathbb{R}^{\eta}\hookrightarrow X^{\eta}\sim X^{\xi},$$

which is absurd by Lemma 3.4.

Conversely, if $\xi\leqslant\eta$, $\bar{\xi}=\bar{\eta}=\bar{\alpha}$ and $\eta<\xi^{\omega}$, then from Theorem 1 in [**10**] follows that $\mathbb{R}^{\xi}\sim\mathbb{R}^{\eta}$, so $\mathbb{R}^{\xi}\hat{\hat{\otimes}}X\sim\mathbb{R}^{\eta}\hat{\hat{\otimes}}X$, that is, $X^{\xi}\sim X^{\eta}$.

(2) Let α be regular.

(a) If $X^{\alpha}\sim X^{\alpha\eta}$ with $1\leqslant\eta\leqslant\omega_1$, then we consider two cases. If $\alpha=\omega_1$, then, from Remark 2.3 in [**7**], we have

$$\frac{(X^{\omega_1})^{\omega}_{\omega_1}}{cX^{\omega_1}}\sim\frac{(X^{\omega_1\eta})^{\omega}_{\omega_1}}{cX^{\omega_1\eta}}.$$

Then, by Lemma 3.2, $C_0(\wedge^{\omega_1}_{\omega_1},X)\sim C_0(\wedge^{\omega_1}_{\omega_1\eta},X)$, that is $X\sim C_0(\Gamma,X)$, where $\Gamma=[1,\eta]$ (see [**10**]). Since X is separable, we conclude that $\eta<\omega_1$.

If $\alpha>\omega_1$, then again from Remark 2.3 in [**7**] it follows that

$$\frac{(X^{\alpha})^{\omega_1}_{\alpha}}{cX^{\alpha}}\sim\frac{(X^{\alpha\eta})^{\omega_1}_{\alpha}}{cX^{\alpha\eta}}.$$

Since we have the hypothesis that $\dim X^*=2^{\aleph_0}=\aleph_1<\bar{\alpha}$, we can apply Corollary 2.8 of [**7**] to obtain $C_0(\wedge^{\alpha}_{\alpha},X)\sim C_0(\wedge^{\alpha}_{\alpha\eta},X)$, that is $X\sim C_0(\Gamma,X)$, where $\Gamma=[1,\eta]$; the separability of X implies $\eta<\omega_1$.

Conversely, let η be $1\leqslant\eta<\omega_1$. It suffices to prove that

$$X^{\alpha}\sim X^{\alpha\theta}\quad\forall\theta,\ \omega\leqslant\theta<\omega_1. \tag{3.1}$$

Indeed, if $1 \leqslant n < \omega$, from $X^\alpha \sim X^{\alpha\omega}$, Property III in [**1**, p. 54] and Theorem 2 in [**10**], it follows that

$$X^{\alpha n} \sim (X^\alpha)^n \sim (X^{\alpha\omega})^n \sim X^{(\alpha\omega)n} \sim X^{\alpha(\omega n)} \sim \mathbb{R}^{\alpha(\omega n)} \hat{\hat{\otimes}} X \sim \mathbb{R}^{\alpha\omega} \hat{\hat{\otimes}} X \sim X^\alpha.$$

To see (3.1), firstly we note that

$$\mathbb{R}^{\alpha\omega} \sim \mathbb{R}_0^{\alpha\omega} \sim (\mathbb{R}_0^\alpha)_0^\omega \sim (\mathbb{R}^\alpha)^\omega \sim \mathbb{R}^\alpha \hat{\hat{\otimes}} \mathbb{R}^\omega.$$

The first and the third isomorphisms are Remark 2.1 in [**1**, p. 55], the second isomorphism follows from Corollary 3.1 in [**10**], and the fourth isomorphism follows from Corollary 7.7.6 and Theorem 20.5.6 in [**18**].

Finally, let θ be $\omega \leqslant \theta < \omega_1$, thus

$$\begin{aligned} X^{\alpha\theta} &\sim \mathbb{R}^{\alpha\theta} \hat{\hat{\otimes}} X \sim \mathbb{R}^{\alpha\omega} \hat{\hat{\otimes}} X \sim (\mathbb{R}^\alpha \hat{\hat{\otimes}} \mathbb{R}^\omega) \hat{\hat{\otimes}} X \\ &\sim \mathbb{R}^\alpha \hat{\hat{\otimes}} (\mathbb{R}^\omega \hat{\hat{\otimes}} X) \sim \mathbb{R}^\alpha \hat{\hat{\otimes}} X^\omega \sim \mathbb{R}^\alpha \hat{\hat{\otimes}} X \sim X^\alpha. \end{aligned}$$

The second isomorphism follows from Theorem 2 in [**10**].

(b) If $X^{\alpha\xi} \sim X^{\alpha\eta}$ with $\omega_1 \leqslant \xi \leqslant \eta \leqslant \alpha$, we can suppose $\alpha > \omega_1$, because, if $\alpha = \omega_1$, then $\xi = \eta = \omega_1$ and we have nothing to prove. So as in the proof of the second case in (a) we obtain that $C_0(\wedge_{\alpha\xi}^\alpha, X) \sim C_0(\wedge_{\alpha\eta}^\alpha, X)$, that is $C_0(\Gamma_1, X) \sim C_0(\Gamma_2, X)$, where $\Gamma_1 = [1, \xi]$ and $\Gamma_2 = [1, \eta]$, thus, the separability of X implies $\bar{\xi} = \bar{\eta}$.

Conversely, if $\omega_1 \leqslant \xi \leqslant \eta \leqslant \alpha$ with $\bar{\xi} = \bar{\eta}$, then Theorem 2 in [**10**] implies that $\mathbb{R}^{\alpha\xi} \sim \mathbb{R}^{\alpha\eta}$, so $X \hat{\hat{\otimes}} \mathbb{R}^{\alpha\xi} \sim X \hat{\hat{\otimes}} \mathbb{R}^{\alpha\eta}$.

(c) If $X^\xi \sim X^\eta$ with $\xi \leqslant \eta$, then, as it was proved in the case in which α is singular, we have $\eta < \xi^\omega$. Conversely, if $\alpha^2 \leqslant \xi \leqslant \eta$, $\bar{\xi} = \bar{\eta} = \bar{\alpha}$, then Theorem 2 in [**10**] implies that $\mathbb{R}^\xi \sim \mathbb{R}^\eta$, so $X \hat{\hat{\otimes}} \mathbb{R}^\xi \sim X \hat{\hat{\otimes}} \mathbb{R}^\eta$. □

Question 3.5. Give an isomorphic classification of the Banach spaces $C(I)^\xi$, $\xi \geqslant \omega_1$, without using the continuum hypothesis.

Remark 3.6. For each γ, $1 \leqslant \gamma < \omega_{\xi+1}$, where $\omega_{\xi+1}$ is the first ordinal of cardinality $\aleph_{\xi+1}$, we define $K_\gamma = [1, \omega_\xi^{\omega^\gamma}] \times I$, ω_ξ being the first ordinal of cardinality $\aleph_\xi$. It follows from Lemma 3.4 that $C(K_{\eta_1}) \not\hookrightarrow C(K_{\xi_1})$, for every $1 \leqslant \xi_1 < \eta_1 < \omega_{\xi+1}$.

Indeed, let $\theta_{\xi_1} = \omega_\xi^{\omega^{\xi_1}}$ and $\theta_{\gamma_1} = \omega_\xi^{\omega^{\gamma_1}}$, thus $\theta_{\xi_1}^\omega = \omega_\xi^{\omega^{\xi_1+1}} \leqslant \theta_{\gamma_1}$. If $C(K_{\eta_1}) \hookrightarrow C(K_{\xi_1})$, then

$$\mathbb{R}^{\theta_{\xi_1}^\omega} \hookrightarrow \mathbb{R}^{\theta_{\gamma_1}} \hookrightarrow C(K_{\eta_1}) \hookrightarrow C(K_{\xi_1}) = C(I)^{\theta_{\xi_1}},$$

which is a contradiction.

So, for each $\aleph_\xi \geqslant 2^{\aleph_0}$, there exists at least $\aleph_{\xi+1}$ perfect compacts K of the cardinality $\aleph_\xi$, such that $C(K)$ are isomorphically different.

Question 3.7. Under the continuum hypothesis, are there more than $\aleph_2$ perfect compacts K of cardinality $2^{\aleph_0}$, such that $C(K)$ are isomorphically different?

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4. Characterization of the separable Banach spaces satisfying $X^\xi \sim X$, $\forall \xi$, $\omega \leqslant \xi < \omega_1$

If X is a Banach space and K a compact, $C(K, X)$ will indicate the Banach space of the continuous X-valued functions defined on K and equipped with the supremum norm.

It follows from the Milutin's Theorem that if X is isomorphic to $C(I)$, then X satisfies the following equation: $X^\xi \sim X$, $\forall \xi$, $\omega \leqslant \xi < \omega_1$. In this section we will prove Theorem 4.1, which gives an isomorphic characterization of the separable Banach spaces satisfying such an equation.

Theorem 4.1. *Let X be a separable Banach space. X satisfies the equation $X^\xi \sim X$, $\forall \xi$, $\omega \leqslant \xi < \omega_1$, if and only if $C(I, X) \sim X$.*

Proof. If X is a Banach space satisfying

$$C(I, X) \sim X \quad \text{and} \quad \xi, \quad \omega \leqslant \xi < \omega_1, \tag{4.1}$$

then, from Lemma 21.5.1 of [**18**], we have $\mathbb{R}^\xi \hat{\hat{\otimes}} C(I, X) \sim \mathbb{R}^\xi \hat{\hat{\otimes}} X$. Now, from Theorem 20.5.6 of [**18**], we obtain $\mathbb{R}^\xi \hat{\hat{\otimes}} C(I) \hat{\hat{\otimes}} X \sim X^\xi$. So, from Milutin's Theorem, $C(I, X) \sim X^\xi$. But, bearing in mind (4.1), we conclude that $X \sim X^\xi$. $\square$

The converse follows immediately from the following proposition.

Proposition 4.2. *Let X and B be separable Banach spaces with $X^\xi \sim B$, $\forall \xi$, $\omega \leqslant \xi < \omega_1$, then $C(I, X) \sim B$.*

Proof. (Inspired by [15].) Let 2^I be the space of all compact subsets of I endowed with the Hausdorff metric

$$d(A, B) = \max\{\max_{a \in A} \operatorname{dist}(a, B), \max_{b \in B} \operatorname{dist}(b, A)\}.$$

Let $Y = \{K \in 2^I : C(K, X) \sim B\}$. For each $n < \omega$, $Y_n = \{K \in 2^I : \exists\, \bar{T} : C(I, X) \to B$ a bounded linear operator, $\|\bar{T}\| \leqslant 1$ and $\bar{L} : B \to C(I, X)$ a bounded linear operator satisfying $(1/n)\|f_{|K}\| \leqslant \|\bar{T}(f)\| \leqslant \|f_{|K}\|$, $\forall f \in C(I, X)$ and $\bar{T}\bar{L}(b) = b$, $\forall b \in B$, $\|\bar{L}\| \leqslant n\}$.

Firstly, we remark that $Y = \bigcup_{n<\omega} Y_n$. Indeed, supposing that $K \in Y$, there exists $T : C(K, X) \to B$, an isomorphism onto the image (we can suppose $\|T\| \leqslant 1$), and $L : B \to C(K, X)$, a bounded linear operator (L is the inverse of T), satisfying $TL(b) = b$, $\forall b \in B$.

Let $n < \omega$ be such that $\|L\| \leqslant n$. We define $\bar{T} : C(I, X) \to B$ by $\bar{T}(g) = T(g_{|K})$ and $L : B \to C(I, K)$, by $\bar{L}(b) = EL(b)$, where E is a linear extension operator (see [**18**, p. 365]), so $\|g_{|K}\| = \|LT(g_{|K})\| \leqslant n\|T(g_{|K})\|$,

(I) $(1/n)\|g_{|K}\| \leqslant \|\bar{T}(g)\| = \|T(g_{|K})\| \leqslant \|g_{|K}\|$, $\forall g \in C(I, X)$, and

(II) $\bar{T}\bar{L}(b) = \bar{T}EL(b) = T(EL(b)_{|K}) = TL(b) = b$, $\forall b \in b$, that is $K \in Y_n$.

Conversely, supposing $K \in Y_n$ for some $n < \omega$, we define $T : C(K,X) \to B$ by $T = \bar{T}E$, where E is a linear extension operator and $L : B \to C(K,X)$ by $L = R\bar{L}$, where R is an operator defined by $R(f) = f_{|K}$, $\forall f \in C(I,X)$. Let $f \in C(K,X)$ and let $b \in B$, then

(III) $(1/n)\|f\| = (1/n)\|E(f)_{|K}\| \leqslant \|T(f)\| = \|\bar{T}E(f)\| \leqslant \|E(f)_{|K}\| = \|f\|$, that is T is an isomorphism onto the image.

(IV) $TL(b) = \bar{T}ER\bar{L}(b)$ and, since $\bar{L}(b)_{|K} = (ER\bar{L}(b))_{|K}$ and $\|\bar{T}(f)\| \leqslant \|f_{|K}\|$, $\forall f \in C(I,K)$, it follows that $\bar{T}(ER\bar{L}(b)) = \bar{T}(\bar{L}(b)) = b$ and, therefore, $TL(b) = b$, $\forall b \in B$, so the image of T is B, consequently $B \in Y$.

Next we will remark that each Y_n is analytic. Let A be the unit ball of $L(C(I,X),B)$ in the pointwise convergence topology, and let J be the ball of radius n of $L(B,C(I,X))$ also in the pointwise convergence topology.

We consider the Polish space $Z = 2^I \times A \times J$ (see [**3**, p. 195]). Let $Q = \{(K,T,L) \in Z : (1/n)\|f_{|K}\| \leqslant \|T(f)\| \leqslant \|f_{|K}\|, \forall f \in C(I,X) \text{ and } TL(b) = b, \forall b \in B\}$. Q is a closed subset of Z, because, if $(K_\gamma, T_\gamma, L_\gamma)$ is a net of Q converging to (K,T,L) in Z, it follows that $(1/n)\|f_{|K}\| \leqslant \|T(f)\| \leqslant \|f_{|K}\|$, $\forall f \in C(I,X)$, from the pointwise convergence of T_γ.

Next we show that $TL(b) = b$, $\forall b \in B$.

Let $b \in B$, from $T_\gamma \to T$, it follows that

(1) $T_\gamma L(b) \to TL(b)$,

and, from $L_\gamma \to L$, it follows that $L_\gamma(b) \to L(b)$ and, therefore,

(2) $TL_\gamma(b) \to TL(b)$.

Now, using

$$\|T_\gamma L_\gamma(b) - TL_\gamma(b)\| \leqslant \|T_\gamma\|\,\|L_\gamma(b) - L(b)\| + \|T\|\,\|L_\gamma(b) - L(b)\| + \|T_\gamma L(b) - TL(b)\|$$

with (1) and (2), we conclude that $b = T_\gamma L_\gamma(b) \to TL(b)$, that is $TL(b) = b$. So $(K,T,L) \in Q$ and, therefore, Q is closed, consequently (see [**3**, p. 195]) Q is a Polish space. Since Y_n is a projection of Q onto the 2^I axis, Y_n is analytic and, therefore, so is Y (see [**3**, p. 195]).

Let $D = \{K \in 2^I : K \text{ is countable}\}$, by the hypothesis of the proposition, it follows that $D \subset Y$ (see [**18**, p. 155]), and, since D is non-analytic (see [**8**]), there must be a non-denumerable compact subset K of I, such that $C(K,X) \sim B$, and, from Milutin's Theorem and Corollary 21.5.2 of [**18**], it follows that $C(I,X) \sim B$. □

Remark 4.3. Let Y be a Banach space. It follows from Milutin's Theorem and from properties of injective tensor products that $X = C(I,Y)$ satisfies

$$C(I,X) \sim X. \tag{4.2}$$

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It will be shown (Corollary 5.6) that there exists a separable Banach space $W, C(I) \not\twoheadrightarrow W$, such that $X = C(I) \oplus W$ does not satisfy (4.2). (To see this, bear in mind that if X satisfies (4.2), then $X^\xi \sim X^\eta$, $\forall \xi, \eta$, $\omega \leqslant \xi \leqslant \eta < \omega_1$.)

Now, since $C(I) \oplus C(I, Y) \sim C(I, Y)$, we have that $X = C(I) \oplus Y$ satisfies (4.2) if and only if

$$C(I, Y) \sim C(I) \oplus Y. \tag{4.3}$$

If Y is isomorphic to a complemented subspace of $C(I)$ and $C(I) \not\twoheadrightarrow Y$, then Y satisfies (4.3). Indeed, from Corollary 1 of [**12**], we have $C(I) \oplus Y \sim C(I)$ and, therefore, $C(I, C(I) \oplus Y) \sim C(I, C(I))$, that is $C(I, Y) \sim C(I)$.

Question 4.4. Let Y be a separable Banach space, $C(I) \not\twoheadrightarrow Y$, such that $C(I, Y) \sim C(I) \oplus Y$. Is it true that Y is isomorphic to a complemented subspace of $C(I)$?

Question 4.5. Give a Banach space X such that $X^{\omega_1} \sim X$.

5. ω_1 cancellable Banach spaces

Next we present two Banach spaces X containing subspaces isomorphic to $C(I)$ such that the isomorphic classifications of X^ξ, $\omega \leqslant \xi < \omega_1$ are similar to that of $\mathbb{R}^\xi$ given by Bessaga and Pelczynski in [**1**]. The first space is $C(I) \oplus F^*$, where F is the Banach space considered by Figiel in [**6**] (Corollary 5.6), and the second is $\ell_\infty(\mathbb{N})$ (Corollary 5.8).

These results will be consequences of Theorem 5.1, and, in order to prove it, we will need some auxiliary results.

Theorem 5.1. *Let X^* and F be totally incomparable Banach spaces satisfying $X \twoheadrightarrow X^\xi$, $\forall \xi$, $\omega \leqslant \xi < \omega_1$, F^* uniformly convex, $F^{n+1} \not\twoheadrightarrow F^n$, $\forall n$, $n < \omega$. If $(X \oplus F^*)^\xi \twoheadrightarrow (X \oplus F^*)^\eta$, with $\omega \leqslant \xi \leqslant \eta < \omega_1$, then $\eta < \xi^\omega$.*

Lemma 5.2. *Let X, Y and Z Banach spaces, $T : X \oplus Y \to Z$ is a bounded linear operator such that $i_1^* T^* : Z^* \to X^*$ is an isomorphism onto the image, where i_1 is the canonical inclusion from X in $X \oplus Y$. Then $T_1 : X \to Z$ defined by $T_1(x) = T(x, 0)$, $\forall x \in X$ is onto Z.*

Proof. From $T_1^*(z^*)(x) = z^*(T_1(x)) = z^*(T(i_1(x))) = i_1^*(T^*(z^*(x)))$, $\forall z^* \in Z$, and $\forall x \in X$, it follows that T_1^* is one-to-one and has a closed image. Then from Lemma 3 of [**5**, p. 488], we have that T_1 is onto Z. □

Lemma 5.3. *Let γ be a denumerable ordinal, X and Y Banach spaces and $T : X_0^\gamma \oplus Y \to Z$ a surjective bounded linear operator. Let $\beta < \gamma$ be such that $T|_{X_0^\beta \oplus Y} : X_0^\beta \oplus Y \to Z$ is not surjective and $r > r_0(T)$, then, for every ε, $0 < \varepsilon < 1$, there exists $g \in X_0^\gamma$ with $g(\xi) = 0$, $\forall \xi$, $\xi \leqslant \beta$, $\|g\| \leqslant r$ and $\|T(g)\| \geqslant \varepsilon$.*

Proof. Let ε be such that $0 < \varepsilon < 1$; choosing $\delta = 1 - \varepsilon$ and writing $X_0^\gamma \oplus Y = X_0^\beta \oplus W \oplus Y$, where $W = \{f \in X_0^\gamma : f(\xi) = 0, \forall \xi, \xi \leqslant \beta\}$ and indicating by i_1 the canonical inclusion from $X_0^\beta \oplus Y$ to $X_0^\gamma \oplus Y$, it follows from the previous lemma that

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there exists $z^* \in Z^*$, $\|z^*\| = 1$ such that $\|i_1^* T^* z^*\| \leqslant (\delta/2r)$, so, for every $f+y$ in $X_0^\beta \oplus Y$ with $\|f+y\| \leqslant r$, we have $\|z^* T(f+y)\| \leqslant \frac{1}{2}\delta$.

Let $z \in Z$, $\|z\| = 1$ such that $\|z^*(z)\| \geqslant 1 - \frac{1}{2}\delta$, since T is surjective, then there exists $g^1 + y_1$ in $X_0^\gamma \oplus Y$, $\|g^1 + y_1\| \leqslant r$ such that $T(g^1 + y_1) = z$. Let $g_1 = g^1_{|_{[\beta+1,\gamma]}}$, thus:

$$1 - \tfrac{1}{2}\delta \leqslant \|z^* T(g^1 + y_1)\| \leqslant \|z^* T(g^1 - g_1 + y_1)\| + \|z^* T(g_1)\| \leqslant \tfrac{1}{2}\delta + \|T(g_1)\|,$$

so $\|T(g_1)\| \geqslant \varepsilon$. □

Proposition 5.4. *Let X, Y and Z be Banach spaces, Y uniformly convex, $Z \twoheadrightarrow Y$ and α an infinite denumerable ordinal such that $\forall \beta$, $\beta < \alpha$, $Z^\beta \oplus X \not\twoheadrightarrow Y^\alpha$. Then $Z^\alpha \oplus X \not\twoheadrightarrow Y^{\alpha^\omega}$.*

Proof. (Inspired by [16].) Y being uniformly convex, it follows from Pisier's Theorem (see [**14**, p. 803]) that Y admits an equivalent norm (that will be denoted by $\|\cdot\|$) and there exists $\delta > 0$ and $p \in \mathbb{R}$, $2 < p < +\infty$ such that if $b \in \mathbb{R}_+$ and y_1, $y_2 \in Y$ with $\|y_1\| \geqslant 1$ and $\|y_2\| \geqslant \sqrt[p]{b}$, then either $\|y_1 + y_2\| \geqslant \sqrt[p]{1+\delta b}$ or $\|y_1 - y_2\| \geqslant \sqrt[p]{1+\delta b}$. So, given $y_1, y_2, \dots, y_n \in Y$, with $\|y_1\| = 1$, $\|y_i\| \geqslant \sqrt[p]{b}$, $i = 2, 3, \dots, N$, there exists $c_i \in \mathbb{R}$, $|c_i| = 1$, $i = 1, 2, \dots, N$, such that $\|\sum_{i=1}^N c_i y_i\| \geqslant \sqrt[p]{1 + (N-1)b\delta}$.

Let $\alpha = \omega^{\alpha_1} n_1 + \omega^{\alpha_2} n_2 + \cdots + \omega^{\alpha_k} n_k$ be in the Cantor normal form (see [**18**, p. 153]), so $Z^\alpha \sim Z^{\omega^{\alpha_1}}$ (see [**1**]), and, therefore, $Z^{\omega^{\alpha_1}} \oplus X \twoheadrightarrow Z^{\omega^{\alpha_1}} \twoheadrightarrow Y^{\omega^{\alpha_1}} = Y^\alpha$. By the hypothesis of the proposition, we cannot have $\omega^{\alpha_1} < \alpha$, i.e. $\alpha = \omega^{\alpha_1}$, and so α is a prime component ordinal (see [**18**, p. 153]).

We also know that $\alpha^\omega = \omega^{\alpha_1 \omega}$ and $Z^{\omega^{\alpha_1}} \sim Z_0^{\omega^{\alpha_1}}$ (see [**1**]). Then denying the thesis of the proposition, there will be T, a surjective bounded linear operator from $Z_0^{\omega^{\alpha_1}} \oplus X$ onto $Y^{\omega^{\alpha_1 \omega}}$.

Let $r > r_0(T)$ and ε, $0 < \varepsilon < 1$; we choose s, $s > 0$ and N, $1 \leqslant N < \omega$ such that $\varepsilon + s < 1$ and $\|T\|(r+\varepsilon) < \sqrt[p]{1 + (N-1)\varepsilon\delta}$. Let $y_1 \in Y$, $\|y_1\| = 1$ and $h \in Y^{\omega^{\alpha_1 \omega}}$ defined by $h(\gamma) = y_1$, $\forall \gamma$. There exists g^1, $g^1 \in Z_0^{\omega^{\alpha_1}} \oplus X$ such that $\|g^1\| \leqslant r$ and $T(g^1) = h_1$. Writing $g^1 = g_1 + f_1$, with $g_1 \in Z_0^{\omega^{\alpha_1}}$ and $f_1 \in X$, we get γ_1, $\gamma_1 < \omega^{\alpha_1}$ such that $\|g_1(\gamma)\| \leqslant (\varepsilon/N)$, $\forall \gamma \in [\gamma_1 + 1, \omega^{\alpha_1}]$.

Let us denote for every $\beta \in [0, \omega^{\alpha_1}[$, $\Delta_\beta^1 = [\omega^{\alpha_1(N-1)}\beta + 1, \omega^{\alpha_1(N-1)}(\beta+1)]$ and writing $W_1 = \{f \in Y^{\omega^{\alpha_1 N}} : \forall \beta,\ \beta \in [0, \omega^{\alpha_1}[,\ f \text{ is constant in } \Delta_\beta^1\}$. Let $P_1 : Y^{\omega^{\alpha_1 \omega}} \to W_1$ be the bounded linear operator defined by

$$P_1(f)(\gamma) = \begin{cases} 0, & \text{if } \gamma \in [\omega^{\alpha_1 N} + 1, \omega^{\alpha_1 \omega}], \\ f(\gamma), & \text{if } \gamma = \omega^{\alpha_1(N-1)}\beta, \text{ with } 1 \leqslant \beta \leqslant \omega^{\alpha_1}, \\ f(\omega^{\alpha_1(N-1)}(\beta+1)), & \text{if } \gamma = \omega^{\alpha_1(N-1)}\beta + \xi, \\ & \quad \text{with } 0 \leqslant \beta < \omega^{\alpha_1} \text{ and } 1 \leqslant \xi < \omega^{\alpha_1(N-1)}, \end{cases}$$

$\forall f \in Y^{\omega^{\alpha_1 \omega}}$.

Clearly, W_1 is a subspace of $Y^{\omega^{\alpha_1 \omega}}$ isometric to $Y^{\omega^{\alpha_1}}$, and $P_1 T$ is a surjective bounded linear operator from $Z_0^{\omega^{\alpha_1}} \oplus X$ to W_1 such that $r_0(P_1 T) \leqslant r_0(T)$.

The previous lemma applied to γ_1 and $\sqrt[p]{\varepsilon+s}$ implies that there exists $g_2 \in Z_0^{\omega^{\alpha_1}}$, with $g_2(\xi)=0$, $\forall\xi$, $\xi\leqslant\gamma_1$, $\|g_2\|\leqslant r$ and $\|P_1T(g_2)\|\leqslant\sqrt[p]{\varepsilon+s}$; so there exists $\beta_1\in[1,\omega^{\alpha_1}[$ such that (1.1) $\|Tg_2(\omega^{\alpha_1(N-1)}\beta_1)\|\geqslant\sqrt[p]{\varepsilon+s}$, and we can suppose that β_1 is not a limit ordinal satisfying $\|Tg_2(\omega^{\alpha_1(N-1)}\beta_1)\|\geqslant\sqrt[p]{\varepsilon}$.

Let $\beta_1=\beta_1'+1$. Since $T(g_2)\in Y^{\omega^{\alpha_1\omega}}$ we can find λ_1, $\lambda_1\in[0,\omega^{\alpha_1}[$, such that for every γ, $\gamma\in[\omega^{\alpha_1(N-1)}\beta_1'+\omega^{\alpha_1(N-2)}\lambda_1+1,\ \omega^{\alpha_1(N-1)}\beta_1]$, we have $\|Tg_2(\gamma)\|\geqslant\sqrt[p]{\varepsilon}$. Since $g_2\in Z_0^{\omega^{\alpha_1}}$, there exists $\gamma_2\in[\gamma_1+1,\omega^{\alpha_1}]$ such that $\|g_2(\gamma)\|\leqslant(\varepsilon/N)$, $\forall\gamma$, $\gamma\in[\gamma_2+1,\omega^{\alpha_1}]$. Let us denote for every β, $\beta\in[\lambda_1,\omega^{\alpha_1}[$, $\Delta_\beta^2=[\omega^{\alpha_1(N-1)}\beta_1'+\omega^{\alpha_1(N-2)}\beta+1,\omega^{\alpha_1(N-1)}\beta_1'+\omega^{\alpha_1(N-2)}(\beta+1)]$, and, writing $W_2=\{f\in Y^{\omega^{\alpha_1\omega}}:\forall\beta,\ \beta\in[\lambda_1,\omega^{\alpha_1}[,\ f$ is constant in Δ_β^2, and $\forall\gamma,\ \gamma\notin[\omega^{\alpha_1(N-1)}\beta_1'+\omega^{\alpha_1(N-2)}\lambda_1+1,\omega^{\alpha_1(N-1)}\beta_1],\ f(\gamma)=0\}$. Let $P_2:Y^{\omega^{\alpha_1\omega}}\to W_2$ be the bounded linear operator defined by

$$P_2(f)(\gamma)=\begin{cases}0, & \text{if } \gamma\notin[\omega^{\alpha_1(N-1)}\beta_1'+\omega^{\alpha_1(N-2)}\lambda_1+1,\omega^{\alpha_1(N-1)}\beta_1],\\[2mm] f(\gamma), & \text{if } \gamma=\omega^{\alpha_1(N-1)}\beta_1'+\omega^{\alpha_1(N-2)}\beta,\ \text{with } \lambda_1+1\leqslant\beta\leqslant\omega^{\alpha_1},\\[2mm] f(\omega^{\alpha_1(N-1)}\beta_1'+\omega^{\alpha_1(N-2)}(\beta+1)), & \\ & \text{if } \gamma=\omega^{\alpha_1(N-1)}\beta_1'+\omega^{\alpha_1(N-2)}\beta+\xi,\\ & \qquad\text{with } \lambda_1\leqslant\beta<\omega^{\alpha_1} \text{ and } 1\leqslant\xi\leqslant\omega^{\alpha_1(N-2)},\end{cases}$$

$\forall f\in Y^{\omega^{\alpha_1\omega}}$ and $\forall\gamma$, $\gamma\in[1,\omega^{\alpha_1\omega}]$.

Since ω^{α_1} is a prime component ordinal, it follows that $[\lambda_1,\omega^{\alpha_1}]$ is homeomorphic to $[1,\omega^{\alpha_1}]$ and, therefore, W_2 is a subspace of $Y^{\omega^{\alpha_1\omega}}$ isometric to $Y^{\omega^{\alpha_1}}$ and P_2T is a surjective bounded linear operator from $Z_0^{\omega^{\alpha_1}}\oplus X$ onto W_2 such that $r_0(P_2T)\leqslant r_0(T)$.

The previous lemma applied to γ_2 and $\sqrt[p]{\varepsilon+s}$ implies that there exists $g_3\in Z_0^{\omega^{\alpha_1}}$ with $g_3(\xi)=0$, $\forall\gamma$, $\gamma\leqslant\gamma_2$, $\|g_3\|\leqslant r$ and $\|P_2Tg_3\|\leqslant\sqrt[p]{\varepsilon+s}$; so there exists $\beta_2\in[\lambda_1+1,\omega^{\alpha_1}]$ such that $\|T(g_3)(\omega^{\alpha_1(N-1)}\beta_1'+\omega^{\alpha_1(N-2)}\beta_2\|\geqslant\sqrt[p]{\varepsilon+s}$ and we can suppose that β_2 is not a limit ordinal satisfying $\|T(g_3)(\omega^{\alpha_1(N-1)}\beta_1'+\omega^{\beta_1(N-2)}\beta_2)\|\geqslant\sqrt[p]{\varepsilon}$. Let $\beta_2=\beta_2'+1$. Since $T(g_3)\in Y^{\omega^{\alpha_1\omega}}$, we can find λ_2, $\lambda_2\in[1,\omega^{\alpha_1}]$ such that $\forall\gamma$, $\gamma\in[\omega^{\alpha_1(N-1)}\beta_1'+\omega^{\alpha_1(N-2)}\beta_2'+\omega^{\alpha_1(N-3)}\lambda_2+1,\omega^{\alpha_1(N-1)}\beta_1'+\omega^{\alpha_1(N-2)}\beta_2'+1]$. We have $\|T(g_3)\|\geqslant\sqrt[p]{\varepsilon}$.

Repeating this procedure N times, we can find $g^1=g_1+f_1$, $g_1\in Z_0^{\omega^{\alpha_1}}$, $f_1\in X$, $g_2,g_3,\dots,g_N\in Z_0^{\omega^{\alpha_1}}$, ordinals $\gamma_1<\gamma_2<\dots<\gamma_{N-1}<\omega^{\alpha_1}$, non-empty intervals $\Delta_1=[1,\omega^{\alpha_1N}]$, $\Delta_2=[\omega^{\alpha_1(N-1)}\beta_1'+\omega^{\alpha_1(N_1)}\lambda_1+1,\omega^{\alpha_1(N-1)}\beta_1'+1],\dots,\Delta_n$, such that

(1) $\Delta_1\supset\Delta_2\supset\dots\supset\Delta_N$;

(2) $\|g^1\|\leqslant r$ and $\|g_i\|\leqslant r$, $i=1,2,\dots,N$;

(3) $\|g_i(\gamma)\|\leqslant(\varepsilon/N)$, $\forall\gamma$, $\gamma\in[\gamma_{i+1},\omega^{\alpha_1}]$, $i=1,2,\dots,N-1$;

(4) $g_i(\gamma)=0$, $\forall\gamma$, $\gamma\in[1,\gamma_{i+1}]$, $i=2,3,\dots,N$;

(5) $Tg^1(\xi)=y_1$, $\forall\xi\in\Delta_1$, $\|Tg_1(\gamma)\|\geqslant\sqrt[p]{\varepsilon}$, $\forall i=2,\dots,N$, and $\forall\gamma$, $\gamma\in\Delta_i$;

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(6) we take $\gamma \in \bigcap_{i=1}^{n} \Delta_i$, so, by the initial remark of this proof, there exists $c_i \in \mathbb{R}$, with $|c_i| = 1$, $i = 1, 2, \dots, N$, such that $\|c_1 T(g_1) + \cdots + c_N T(g_N)\| \geqslant \sqrt[p]{1 + (N-1)\delta\varepsilon}$;

(7) $\|c_1 g_1 + \cdots + c_N g_N\| = \max\{\|f_1\|, \|c_1 g_1 + \cdots + c_N g_N\|\} \leqslant r + \varepsilon$.

From (6) and (7) we conclude that $\sqrt[p]{1 + (N-1)\delta\varepsilon} \leqslant \|T\|(r + \varepsilon)$, which is absurd because of the choices of ε and N. $\square$

Corollary 5.5. *Let X and F be Banach spaces and α an infinite denumerable ordinal, F uniformly convex. If $F^\alpha \oplus X \twoheadrightarrow F^{\alpha^\omega}$, then there exists n, $m < \omega$, such that $F^n \oplus X^m \twoheadrightarrow F^{\alpha^\omega}$.*

Proof. Let $\alpha_0 = \min\{\xi : \exists\ m,\ m < \omega,\ X^m \oplus F^\xi \twoheadrightarrow F^\alpha\}$. So $\alpha_0 \leqslant \alpha$, and if we suppose that $\alpha_0 \geqslant \omega$, then there exists $n_0 < \omega$ such that (a) $X^{n_0} \oplus F^{\alpha_0} \twoheadrightarrow F^\alpha$, and (b) $\forall\beta$, $\beta < \alpha_0$, $F^\beta \oplus X^{n_0+1} \not\twoheadrightarrow F^{\alpha_0}$, otherwise there exists β, $\beta < \alpha_0$, such that $X^{2n_0+1} \oplus F^\beta \twoheadrightarrow X^{n_0+1} \oplus F^\beta \oplus X^{n_0} \twoheadrightarrow F^{\alpha_0} \oplus X^{n_0} \twoheadrightarrow F^\alpha$, which is absurd because of the choice of α_0. Therefore, from Proposition 5.4, it follows that $F^{\alpha_0} \oplus X^{n_0+1} \not\twoheadrightarrow F^{\alpha_0^\omega}$.

However, from (a) and our hypothesis it follows that $F^{\alpha_0} \oplus X^{n_0+1} = F^{\alpha_0} \oplus X \oplus X^{n_0} \twoheadrightarrow X \oplus F^\alpha \twoheadrightarrow F^{\alpha^w} \twoheadrightarrow F^{\alpha_0^\omega}$, which is absurd. Consequently, $\alpha_0 < \omega$, and again from (a) we have $X^{n_0} \oplus F^{\alpha_0} \oplus X \twoheadrightarrow F^\alpha \oplus X \twoheadrightarrow F^{\alpha^\omega}$. $\square$

Proof of Theorem 5.1. If $\xi^\omega \leqslant \eta$, then writing $G = F^*$, we have

$$X \oplus G^\xi \twoheadrightarrow X^\xi \oplus G^\xi \sim (X \oplus G)^\xi \twoheadrightarrow (X \oplus G)^\eta \twoheadrightarrow G^\eta \twoheadrightarrow G^{\xi^\omega}.$$

From Corollary 5.5 it follows that there exists n, $m < \omega$, such that $G^n \oplus X^m \twoheadrightarrow G^{\xi^\omega} \twoheadrightarrow G^\omega$. Now, bearing in mind that every uniformly convex Banach space is reflexive, see Proposition 1.e.3 of [**13**], we have $F^{n+2} \hookrightarrow (G^\omega)^* \hookrightarrow F^n \oplus (X^*)^m$, that is there exists $T : F^{n+2} \to F^n \oplus (X^*)^m$ an isomorphism onto the image; $T(F^{n+2})$ and $(X^*)^m$ being totally incomparable Banach spaces, it follows that $T(F^{n+2}) \cap (X^*)^m = V$, where $\dim V = p$, $p < \omega$ and, therefore, $T(F^{n+2}) = Z \oplus V$ for some Banach space Z. Noticing that $Z \subset (X^*)^m \oplus F^n$, $Z \cap (X^*)^m = \{0\}$ and Z and X^* are totally incomparable, we have from Lemma 1.1 in [**20**] that $Z \hookrightarrow F^n$, and so $T(F^{n+2}) = Z \oplus V \hookrightarrow F^n \oplus \mathbb{R}^p \hookrightarrow F^{n+1}$, which is absurd. $\square$

Let p be a real number, $p > 2$. It follows from the main result of [**6**] that there exists finite-dimensional uniformly convex Banach spaces X_i, $i = 1, 2, \dots$, such that if F is the p sum of these spaces, then $F^{n+1} \not\hookrightarrow F^n$, $\forall n$, $1 \leqslant n < \omega$.

It is well known that every infinite-dimensional subspace of F contains a subspace isomorphic to ℓ_p and that $\ell_p \not\hookrightarrow C(I)^*$ (see [**2**, p. 207]). So F and $C(I)^*$ are totally incomparable and we have the following corollary.

Corollary 5.6. $(C(I) \oplus F^*)^\xi \sim (C(I) \oplus F^*)^\eta$, *with $\omega \leqslant \xi \leqslant \eta < \omega_1$, if and only if $\eta < \xi^\omega$.*

Theorem 5.7. *Let ξ, $\omega \leqslant \xi < \omega_1$, and let X be a Banach space. If $X^\xi \twoheadrightarrow \mathbb{R}^{\xi^\omega}$, then $X \twoheadrightarrow \mathbb{R}^\omega$.*

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Proof. Let $\xi_0 = \min\{\eta \geqslant \omega : X^\eta \twoheadrightarrow \mathbb{R}^{\eta^\omega}\}$. If $\xi_0 > \omega$, then $X^\beta \not\twoheadrightarrow \mathbb{R}^{\xi_0}$, $\forall \beta, \omega \leqslant \beta < \xi_0$, otherwise there exists β, $\omega \leqslant \beta < \xi_0$, such that $X^\beta \twoheadrightarrow \mathbb{R}^{\xi_0}$.

If $\beta^\omega < \xi_0^\omega$, it follows from Theorem 1 of [**1**] that $X^\beta \twoheadrightarrow \mathbb{R}^{\xi_0} \twoheadrightarrow \mathbb{R}^{\beta^\omega}$, which is absurd because of the choice of ξ_0.

If $\beta^\omega = \xi_0^\omega$, again from Theorem 1 of [**1**] we have $X^\beta \sim X^{\xi_0} \twoheadrightarrow \mathbb{R}^{\xi_0^\omega} = \mathbb{R}^{\beta^\omega}$, which, again, is absurd.

From Proposition 5.4 it follows that $X^{\xi_0} \not\twoheadrightarrow \mathbb{R}^{\xi_0^\omega}$, which is a contradiction.

Consequently, $\xi_0 = \omega$ and, therefore, $X^\omega \twoheadrightarrow \mathbb{R}^{\omega^\omega}$, and again from Proposition 5.4 it follows that there exists $n < \omega$ such that $X^\omega \twoheadrightarrow \mathbb{R}^\omega$, and from Theorem 2 of [**17**] we have $X \twoheadrightarrow \mathbb{R}^\omega$. $\square$

Corollary 5.8. *$\ell_\infty(\mathbb{N})^\xi \sim \ell_\infty(\mathbb{N})^\eta$, with $\omega \leqslant \xi \leqslant \eta < \omega_1$, if and only if $\eta < \xi^\omega$.*

Proof. If $\xi^\omega \leqslant \eta$, then $\ell_\infty(\mathbb{N})^\xi \twoheadrightarrow \ell_\infty(\mathbb{N})^\eta \twoheadrightarrow \mathbb{R}^\eta \twoheadrightarrow \mathbb{R}^{\xi^\omega}$, so, by the above theorem, $\ell_\infty(\mathbb{N}) \twoheadrightarrow \mathbb{R}^\omega$, which is absurd because $\mathbb{R}^\omega$ is not reflexive, see the theorem on p. 304 in [**18**]. $\square$

Our results suggest the following.

Definition 5.9. We say that the Banach space X is ω_1 cancellable if $X^\xi \sim X^\eta$ with $\xi \leqslant \eta < \omega_1$ implies $\eta < \xi^w$.

Question 5.10. Give an isomorphic characterization of the separable ω_1 cancellable Banach spaces.

6. Remarks and questions about the Banach spaces $\mathbb{R}^\xi$, $\omega \leqslant \xi < \omega_1$

Corollary 6.1 follows from Corollary 5.5, so we put Question 6.2.

Corollary 6.1. *Let α be an infinite denumerable ordinal and X a Banach space. If $\mathbb{R}^\alpha \oplus X \twoheadrightarrow \mathbb{R}^{\alpha^\omega}$, then there exists m, $m < \omega$, such that $X^m \twoheadrightarrow \mathbb{R}^{\alpha^\omega}$.*

Question 6.2. If X is a Banach space such that there exists $m < \omega$ and $\alpha, \omega \leqslant \alpha < \omega_1$ with $X^m \twoheadrightarrow \mathbb{R}^{\alpha^\omega}$, then is it true that $X \twoheadrightarrow \mathbb{R}^{\alpha^\omega}$?

Definition 6.3. Let α be an infinite ordinal. We say that the Banach space X has the $SQ(\alpha)$ property, if, for every γ, $\omega \leqslant \gamma \leqslant \alpha$, such that $X \twoheadrightarrow \mathbb{R}^\gamma$, we have $X \twoheadrightarrow \mathbb{R}^\gamma \oplus X$.

Remark 6.4. It is clear that if the Banach space X satisfies $X \twoheadrightarrow X^2$, then X has the $SQ(\alpha)$ property $\forall \alpha$, $\alpha \geqslant \omega$, and, if $G = F^*$, where F is the space of Figiel, then G has the $SQ(\alpha)$ property $\forall \alpha$, $\alpha \geqslant \omega$, but $G \not\twoheadrightarrow G^2$.

Question 6.5. Give a Banach space that does not have the $SQ(\alpha)$ property for some α, $\omega \leqslant \alpha < \omega_1$.

Theorem 6.6. *Let ξ and η be infinite denumerable ordinals and let X be a Banach space having the $SQ(\xi)$ property. If $\mathbb{R}^\xi \oplus X \twoheadrightarrow \mathbb{R}^\eta$, then either $X \twoheadrightarrow \mathbb{R}^\eta$ or $\eta < \xi^\omega$.*

Proof. We will prove by transfinite induction on η that: $\forall \xi$, $\omega \leqslant \xi < \omega_1$, and for every Banach space X having the $SQ(\xi)$ property, with $\mathbb{R}^\xi \oplus X \twoheadrightarrow \mathbb{R}^\eta$, then either $X \twoheadrightarrow \mathbb{R}^\eta$ or $\eta < \xi^\omega$.

If $\eta = \omega$, then, since $\xi \geqslant \omega$, we have $\eta < \xi^\omega$.

Now, we suppose that this result is true for every ordinal φ, $\omega \leqslant \varphi < \theta$, and we consider $\mathbb{R}^\xi \oplus X \twoheadrightarrow \mathbb{R}^\theta$, with X having the $SQ(\xi)$ property. If $\xi^\omega \leqslant \theta$, then $\mathbb{R}^\xi \oplus X \twoheadrightarrow \mathbb{R}^{\xi^\omega}$. Let $\gamma = \min\{\beta : \mathbb{R}^\beta \twoheadrightarrow \mathbb{R}^\xi\}$, so $\omega \leqslant \gamma \leqslant \xi$, $\gamma < \gamma^\omega \leqslant \xi^\omega \leqslant \theta$ and, since $\mathbb{R}^\gamma \twoheadrightarrow \mathbb{R}^\xi$, it follows that $\mathbb{R}^\gamma \oplus X \twoheadrightarrow \mathbb{R}^\xi \oplus X \twoheadrightarrow \mathbb{R}^{\xi^\omega} \twoheadrightarrow \mathbb{R}^{\gamma^\omega}$, and by Proposition 5.4 we have

$$X \oplus \mathbb{R}^{\gamma_1} \twoheadrightarrow \mathbb{R}^\gamma \quad \text{for some } \omega \leqslant \gamma_1 < \gamma. \tag{6.1}$$

By the choice of γ we conclude that $\mathbb{R}^{\gamma_1} \not\twoheadrightarrow \mathbb{R}^\gamma$, so from [**1**] it follows that $\gamma_1^\omega \leqslant \gamma$, and, since X has the $SQ(\gamma_1)$ property, using the hypothesis of induction at (6.1) we have that $X \twoheadrightarrow \mathbb{R}^\gamma \twoheadrightarrow \mathbb{R}^\xi$, and, since X has the $SQ(\xi)$ property, we conclude that $X \twoheadrightarrow \mathbb{R}^\xi \oplus X \twoheadrightarrow \mathbb{R}^\theta$. □

Question 6.7. Let X and Y be separable Banach spaces and ξ, $\omega^\omega \leqslant \xi < \omega_1$. If $X \oplus Y \twoheadrightarrow \mathbb{R}^\xi$, then is it true that either $X \twoheadrightarrow \mathbb{R}^\xi$ or $Y \twoheadrightarrow \mathbb{R}^\xi$?

Since $\mathbb{R} \oplus \mathbb{R}^\xi \sim \mathbb{R}^\xi$, $\forall \xi$, $\xi \geqslant \omega$, $\mathbb{R}^\xi$ is isomorphic to each of its closed hyperplanes. The following lemma gives a positive answer to the above question in the case in which either X or Y is a finite-dimensional space.

Lemma 6.8. *Let X and H be Banach spaces such that H is isomorphic to each of its closed hyperplanes. If $\mathbb{R} \oplus X \twoheadrightarrow H$, then $X \twoheadrightarrow H$.*

Proof. Let $T : \mathbb{R} \oplus X \to H$ be a surjective bounded linear operator. If $T(1,0) = 0$, then $T|_X : X \to H$ is surjective. If $T(1,0) = h_1 \neq 0$, then writing $H = [h_1] \oplus H_1$ for some closed hyperplane H_1 of H and indicating by P the canonical projection from H onto H_1, we have $PT(1,0) = 0$, therefore $PT|_{H_1} : X \to H_1$ is onto H_1, and, from the hypothesis $H \sim H_1$, we have $X \twoheadrightarrow H$. □

Question 6.9. If X and Y are Banach spaces such that $\mathbb{R} \oplus X \twoheadrightarrow H$, and H is of infinite dimension, then is it true that $X \twoheadrightarrow H$?

Corollary 6.10. *Let α be an infinite denumerable ordinal and let X be a Banach space. If $X^\alpha \twoheadrightarrow \mathbb{R}^{\alpha^\omega}$, then there exists n, $m < \omega$, such that $(\hat{\hat{\otimes}}_m X)^n \twoheadrightarrow \mathbb{R}^{\alpha^\omega}$.*

Proof. It suffices to take $\alpha_0 = \min\{\xi : \exists\, m,\, m < \omega,\, (\hat{\hat{\otimes}}_m X)^\xi \twoheadrightarrow \mathbb{R}^\alpha\}$ and to proceed as in the Lemma 3.4 using Proposition 5.4. □

Question 6.11. If X is a Banach space such that there exists n, $n < \omega$ and α, $\omega \leqslant \alpha < \omega_1$ with $\hat{\hat{\otimes}}_n X \twoheadrightarrow \mathbb{R}^{\alpha^\omega}$, then is it true that $X \twoheadrightarrow \mathbb{R}^\alpha$?

Definition 6.12. Let α be an infinite ordinal. We say that the Banach space X has the $TQ(\alpha)$ property if, for every γ, $\omega \leqslant \gamma \leqslant \alpha$, such that $X \twoheadrightarrow \mathbb{R}^\gamma$, we have $X \twoheadrightarrow X^\gamma$.

Remark 6.13. It is clear that if the Banach space X satisfies $X \twoheadrightarrow X \hat{\hat{\otimes}} X$, then X has the $TQ(\alpha)$ property $\forall \alpha$, $\alpha \geqslant \omega$.

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Theorem 6.14. *Let ξ and η be infinite denumerable ordinals and let X be a Banach space having the $TQ(\xi)$ property. If $X^\xi \twoheadrightarrow \mathbb{R}^\eta$, then either $X \twoheadrightarrow \mathbb{R}^\eta$ or $\eta < \xi^\omega$.*

Proof. Analogous to that for Theorem 6.8. □

Question 6.15. Give a Banach space that does not have the $TQ(\alpha)$ property for some α, $\omega \leqslant \alpha < \omega_1$.

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