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On the analyticity of generalized minimal surfaces

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Strongly differentiable solutions of the minimal surface equation are shown to be classical solutions and consequently locally analytic. A global regularity result is also proved.

It follows readily from De Giorgi's interior estimate [2], that *continuous*, strongly differentiable solutions of the minimal surface equation must be locally analytic. A proof of this assertion, utilizing the uniqueness of solutions to the generalized Dirichlet problem, was indicated to the author by Nitsche [6]. The purpose of this note is to establish this result for arbitrary generalized solutions, not necessarily assumed continuous beforehand. Our method involves an extension of Nitsche's uniqueness argument coupled with a bound for generalized solutions obtained in [7]. Regularity results for fairly large classes of divergence form, quasilinear elliptic equations are proved in the book [5]; however the permissible nonlinear structures considered there cannot be stretched to embrace the minimal surface equation.

Let us begin by writing the minimal surface equation in its divergence form

$$\operatorname{div} A(Du) = 0 \quad , \tag{1}$$

where Du denotes the gradient vector of the function, u , and the mapping $A : E^n \to E^n$ is given by

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(2) $$A(p) = \frac{p}{(1+p^2)^{1/2}}\ , \quad p^2 = |p|^2\ .$$

Equation (1) is elliptic and consequently strictly monotone. In fact we have for all $p, q \in E^n$

(3) $$(p-q)(A(p)-A(q)) \geq \frac{(p-q)^2}{(1+p^2+q^2)^{3/2}}\ .$$

A *classical solution* of equation (1) in a domain Ω will simply be a $C^2(\Omega)$ solution. By a *generalized solution*, we will mean a strongly differentiable function u, satisfying

(4) $$\int_\Omega A(Du)D\phi dx = 0$$

for all ϕ continuously differentiable with compact support in Ω, that is belonging to the space $C_0^1(\Omega)$. Let us recall that a strongly differentiable function in Ω is a function whose distributional derivatives are locally integrable in Ω. The Sobolev space $W_1^1(\Omega)$ consists of strongly differentiable functions, u, for which the norm

(5) $$\|u\|_{W_1^1(\Omega)} = \int_\Omega (|u|+|Du|)dx$$

is finite, and $\overset{\circ}{W}{}_1^1(\Omega)$ denotes the closure of $C_0^1(\Omega)$ in $W_1^1(\Omega)$. Since A is bounded, the equation (4) will then hold for all ϕ lying in $\overset{\circ}{W}{}_1^1(\Omega)$. We will prove the following result.

THEOREM 1. *A generalized solution of equation* (1) *coincides, almost everywhere, with a classical solution.*

Prior to giving the proof, we collect together some basic results concerning equation (1) for later reference.

THEOREM A. *Let* Ω *be a bounded domain in* E^n *whose* C^2 *boundary,* $\partial\Omega$, *has non-negative mean curvature everywhere. Then for any continuous function* ϕ *on* $\partial\Omega$, *there exists a unique classical solution,* u, *of*

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equation (1) *in* Ω *assuming the boundary values* ϕ *continuously on* $\partial\Omega$. *Furthermore* u *is analytic in* Ω , *and for any compact subset,* K , *of* Ω *and multi-index,* α , *we have the estimate*

$$\sup_K |D^\alpha u| \le C , \tag{6}$$

where the constant C *depends on* n, α, $\mathrm{dist}(K, \partial\Omega)$ *and* $\sup_{\partial\Omega} |\phi|$. *If* ϕ *is twice continuously differentiable, then* u *is continuously differentiable in* $\overline{\Omega}$.

Theorem A is a big theorem and embodies not only the interior gradient bound [2], but also, among other things, the De Giorgi-Nash Hölder estimates [3], the Schauder theory [1] and Jenkins and Serrin's boundary gradient estimate [4]. The next result was derived by Serrin in [7].

THEOREM B. *Let* u *be a generalized solution of equation* (1) *in* Ω . *Then* u *is locally bounded in* Ω *and for any compact subset* K *of* Ω , *we have*

$$\sup_K |u| \le C\left(\int_\Omega |u|dx + 1\right) \tag{7}$$

where the constant C *depends on* n *and* $\mathrm{dist}(K, \partial\Omega)$.

Proof of Theorem 1. Let B and B_0 be balls in Ω such that B is strictly contained in B_0 which is strictly contained in Ω . Let u be a generalized solution of equation (1) in Ω and ρ a mollifier. Consequently the mollified function u_h , $h > 0$, given by

$$u_h(x) = h^{-n} \int_\Omega \rho\left(\frac{x-y}{h}\right)u(y)dy \tag{8}$$

will converge in $W_1^1(B)$ to u as h tends to zero. But also for $h < \mathrm{dist}\left(B, \partial B_0\right)$,

$$\begin{aligned} \sup_B |u_h| &\le \sup_{B_0} |u| \\ &\le C\left(\int_\Omega |u|dx + 1\right) \end{aligned} \tag{9}$$

by Theorem B.

Define now v_h to be the classical solution of equation (1) in B with $v_h = u_h$ on ∂B. By Theorem A, $v_h \in C^1(\bar{\Omega})$, and using also the estimate (9) we obtain, for any α and compact $K \subset B$,

$$\sup_K \left|D^\alpha v_h\right| \leq C , \tag{10}$$

where the constant C is independent of h. By a standard argument, involving Ascoli's Theorem, we then obtain a subsequence v_{h_j}, $j = 1, 2, \ldots$, converging, together with its derivatives, normally in B. The limit function, v, will consequently be a classical solution of (1).

To complete the proof we show that v coincides with u, almost everywhere in B. Let us write $v_j = v_{h_j}$, $u_j = u_{h_j}$. Since u and v_j, for any j, are both generalized solutions of (1), we have by subtraction,

$$\int_B \left\{A(Du)-A\left(Dv_j\right)\right\}D\phi dx = 0 \tag{11}$$

for all $\phi \in \overset{\circ}{W}{}^1_1(B)$. We choose $\phi = u_j - v_j = \left(u-v_j\right) - \left(u-u_j\right)$ and substitute in (11) to obtain

$$\begin{aligned}\int_B \left\{A(Du)-A\left(Dv_j\right)\right\}\left(Du-Dv_j\right)dx &\leq \int_B \left|A(Du)-A\left(Dv_j\right)\right|\left|Du-Du_j\right|dx \\ &\leq 2\int_B \left|Du-Du_j\right|dx .\end{aligned} \tag{12}$$

Letting j tend to infinity, we obtain by Fatou's Lemma,

$$\int_B \left(A(Du)-A(Dv)\right)(Du-Dv)dx = 0 , \tag{13}$$

and hence $Du = Dv$ almost everywhere in B by the strict monotonicity (3). It then follows easily that $u = v$ almost everywhere in B and the theorem is proved. //

In addition to Theorem 1, a global regularity result is readily derived.

THEOREM 2. *Let* Ω *be a bounded domain in* E^n *, whose* C^2 *boundary,* $\partial\Omega$ *, has nonnegative mean curvature everywhere. Let* u *be a* $W_1^1(\Omega)$ *solution of equation* (1) *in* Ω *,* v *a continuous function in* $\bar{\Omega}$ *and suppose that the difference* $u - v$ *belongs to* $\overset{\circ}{W}{}_1^1(\Omega)$ *. Then* u *is continuous in* $\bar{\Omega}$ *.*

Proof. Define w to be the classical solution of equation (1) satisfying $w = v$ on $\partial\Omega$. If we can show $w \in W_1^1(\Omega)$, we are done, for then $w - u \in \overset{\circ}{W}{}_1^1(\Omega)$ and consequently the equation (13) holds for w and u . In other words, the generalized Dirichlet problem for equation (1) in $W_1^1(\Omega)$ can only have unique solutions. Let us now choose, for $\varepsilon > 0$,

$$\phi = \operatorname{sign}(w-v)\sup(|w-v|-\varepsilon,\ 0) \tag{14}$$

as a test function in (2). It is easily seen that $\phi \in \overset{\circ}{W}{}_1^1(\Omega)$, and consequently by substitution we obtain

$$\int_{\text{support } \phi} \frac{|Dw|^2}{(1+|Dw|^2)^{1/2}}\,dx \le \int_\Omega |A(Dw)|\,|Dv| \le \int_\Omega |Dv| \,. \tag{15}$$

Hence as ε tends to zero, we get

$$\int_\Omega |Dw|dx \le \int_\Omega (1+|Dv|)dx \,, \tag{16}$$

so that $w \in W_1^1(\Omega)$. //

Further regularity at $\partial\Omega$, along with local regularity at $\partial\Omega$, follows by standard methods. We also mention that the above proofs automatically carry over to more general classes of quasilinear elliptic equations. In particular, Theorem 1 holds for the equation of prescribed mean curvature

$$\operatorname{div} A(Du) = H(x) \quad , \tag{17}$$

provided H is Hölder continuous in Ω .

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