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Rings in which Every Element is a Sum of Two Tripotents

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Abstract. Let R be a ring. The following results are proved. (1) Every element of R is a sum of an idempotent and a tripotent that commute if and only if R has the identity $x^6 = x^4$ if and only if $R \cong R_1 \times R_2$, where $R_1/J(R_1)$ is Boolean with $U(R_1)$ a group of exponent 2 and R_2 is zero or a subdirect product of $\mathbb{Z}_3$'s. (2) Every element of R is either a sum or a difference of two commuting idempotents if and only if $R \cong R_1 \times R_2$, where $R_1/J(R_1)$ is Boolean with $J(R_1) = 0$ or $J(R_1) = \{0, 2\}$ and R_2 is zero or a subdirect product of $\mathbb{Z}_3$'s. (3) Every element of R is a sum of two commuting tripotents if and only if $R \cong R_1 \times R_2 \times R_3$, where $R_1/J(R_1)$ is Boolean with $U(R_1)$ a group of exponent 2, R_2 is zero or a subdirect product of $\mathbb{Z}_3$'s, and R_3 is zero or a subdirect product of $\mathbb{Z}_5$'s.

1 Introduction

In 1988, Hirano and Tominaga [2] investigated the rings for which every element is a sum of two idempotents and proved that every element of a ring R is a sum of two commuting idempotents if and only if R has the identity $x^3 = x$; that is, $R = A \times B$ where A is a Boolean ring and B is zero or a subdirect product of $\mathbb{Z}_3$'s. It can be shown that every element of a ring is a sum of two commuting idempotents if and only if every element is a difference of two commuting idempotents. Thus, one is naturally led to ask which rings have the property that every element is a sum or a difference of two commuting idempotents. For any idempotent e, both e and $-e$ are tripotents, *i.e.,* the elements equal to their cubes. One is further led to two more general questions. Which rings have the property that every element is a sum of an idempotent and a tripotent that commute? Which rings have the property that every element is a sum of two commuting tripotents? The goal of this paper is to present complete answers to those three questions. So far, no structure theorem exists for the rings for which every element is a sum of two idempotents. We prove a structural result to reduce that situation to the case of characteristic 2.

Throughout, rings R are associative with 1. For a ring R, the characteristic, the Jacobson radical, the set of units and the set of nilpotents of a ring R are denoted by $\mathrm{ch}(R)$, $J(R)$, $U(R)$, and $\mathrm{Nil}(R)$, respectively. As usual, $\mathbb{M}_n(R)$ and $\mathbb{T}_n(R)$ stand for the $n \times n$ matrix ring and $n \times n$ upper triangular matrix ring, respectively, over R. We write $\mathbb{Z}_n$ for the ring of integers modulo n. A *reduced* ring is a ring without nonzero nilpotents. An *abelian* ring is a ring for which every idempotent is central.

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2 Motivation and Questions

In [2], Hirano and Tominaga proved the following interesting result.

Theorem 2.1 ([2]) *The following are equivalent for a ring R.*

(i) *Every element of R is a sum of two commuting idempotents.*
(ii) *R is commutative and every element of R is a sum of two idempotents.*
(iii) *R has the identity $x^3 = x$.*

We can add one condition to the equivalence list.

Proposition 2.2 *Let R be a ring. Then every element of R is a difference of two commuting idempotents if and only if R has the identity $x^3 = x$.*

Proof $(\Rightarrow)$. Let $b \in \mathrm{Nil}(R)$. Write $b = e - f$ where e, f are commuting idempotents. Then $b+f = (b+f)^2 = b^2+2bf+f$, showing that $b(1-2f) = b^2$. So $b = b^2(1-2f)^{-1} = b^2(1-2f)$. As b is nilpotent, it follows that $b = 0$. So R is a reduced ring. Thus, R is a subdirect product of the domains $\{R_\alpha\}$. As an image of R, each R_α has the same property as R. That is, every element of R_α is a difference of two commuting idempotents. But R_α has only the trivial idempotents, so $R_\alpha = \{0, 1, -1\}$ (and possibly $-1 = 1$). Hence, R_α has the identity $x^3 = x$. It follows that R has the identity $x^3 = x$.

$(\Leftarrow)$. If R has the identity, then R is a subdirect product of $\mathbb{Z}_2$'s and $\mathbb{Z}_3$'s. Hence, $R = R_1 \times R_2$, where R_1 is a Boolean ring (a subdirect product of $\mathbb{Z}_2$'s) and R_2 is a subdirect product of $\mathbb{Z}_3$'s. Clearly, every element of R_1 is a difference of two commuting idempotents. For $y \in R_2$, as $3 = 0$, $y = \frac{1}{2}(y^2 + y) - \frac{1}{2}(y^2 - y)$ is a difference of two commuting idempotents. It follows that every element of R is a difference of two commuting idempotents. ∎

The following question is naturally motivated.

Question 2.3 *What can be said about the rings for which every element is either a sum or a difference of two commuting idempotents?*

An element e of a ring is called a *tripotent* if $e^3 = e$. For any idempotent e, both e and $-e$ are tripotents. Thus, the following questions are also motivated.

Question 2.4 *What can be said about the rings for which every element is a sum of an idempotent and a tripotent that commute?*

Question 2.5 *What can be said about the rings for which every element is a sum of two commuting tripotents?*

In this paper, we give answers to these three questions.

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3 Elements as Sums of an Idempotent and a Tripotent that Commute

Definition 3.1 A ring is called a (strong) SIT-ring if every element is a sum of an idempotent and a tripotent (that commute).

Proposition 3.2 *Any direct product of (strong) SIT-rings is a (strong) SIT-ring. Any factor ring of a (strong) SIT-ring is a (strong) SIT-ring.*

Lemma 3.3 *If R is a ring for which $3 = e + f$ where $e^2 = e$ and $f^3 = f$, then $24 = 0$ in R.*

Proof From $3 = e+f$, we see $ef = fe$, so $9 = (e+f)^2 = e+2ef+f^2$. Thus, $2(e+f) = 6 = 9-3 = (e+2ef+f^2)-(e+f) = 2ef+f^2-f$. It follows that $2e+3f-2ef-f^2 = 0$. Thus, $0 = (2e+3f-2ef-f^2)ef = 2ef+3ef^2-2ef^2-ef^3 = ef+ef^2$. So, $24 = 3^3-3 = (e+f)^3-3 = (e+3ef+3ef^2+f)-3 = [(e+f)-3]+3(ef+ef^2) = 0$. ∎

Lemma 3.4 *A ring R is a (strong) SIT-ring if and only if $R \cong R_1 \times R_2$ where R_1, R_2 are (strong) SIT-rings, $2^3 = 0$ in R_1, and $3 = 0$ in R_2.*

Proof The sufficiency is clear by Proposition 3.2. For the necessity, assume that R is a (strong) SIT-ring. Then, by Lemma 3.3, $2^3 3 = 0$. Thus, $2^3 R \cap 3R = 0$ and $R = 2^3 R + 3R$. By the Chinese Remainder Theorem, $R \cong R/2^3 R \times R/3R$. Let $R_1 = R/2^3 R$ and $R_2 = R/3R$. Then R_1, R_2 are (strong) SIT-rings by Proposition 3.2 with $2^3 = 0$ in R_1 and $3 = 0$ in R_2, and $R \cong R_1 \times R_2$. ∎

The argument in the proof of the next lemma is well known for lifting idempotents modulo a nil ideal (see [4, p. 319]). As the lemma is stated slightly differently than usual, we include its proof for the reader's convenience.

Lemma 3.5 *Let $a \in R$. If $a^2 - a$ is nilpotent, then there exists a monic polynomial $\theta(t) \in \mathbb{Z}[t]$ such that $\theta(a)^2 = \theta(a)$ and $a - \theta(a)$ is nilpotent.*

Proof Let $b = 1-a$. We have $ab = ba = a-a^2$, so $(ab)^m = 0$ for some integer $m \geq 1$. Then

$$1 = (a+b)^{2m} = a^{2m} + r_1 a^{2m-1} b + \cdots + r_m a^m b^m + r_{m+1} a^{m-1} b^{m+1} + \cdots + b^{2m},$$

where the r_i's are integers. Let

$$e = a^{2m} + r_1 a^{2m-1} b + \cdots + r_m a^m b^m \quad \text{and} \quad f = r_{m+1} a^{m-1} b^{m+1} + \cdots + b^{2m}.$$

Since $a^m b^m = b^m a^m = 0$, we have $ef = 0$, and so $e = e(e+f) = e^2$. So far, all the arguments are the same as in [4, p. 319]. It is clear that $e = \theta(a)$ for a monic polynomial $\theta(t)$ over $\mathbb{Z}$. Since $ab = ba$ is nilpotent, $e - a^{2m} = r_1 a^{2m-1} b + \cdots + r_m a^m b^m$ is nilpotent. As $a - a^2$ is nilpotent, we infer that $a - e = (a - a^{2m}) - (e - a^{2m}) = (a-a^2) + (a^2-a^3) + \cdots + (a^{2m-1} - a^{2m}) - (e - a^{2m})$ is nilpotent. ∎

Following [1], an element of a ring is called *strongly nil clean* if it is the sum of an idempotent and a nilpotent element that commute with each other, and the ring is called *strongly nil clean* if each of its elements is strongly nil clean.

Theorem 3.6 *The following are equivalent for a ring R.*

(i) *R is a strong SIT-ring with $2 \in J(R)$.*
(ii) *R is a strong SIT-ring with $2^3 = 0$.*
(iii) *R has the identity $x^6 = x^4$ and $2 \in J(R)$.*
(iv) *$R/J(R)$ is Boolean and $j^2 = 2j$ for all $j \in J(R)$.*
(v) *$R/J(R)$ is Boolean and $U(R)$ is a group of exponent 2.*

Proof (i)⇒(ii). This is clear by Lemma 3.4.

(ii)⇒(iii). For $a \in R$, write $a = e + f$ where $e^2 = e$, $f^3 = f$, and $ef = fe$. Then

$$\begin{aligned} a^4 &= (e+f)^4 = e^4 + 4e^3 f + 6e^2 f^2 + 4ef^3 + f^4 \\ &= e + 4ef + 6ef^2 + 4ef + f^2 = e + 6ef^2 + f^2, \\ a^6 &= a^4 a^2 = (e + 6ef^2 + f^2)(e + 2ef + f^2) \\ &= (e + 6ef^2 + ef^2) + (2ef + 12ef + 2ef) + (ef^2 + 6ef^2 + f^2) \\ &= e + 6ef^2 + f^2. \end{aligned}$$

So $a^6 = a^4$ holds.

(iii)⇒(iv). For $j \in J(R)$, we have $(1-j)^6 = (1-j)^4$, so $(1-j)^2 = 1$ as $(1-j)^4 \in U(R)$. It follows that $j^2 = 2j$. Hence, we have proved that $j^2 = 2j$ for all $j \in J(R)$. From $2^6 = 2^4$, we obtain $2^4 3 = 0$. As $2 \in J(R)$, $3 \in U(R)$, so we infer $2^4 = 0$. For $a \in R$, we have $a^6 = a^4$, so $(a - a^2)^4 = a^4(1-a)^4 = a^4(1 - 4a + 6a^2 - 4a^3 + a^4) = a^4 - 4a^5 + 6a^6 - 4a^7 + a^8 = a^4 - 4a^5 + 6a^4 - 4a^5 + a^4 = 8(a^4 - a^5)$, which is nilpotent as 2 is nilpotent. Thus, $a - a^2$ is nilpotent. By Lemma 3.5, there exists $e^2 = e$ such that $ae = ea$ and $a - e$ is nilpotent. This shows that $a = e + (a - e)$ is strongly nil clean. Therefore, we have proved that R is strongly nil clean. By [3], $R/J(R)$ is Boolean.

(iv)⇒(v). For $u \in U(R)$, $u^2 - u \in J(R)$, since $R/J(R)$ is Boolean, so $u \in 1 + J(R)$; hence, $U(R) = 1 + J(R)$. Write $u = 1 - j$ for $j \in J(R)$. Then $u^2 = (1-j)^2 = 1 - 2j + j^2 = 1$, as $j^2 = 2j$. Hence, $U(R)$ is a group of exponent 2.

(v)⇒(i). For $j \in J(R)$, we have $(1-j)^2 = 1$ by (v), so $j^2 = 2j$. Replacing j by $j(1+j)$, we have $(j(1+j))^2 = 2j(1+j)$. We infer that $j(1+j)j = 2j$; that is, $j^2 + j^3 = 2j$. It follows that $j^3 = 0$. Hence, $J(R)$ is nil. Since $R/J(R)$ is Boolean, R is strongly nil clean by [3]. Therefore, for any $a \in R$, there exist $b \in \mathrm{Nil}(R)$ and $e^2 = e$ such that $eb = be$ and $a - 1 = e + b$. By (v), $(1+b)^2 = 1$, so $1 + b$ is a tripotent. Hence, $a = e + (1+b)$ is a sum of an idempotent and a tripotent that commute. So R is a strong SIT-ring. Moreover, since $R/J(R)$ is Boolean, $2 \in J(R)$ as required. ■

Example 3.7 The ring $\mathbb{T}_2(\mathbb{Z}_2)$ is a strong SIT-ring, but it is not commutative.

Corollary 3.8 *Every strong SIT-ring with $2 \in J(R)$ is strongly nil clean.*

Proposition 3.9 *A ring R is a strong SIT-ring with $3 = 0$ if and only if R is a subdirect product of $\mathbb{Z}_3$'s.*

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Proof The sufficiency is clear. For the necessity, let $a \in R$, and write $a = e + f$ where $e^2 = e$, $f^3 = f$, and $ef = fe$. Then $a^3 = (e+f)^3 = e^3 + 3e^2f + 3ef^2 + f^3 = e + f = a$. Hence, R has the identity $x^3 = x$. Since $3 = 0$ in R, R is a subdirect product of $\mathbb{Z}_3$'s. ■

Theorem 3.10 *The following are equivalent for a ring R.*

(i) *R is a strong SIT-ring.*
(ii) *R has the identity $x^6 = x^4$.*
(iii) *R is one of the following types:*
  (a) *$R/J(R)$ is Boolean and $U(R)$ is a group of exponent 2.*
  (b) *R is a subdirect product of $\mathbb{Z}_3$'s.*
  (c) *$R \cong A \times B$, where $A/J(A)$ is Boolean with $U(A)$ a group of exponent 2, and B is a subdirect product of $\mathbb{Z}_3$'s.*

Proof (i)$\Leftrightarrow$(iii). This follows from Lemma 3.4, Theorem 3.6, and Proposition 3.9.

(iii)$\Rightarrow$(ii). Theorem 3.6 indicates that (iii)(a) implies (ii), and (iii)(b) clearly implies (ii). Therefore, (iii)(c) also implies (ii).

(ii)$\Rightarrow$(i). From $2^6 = 2^4$, we see that $2^4(2^2-1) = 2^4 3 = 0$. From $3^6 = 3^4$, we obtain $2^3 3^4 = 0$. Therefore, $\gcd(2^4 3, 2^3 3^4) = 0$, *i.e.*, $2^3 3 = 0$. This shows that $R = R_1 \times R_2$, where $R_1 \cong R/2^3R$ and $R_2 \cong R/3R$. Thus, R_1 has the identity $x^6 = x^4$ with $2 \in J(R_1)$. So R_1 is a strong SIT-ring by Theorem 3.6. On the other side, R_2 has the identity $x^6 = x^4$ with $3 = 0$. Replacing x by $1 + x$ in $x^6 = x^4$, we can obtain $x^3 = x$. Hence, R_2 has the identity $x^3 = x$, and so R_2 is a strong SIT-ring. Thus, $R = R_1 \times R_2$ is a strong SIT-ring. ■

Example 3.11 Let $n \geq 2$. The matrix ring $\mathbb{M}_n(R)$ is not a strong SIT-ring for any ring R.

Proof Let $A = \left(\begin{smallmatrix} X & 0 \\ 0 & I_{n-2} \end{smallmatrix}\right)$ where $X = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then $A^2 \neq I_n$, so $A^6 \neq A^4$. Hence, $\mathbb{M}_n(R)$ is not a strong SIT-ring by Theorem 3.10. ■

Example 3.12 Let $n \geq 2$. The upper triangular matrix ring $\mathbb{T}_n(R)$ is a strong SIT-ring if and only if R is Boolean and $n = 2$.

Proof ($\Leftarrow$). This follows from Theorem 3.6(iv).

($\Rightarrow$). It follows from the hypothesis that R is a strong SIT-ring. So, by Theorem 3.10, $R = A \times B$, where $A/J(A)$ is Boolean with $j^2 = 2j$ for all $j \in J(A)$, and B is zero or a subdirect product of $\mathbb{Z}_3$'s. Thus, $\mathbb{T}_n(R) \cong \mathbb{T}_n(A) \times \mathbb{T}_n(B)$. As $\mathbb{T}_n(B)$ is a strong SIT-ring with characteristic 3, we infer from Theorem 3.10 that the Jacobson radical of $\mathbb{T}_n(B)$ is zero. This shows that $B = 0$, and so $R/J(R)$ is Boolean with $j^2 = 2j$ for all $j \in J(R)$. By Theorem 3.6, we have $0 = E_{1n}^2 = 2E_{1n}$, showing $2 = 0$ in R. Thus, again by Theorem 3.6, $\alpha^2 = 2\alpha = 0$ for all $\alpha \in J(\mathbb{T}_n(R))$. This clearly shows that $n = 2$. Hence, it must be that $n = 2$. Finally, for $j \in J(R)$, from $\left(\begin{smallmatrix} j & 1 \\ 0 & 0 \end{smallmatrix}\right)^2 = 0$, we obtain that $j = 0$. So R is Boolean. ■

4 Elements as Sums or Differences of Two Commuting Idempotents

Next we apply the results above to determine the rings for which every element is either a sum or a difference of two commuting idempotents. The following corollary is an immediate consequence of Theorem 3.6.

Corollary 4.1 *Let $2 = 0$ in R. Then every element of R is either a sum or a difference of two commuting idempotents if and only if R is Boolean.*

Lemma 4.2 *Suppose that every element of R is either a sum or a difference of two commuting idempotents. If $0 \neq 2 \in J(R)$, then the following hold.*

(i) $4 = 0$.
(ii) *For any $j \in J(R)$, $j = 2e$ for some $e^2 = e$. In particular, $J(R) = 2R$.*
(iii) $2J(R) = 0$ *and* $J(R)^2 = 0$.
(iv) $\mathrm{Nil}(R) = J(R)$ *and* $U(R) = \{1 - 2e : e^2 = e \in R\}$.
(v) *R is abelian.*
(vi) $J(R) = \{0, 2\}$.

Proof (i). There exist two commuting idempotents e, f such that $3 = e + f$ or $3 = e - f$.

If $3 = e+f$, then $2(e+f) = 6 = 9-3 = (e+f)^2-(e+f) = (e+2ef+f)-(e+f) = 2ef$. So $2ef = (2ef)e = 2(e + f)e = 2e + 2ef$, showing $2e = 0$. Similarly, $2f = 0$. Hence, $6 = 2(e + f) = 0$, so $2 = 0$ as $3 \in U(R)$.

If $3 = e-f$, then $2(e-f) = 6 = 9-3 = (e-f)^2-(e-f) = -2ef+2f$, so $2(e-f)e = (-2ef + 2f)e$, showing $2e = 2ef$. Thus, $9 = e - 2ef + f = e - 2e + f = f - e = -3$, giving $12 = 0$. So $4 = 0$ as $3 \in U(R)$.

(ii) and (iii). Let $j \in J(R)$. There exist two commuting idempotents e, f such that $j = e + f$ or $j = e - f$. If $j = e + f$, then $j - e = (j - e)^2 = j^2 - 2je + e$, showing that $j - 2e - j^2 + 2je = 0$. That is, $(j - 2e)(1 - j) = 0$. So $j = 2e$ as $1 - j \in U(R)$. If $j = e - f$, then $j + f = (j + f)^2 = j^2 + 2jf + f$, showing that $j - j^2 - 2jf = 0$. That is, $j(1 - 2f - j) = 0$. As $1 - 2f - j \in U(R)$, we have $j = 0 = 2 \cdot 0$. Therefore, we have proved $j = 2g$ for some $g^2 = g$. So $2j = 4g = 0$ by (i). For $j' \in J(R)$, as above, $j' = 2h$ for some $h^2 = h \in R$. Hence, $jj' = (2g)(2h) = 4(gh) = 0$ by (i).

(iv). By Theorem 3.6, $R/J(R)$ is Boolean. So $\mathrm{Nil}(R) = J(R)$ and $U(R) = 1 + J(R)$. If $u \in U(R)$, then $u = 1 - j$ for some $j \in J(R)$. By (ii), $j = 2e$ for some $e^2 = e$. Hence, $u = 1 - 2e$.

(v). Let $e^2 = e \in R$. Note that $eR(1-e), (1-e)Re \subseteq J(R)$, so $eR(1-e) \cdot (1-e)Re = (1 - e)Re \cdot eR(1 - e) = 0$. Consider the Peirce decomposition

$$R = \begin{pmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{pmatrix}$$

with respect to the idempotent e. Let $z \in eR(1 - e)$. By (ii), $\left(\begin{smallmatrix} 0 & z \\ 0 & 0 \end{smallmatrix}\right) = 2\left(\begin{smallmatrix} a & x \\ y & b \end{smallmatrix}\right)$, where

$$\begin{pmatrix} a & x \\ y & b \end{pmatrix} = \begin{pmatrix} a & x \\ y & b \end{pmatrix}^2 = \begin{pmatrix} a^2 & ax + xb \\ ya + by & b^2 \end{pmatrix}.$$

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Thus, $z = 2x = 2(ax + xb) = (2a)x + x(2b) = 0x + x0 = 0$. So $eR(1-e) = 0$ and, similarly, $(1-e)Re = 0$. So e is central.

(vi) Assume $0 \neq j \in J(R)$ with $j \neq 2$. So, by (ii), $j = 2e$ where e is a non-trivial idempotent. Since e is central, $R = A \times B$, where $A = eR$ and $B = (1-e)R$. Note that $0 \neq 2e \in J(A)$ and $0 \neq 2(1-e) \in J(B)$. Therefore, by Theorem 2.1, there exists $a \in A$ such that a is not a sum of two commuting idempotents, and, by Proposition 2.2, there exists $b \in B$ such that b is not a difference of two commuting idempotents. Thus, $r := (a, b) \in R$ is neither a sum nor a difference of two commuting idempotents, a contradiction. ∎

Theorem 4.3 *The following are equivalent for a ring R.*

(i) *Every element of R is either a sum or a difference of two commuting idempotents with $2 \in J(R)$.*
(ii) *$R/J(R)$ is Boolean with $J(R) = 0$ or $J(R) = \{0, 2\}$.*

Proof (i)⇒(ii). Given (i), we see that $R/J(R)$ is Boolean by Theorem 3.6, and that $J(R) = 0$ or $J(R) = \{0, 2\}$ by Corollary 4.1 and Lemma 4.2.

(ii)⇒ (i). By (ii), $2 \in J(R)$ and $4 = 0$. Moreover, R is strongly nil clean by [3]. Let $a \in R$. Then $a = j + e$ where $j \in J(R)$ and $e^2 = e$. We next show that a is either a sum or a difference of two commuting idempotents. We can certainly assume that $j \neq 0$. So $j = 2$ and $a = 2 + e$. As $2e \in J(R)$, we have $2e = 0$ or $2e = 2$.

If $2e = 0$, then $e = -e$, so $a = 2 + e = 1 + (1-e)$ is a sum of two commuting idempotents. If $2e = 2$, then $a = 2 + e = (1-e) + (1+2e) = (1-e) + 3 = (1-e) - 1$ is a difference of two commuting idempotents. ∎

Theorem 4.4 *The following are equivalent for a ring R.*

(i) *Every element of R is either a sum or a difference of two commuting idempotents.*
(ii) *R is one of the following types:*
  (a) *$R/J(R)$ is Boolean with $J(R) = 0$ or $J(R) = \{0, 2\}$.*
  (b) *R is a subdirect product of $\mathbb{Z}_3$'s.*
  (c) *$R \cong R_1 \times R_2$, where $R_1/J(R_1)$ is Boolean with $J(R_1) = 0$ or $J(R_1) = \{0, 2\}$ and R_2 is a subdirect product of $\mathbb{Z}_3$'s.*

Proof (i)⇒(ii). By Theorem 3.10, $R \cong R_1 \times R_2$, where $2 \in J(R_1)$ and R_2 is zero or a subdirect product of $\mathbb{Z}_3$'s. Since every element of R_1 is either a sum or a difference of two commuting idempotents, we infer, by Theorem 4.3, that $R_1/J(R_1)$ is Boolean with $J(R_1) = 0$ or $J(R_1) = \{0, 2\}$.

(ii)⇒(i). This is by Theorems 4.3 and 2.1 and Proposition 2.2. ∎

5 Elements as Sums of Two Commuting Tripotents

The following lemma can easily be proved.

Lemma 5.1 *The $R = \prod R_\alpha$ be direct product of rings. Then every element of R is a sum of two commuting tripotents if and only if, for each α, every element of R_α is a sum of two commuting tripotents.*

Theorem 5.2 *The following are equivalent for a ring R.*

(i) *Every element of R is a sum of two commuting tripotents.*
(ii) *$R \cong R_1 \times R_2 \times R_3$, where R_1 is zero or $R_1/J(R_1)$ is Boolean with $U(R_1)$ a group of exponent 2, R_2 is zero or a subdirect product of $\mathbb{Z}_3$'s, and R_3 is zero or a subdirect product of $\mathbb{Z}_5$'s.*

Proof (i)⇒(ii). Write $3 = e + f$ where e, f are (commuting) tripotents. Then

$$8(e+f) = 24 = 3^3 - 3 = (e+f)^3 - (e+f) = 3e^2 f + 3ef^2.$$

Multiplying both sides by ef gives $8e^2 f + 8ef^2 = 3ef^2 + 3e^2 f$, *i.e.*, $5(e^2 f + ef^2) = 0$. So $2^3 \cdot 3 \cdot 5 = 5 \cdot 24 = 3 \cdot 5(e^2 f + ef^2) = 0$. Hence,

$$R = R_1 \times R_2 \times R_3, \quad \text{where } R_1 \cong R/2^3 R, \quad R_2 \cong R/3R, \quad \text{and} \quad R_3 \cong R/5R.$$

Then $8 = 0$ in R_1. For $a \in R_1$, write $a = e + f$ where e, f are commuting tripotents. Then we have

$$a^4 = e^4 + 4e^3 f + 6e^2 f^2 + 4ef^3 + f^4 = e^2 + 8ef + 6e^2 f^2 + f^2 = e^2 + 6e^2 f^2 + f^2$$

and

$$a^6 = a^4 a^2 = (e^2 + 6e^2 f^2 + f^2)(e+f)^2 = e^2 + 16ef + 14e^2 f^2 + f^2 = e^2 + 6e^2 f^2 + f^2.$$

So $a^6 = a^4$. Hence, R_1 has the identity $x^6 = x^4$. By Theorem 3.6, $R_1/J(R_1)$ is Boolean and $U(R_1)$ is a group of exponent 2.

Assume that $R_2 \neq 0$. We have $3 = 0$ in R_2. If $b^2 = 0$ in R_2, write $b = e + f$ where e, f are commuting tripotents in R_2. Then we have $0 = (e+f)^3 = e^3 + 3e^2 f + 3ef^2 + f^3 = e + f = b$. This shows that R_2 is a reduced ring, so R_2 is a subdirect product of the domains $\{R_\alpha\}$. Since R_α has only the trivial tripotents $0, 1, -1$, we infer that $R_\alpha = \{-2, -1, 0, 1, 2\}$. But $3 = 0$ in R_α, so $-2 = 1$ and $-1 = 2$. Thus, $R_\alpha = \{0, 1, 2\}$, which is isomorphic to $\mathbb{Z}_3$. Hence, R_2 is a subdirect product of $\mathbb{Z}_3$'s.

Assume that $R_3 \neq 0$. We have $5 = 0$ in R_3. If $b^2 = 0$ in R_3, write $b = e + f$ where e, f are commuting tripotents in R_3. Then $0 = (e+f)^5 = e^5 + 5e^4 f + 10e^3 f^2 + 10e^2 f^3 + 5ef^4 + f^5 = e + 5e^2 f + 10ef^2 + 10e^2 f + 5ef^2 + f = e + f = b$. This shows that R_3 is a reduced ring, so R_3 is a subdirect product of the domains $\{R_\alpha\}$. Since R_α has only the trivial tripotents $0, 1, -1$, we infer that $R_\alpha = \{-2, -1, 0, 1, 2\}$. But $5 = 0$ in R_α, so $R_\alpha \cong \mathbb{Z}_5$. Hence, R_3 is a subdirect product of $\mathbb{Z}_5$'s.

(ii)⇒(i). Let R_1, R_2, R_3 be given as in (ii). By Theorem 3.10, every element of $R_1 \times R_2$ is a sum of two commuting tripotents. Thus, we only need to show that every element of R_3 is a sum of two commuting tripotents. Let us assume that R is a subdirect product of $\{R_\alpha : \alpha \in \Lambda\}$ where $R_\alpha = \mathbb{Z}_5$ for all $\alpha \in \Lambda$. So R is a subring of $\prod_{\alpha \in \Lambda} R_\alpha$. Let $x = (x_\alpha) \in R$. Then Λ is a disjoint union of $\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3$, and Λ_4 such that $x_\alpha = i$ if and only if $\alpha \in \Lambda_i$ for $i = 0, 1, 2, 3, 4$. Without loss of generality, we can denote $x = (0_{\Lambda_0}, 1_{\Lambda_1}, 2_{\Lambda_2}, 3_{\Lambda_3}, 4_{\Lambda_4})$. As $x^4 = (0_{\Lambda_0}, 1_{\Lambda_1}, 1_{\Lambda_2}, 1_{\Lambda_3}, 1_{\Lambda_4}) \in R$, $y := x - x^4 = (0_{\Lambda_0}, 0_{\Lambda_1}, 1_{\Lambda_2}, 2_{\Lambda_3}, 3_{\Lambda_4}) \in R$. As $y^4 = (0_{\Lambda_0}, 0_{\Lambda_1}, 1_{\Lambda_2}, 1_{\Lambda_3}, 1_{\Lambda_4}) \in R$, $z := y - y^4 = (0_{\Lambda_0}, 0_{\Lambda_1}, 0_{\Lambda_2}, 1_{\Lambda_3}, 2_{\Lambda_4}) \in R$. As $z^4 = (0_{\Lambda_0}, 0_{\Lambda_1}, 0_{\Lambda_2}, 1_{\Lambda_3}, 1_{\Lambda_4}) \in R$,

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$w := z - z^4 = (0_{\Lambda_0}, 0_{\Lambda_1}, 0_{\Lambda_2}, 0_{\Lambda_3}, 1_{\Lambda_4}) \in R$. Let

$$\begin{aligned} e_4 &= (0_{\Lambda_0}, 0_{\Lambda_1}, 0_{\Lambda_2}, 0_{\Lambda_3}, 1_{\Lambda_4}) \in R, \\ e_3 &= z^4 - e_4 = (0_{\Lambda_0}, 0_{\Lambda_1}, 0_{\Lambda_2}, 1_{\Lambda_3}, 0_{\Lambda_4}) \in R, \\ e_2 &= y^4 - e_3 - e_4 = (0_{\Lambda_0}, 0_{\Lambda_1}, 1_{\Lambda_2}, 0_{\Lambda_3}, 0_{\Lambda_4}) \in R, \\ e_1 &= x^4 - e_2 - e_3 - e_4 = (0_{\Lambda_0}, 1_{\Lambda_1}, 0_{\Lambda_2}, 0_{\Lambda_3}, 0_{\Lambda_4}) \in R. \end{aligned}$$

Then

$$\begin{aligned} e &= e_2 + 4e_3 = (0_{\Lambda_0}, 0_{\Lambda_1}, 1_{\Lambda_2}, 4_{\Lambda_3}, 0_{\Lambda_4}) \in R, \\ f &= e_1 + e_2 + 4e_3 + 4e_4 = (0_{\Lambda_0}, 1_{\Lambda_1}, 1_{\Lambda_2}, 4_{\Lambda_3}, 4_{\Lambda_4}) \in R. \end{aligned}$$

It can be seen that $e^3 = e$, $f^3 = f$, $ef = fe$, and $x = e + f$. This shows that every element of R is a sum of two commuting tripotents. ∎

If $R \cong R_1 \times R_2 \times R_3$ as given in Theorem 5.2, then $R_1 \times R_2$ has the identity $x^6 = x^4$ and R_3 has the identity $x^5 = x$. So R has the identity $x^8 = x^4$. But a ring with identity $x^8 = x^4$ need not be a strong SIT-ring.

Example 5.3 The ring $\mathbb{Z}_{16}$ has the identity $x^8 = x^4$, but it is not a strong SIT-ring.

Proof Let $R = \mathbb{Z}_{16}$. Then R is local with $J(R) = 2R$. As $2^4 = 0$, for any $a \in J(R)$ we have $a^4 = 0$ and so $a^8 = a^4$. For any $a \in R \backslash J(R)$, we have $a^4 = 1$, so $a^8 = a^4$. Hence, R has the identity $x^8 = x^4$. But $4^2 = 0 \neq 8 = 2 \cdot 4$, so R is not a strong SIT-ring by Theorem 3.6. ∎

Proposition 5.4 *A ring R has the identity $x^8 = x^4$ with $2 \in J(R)$ if and only if $R/J(R)$ is Boolean, $j^4 = 0, 2j^2 = 4j$, and $8j = 0$ for all $j \in J(R)$.*

Proof $(\Rightarrow)$. For $j \in J(R)$, $j^8 = j^4$, so $j^4(1 - j^4) = 0$. As $1 - j^4 \in U(R)$, we have $j^4 = 0$. Moreover, $(1 \pm j)^8 = (1 \pm j)^4$, so $(1 \pm j)^4 = 1$ as $1 \pm j \in U(R)$. Thus, $1 + 4j + 6j^2 + 4j^3 + j^4 = 1$ and $1 - 4j + 6j^2 - 4j^3 + j^4 = 1$. That is, $4j + 6j^2 + 4j^3 = 0 = -4j + 6j^2 - 4j^3$. We see that $12j^2 = 0$, so $4j^2 = 0$ as $3 \in U(R)$. It follows that $4j + 2j^2 = 0 = -4j + 2j^2$. Therefore, $2j^2 = 4j$ and $8j = 0$. To see that $R/J(R)$ is Boolean, let $a \in R$. Then $(a - a^2)^4 = a^4 - 4a^5 + 6a^6 - 4a^7 + a^8 = 2(a^4 - 2a^5 + 3a^6 - 2a^7)$, which is nilpotent as $2 \in J(R)$. So $a - a^2$ is a nilpotent. By Lemma 3.5, there exists $e^2 = e \in R$ such that $ae = ea$ and $a - e$ is nilpotent. Thus, a is strongly nil clean, and R is strongly nil clean. By [3], $R/J(R)$ is Boolean.

$(\Leftarrow)$. For $a \in R$, we have $a - a^2 \in J(R)$ by hypothesis, so $a - a^2$ is nilpotent. As argued above, a is strongly nil clean; that is, $a = j + e$, where $j \in \mathrm{Nil}(R)$, $e^2 = e$ and $ea = ae$. As $R/J(R)$ is Boolean, $j \in J(R)$. So $j^4 = 0$, $2j^2 = 4j$, and $8j = 0$, showing $4j^2 = 0$. Thus, we have $a^4 = (j + e)^4 = j^4 + 4j^3e + 6j^2e + 4je + e = 2j^2e + 4je + e = 8je + e = e$, and hence $a^8 = e^2 = e = a^4$. So R has the identity $x^8 = x^4$. Moreover, $R/J(R)$ Boolean implies that $2 \in J(R)$. ∎

Theorem 5.5 *A ring R has the identity $x^8 = x^4$ if and only if $R \cong R_1 \times R_2 \times R_3$, where $R_1/J(R_1)$ is Boolean and $j^4 = 0, 2j^2 = 4j, 8j = 0$ for all $j \in J(R_1)$, R_2 is zero or a subdirect product of $\mathbb{Z}_3$'s, and R_3 is zero or a subdirect product of $\mathbb{Z}_5$'s.*

Proof ($\Leftarrow$). By Proposition 5.4, R_1 has the identity $x^8 = x^4$. As R_2 has the identity $x^3 = x$ and R_3 has the identity $x^5 = x$, they both have the identity $x^8 = x^4$. Hence, R has the identity $x^8 = x^4$.

($\Rightarrow$). We have $2^8 = 2^4$ in R, so $2^4 \cdot 3 \cdot 5 = 0$ in R. Hence, $R = R_1 \times R_2 \times R_3$, where $R_1 \cong R/2^4R$, $R_2 \cong R/3R$ and $R_3 \cong R/5R$. As R_1 has the identity $x^8 = x^4$ and $2 \in J(R_1)$, by Proposition 5.4 we see that $R_1/J(R_1)$ is Boolean, $j^4 = 0, 2j^2 = 4j$, and $8j = 0$ for all $j \in J(R)$.

Assume that $R_2 \neq 0$. We see that R_2 has the identity $x^8 = x^4$ and $3 = 0$. From $(x+1)^8 = (x+1)^4$, we obtain

$$x + x^2 + x^3 + x^4 + 2x^5 + x^6 + 2x^7 = 0. \tag{5.1}$$

From $(x-1)^8 = (x-1)^4$, we obtain

$$-x + x^2 - x^3 + x^4 - 2x^5 + x^6 - 2x^7 = 0. \tag{5.2}$$

Adding (5.1) to (5.2), we obtain $2x^2 + 2x^4 + 2x^6 = 0$. So $x^2 + x^4 + x^6 = 0$, giving $x^3 + x^5 + x^7 = 0$. Subtracting (5.2) from (5.1), we have $0 = 2x + 2x^3 + x^5 + x^7 = (2x + x^3) + (x^3 + x^5 + x^7) = 2x + x^3$. This shows that $x^3 = -2x = x$. So R_2 has the identity $x^3 = x$, and hence R is a subdirect product of $\mathbb{Z}_3$'s.

Assume that $R_3 \neq 0$. We see that R_3 has the identity $x^8 = x^4$ and $5 = 0$. From $(x+1)^8 = (x+1)^4$, we obtain

$$-x + 2x^2 + 2x^3 + x^5 + 3x^6 + 3x^7 = 0. \tag{5.3}$$

From $(x-1)^8 = (x-1)^4$, we obtain

$$x + 2x^2 + 3x^3 + 4x^5 + 3x^6 + 2x^7 = 0. \tag{5.4}$$

Adding (5.3) to (5.4), we obtain $4x^2 + x^6 = 0$; that is,

$$x^6 = x^2. \tag{5.5}$$

Replacing x by $1+x$ in (5.5), we have $1+6x+15x^2+20x^3+15x^4+6x^5+x^6 = 1+2x+x^2$. That is, $x^5 + x^6 = x + x^2$, showing $x^5 = x$. So R_2 has the identity $x^5 = x$, and hence R is a subdirect product of $\mathbb{Z}_5$'s. ■

6 Discussions and Comments

So far, no structure theorem is available for the rings for which every element is a sum of two idempotents, though some partial results were obtained in [2]. Here we present a structural result that reduces the situation to the case of characteristic 2.

Proposition 6.1 *The following are equivalent for a ring R.*

(i) *Every element of R is a sum of two idempotents.*
(ii) *$R \cong R_1 \times R_2$, where $ch(R_1) = 2$ and every element of R_1 is a sum of two idempotents, and R_2 is zero or a subdirect product of $\mathbb{Z}_3$'s.*

Proof (ii)$\Rightarrow$(i). This is clear.

(i)$\Rightarrow$(ii). Given (i), write $3 = e + f$ where e, f are idempotents of R. Then $ef = fe$ and so $9 = (e+f)^2 = e + 2ef + f = 3 + 2ef$. Thus, $2ef = 6 = 2(e+f)$. It follows that $2ef = (2ef)e = 2(e+f)e = 2e + 2ef$, showing that $2e = 0$. Similarly, we have $2f = 0$.

Hence, $6 = 2(e + f) = 0$. By the Chinese Remainder Theorem, $R = R_1 \times R_2$, where $R_1 \cong R/2R$ and $R_2 \cong R/3R$. Of course, every element of R_i is a sum of two idempotents ($i = 1, 2$). Assume that $R_2 \neq 0$. If $a^2 = 0$ where $a \in R_2$, then by [2, Lemma 2], $4a = 0$. As $3R_2 = 0$, we infer $a = 0$. Thus, R_2 is a reduced ring, and hence an abelian ring. So, by Theorem 2.1, R_2 is a subdirect product of $\mathbb{Z}_3$'s. ■

The next result improves [2, Corollary 1] by removing the assumption that R is semiprime. Let $C(R)$ denote the center of a ring R.

Corollary 6.2 *Suppose that every element of R is a sum of two idempotents. Then $C(R) = A \times B$, where A is Boolean and B is zero or a subdirect product of $\mathbb{Z}_3$'s.*

Proof By Proposition 6.1, $R = R_1 \times R_2$, where $\mathrm{ch}(R_1) = 2$ and every element of R_1 is a sum of two idempotents, and R_2 is zero or a subdirect product of $\mathbb{Z}_3$'s. So $C(R) = C(R_1) \times R_2$. Let $a \in C(R_1)$. Write $a = e + f$ where e, f are idempotents of R_1. Then $ef = fe$, so $a^2 = e + 2ef + f = a$ as $2ef = 0$. Hence, $C(R_1)$ is Boolean. ■

A *Morita context* is a 4-tuple $\left(\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right)$, where A, B are rings, ${}_AM_B$ and ${}_BN_A$ are bimodules, and there exist context products $M \times N \to A$ and $N \times M \to B$ written multiplicatively as $(x, y) \mapsto xy$ and $(y, x) \mapsto yx$, such that $\left(\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right)$ is an associative ring with the obvious matrix operations. A Morita context $\left(\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right)$ is called trivial if the context products are trivial, *i.e.,* $MN = 0$ and $NM = 0$. A trivial Morita context $\left(\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right)$ with $N = 0$ is commonly called a *formal triangular matrix ring*. By [2], if A, B are Boolean rings and M is an (A, B)-bimodule, then every element of $\left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$ is a sum of two idempotents. Indeed, if $T = \left(\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right)$ is a trivial Morita context with A, B Boolean, then every element of T is a sum of two idempotents: For $\left(\begin{smallmatrix} a & x \\ y & b \end{smallmatrix}\right) \in T$,

$$\begin{pmatrix} a & x \\ y & b \end{pmatrix} = \begin{pmatrix} 1 & x \\ y & 0 \end{pmatrix} + \begin{pmatrix} a-1 & 0 \\ 0 & b \end{pmatrix}$$

is a sum of two idempotents. Generally, for every element of $\left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$ to be a sum of two idempotents, A, B need not be Boolean. For instance, one can show that, for a Boolean ring B, every element of

$$\begin{pmatrix} \mathbb{T}_2(B) & \mathbb{M}_2(B) \\ 0 & \mathbb{T}_2(B) \end{pmatrix} \quad (\cong \mathbb{T}_4(B))$$

is a sum of two idempotents.

Question 6.3 *Characterize the rings R with* $\mathrm{ch}(R) = 2$ *such that every element of R is a sum of two idempotents.*

Let p be a prime. An element a in a ring is called a p-potent if $a^p = a$. We end the paper by raising the following question.

Question 6.4 *What can be said about the rings for which every element is a sum of two p-potents (that commute)?*

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