ON EULERIAN AND HAMILTONIAN GRAPHS AND LINE GRAPHS

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1. Introduction. A graph G has a finite set V of points and a set X of lines each of which joins two distinct points (called its end-points), and no two lines join the same pair of points. A graph with one point and no line is trivial. A line is incident with each of its end-points. Two points are adjacent if they are joined by a line. The degree of a point is the number of lines incident with it. The line-graph $L(G)$ of G has X as its set of points and two elements x, y of X are adjacent in $L(G)$ whenever the lines x and y of G have a common end-point. A walk in G is an alternating sequence $v_1, x_1, v_2, x_2, \ldots, v_n$ of points and lines, the first and last terms being points, such that x_i is the line joining v_i to v_{i+1} for $i = 1, \ldots, n-1$. We shall call $v_1, v_2, \ldots, v_n$ the point-sequence of this walk. G is connected if every two points of G are connected by a walk. A path is a walk in which the points are distinct. In a closed walk, $v_1 = v_n$. A tour is a closed walk in which no line appears more than once. A spanning tour of G is a tour in which each point of G appears at least once. An eulerian walk of G is a spanning tour containing every line of G. A cycle of G is a closed walk $v_1, x_1, v_2, x_2, \ldots, v_n, x_n, v_1$ in which $v_1, \ldots, v_n$ are distinct and $n \geq 3$. An hamiltonian cycle of G contains every point of G. A graph is eulerian if it has an eulerian walk; it is hamiltonian if it has an hamiltonian cycle.

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The object of this note is to study the relationship between eulerian and hamiltonian graphs and line-graphs. In this connection, we find it convenient to introduce the following formidable formulation which becomes clear after viewing Figure 1. Let G be a graph with p points and q lines. Then, if $n \geq 2$, $L_n(G)$ is a graph with nq points, 2q of which are points $f(v, x)$ corresponding to the pairs v, x such that v is a point of G and x is a line incident in G with v; the construction of $L_n(G)$ is completed by adding a path W_x with $n - 2$ new intermediate points connecting $f(u, x)$ to $f(v, x)$ whenever x is a line joining points u and v in G, and adding a line joining $f(v, x)$ to $f(v, y)$ whenever x, y are distinct lines incident with a point v in G.

In Figure 1, the lines in $L_2(G)$ and $L_3(G)$ which lie within the paths W_x are drawn as broken lines. Two lines x of G are numbered 1, 2 as are the corresponding points of L(G) and the corresponding paths W_x of $L_2(G)$ and $L_3(G)$.

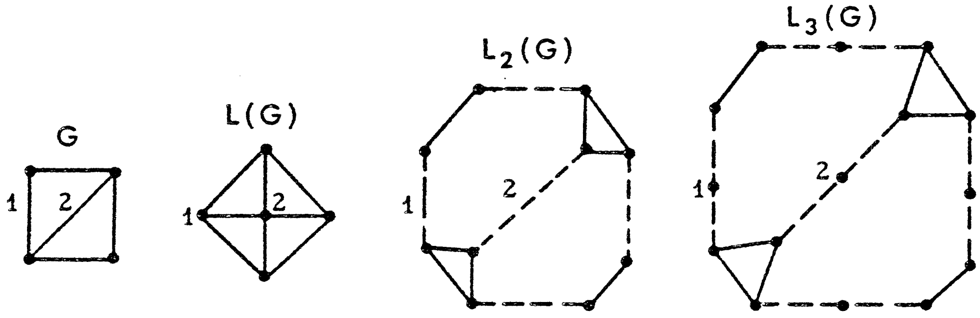

Fig. 1

2. Observations. The first three statements were given in Chartrand [1]; they are easily proved. Sedláček [5] also proved Proposition 3. With no real loss of generality, we assume throughout this section that G is connected and has at least two lines.

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PROPOSITION 1. *If* G *is eulerian, then* L(G) *is eulerian.*

PROPOSITION 2. *If* G *is eulerian, then* L(G) *is hamiltonian.*

PROPOSITION 3. *If* G *is hamiltonian, then* L(G) *is hamiltonian.*

That the converse of each of these three statements is false is easily seen from Figure 2, in which the first graph is neither eulerian nor hamiltonian, and the second, which is the line-graph of the first, is both. But some may object to the counter-example of Figure 2 since Whitney [6] has shown that these two graphs are the only two non-isomorphic graphs whose

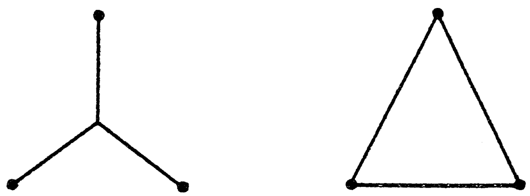

Fig.2

line-graphs are isomorphic. It is easy to supply alternative counter-examples to the converses of these three propositions. In Figure 1 above, L(G) is hamiltonian while G is not eulerian. In Figure 3, L(G) is eulerian while G is not.

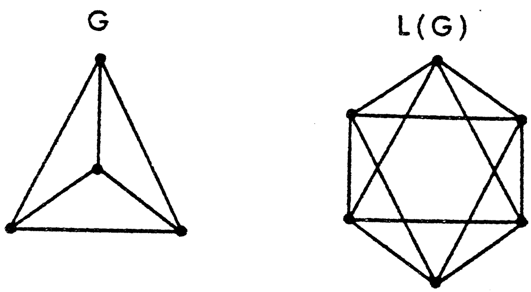

Fig.3

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And in Figure 4, $L(G)$ is hamiltonian while G is not.

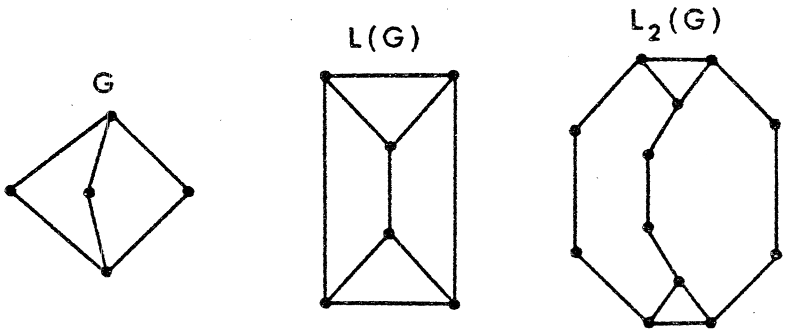

Fig. 4

A refinement of Proposition 3 is provided by the following pair of propositions.

PROPOSITION 4. *If G is hamiltonian, then $L_2(G)$ is hamiltonian.*

PROPOSITION 5. *If $L_2(G)$ is hamiltonian, then $L(G)$ is hamiltonian.*

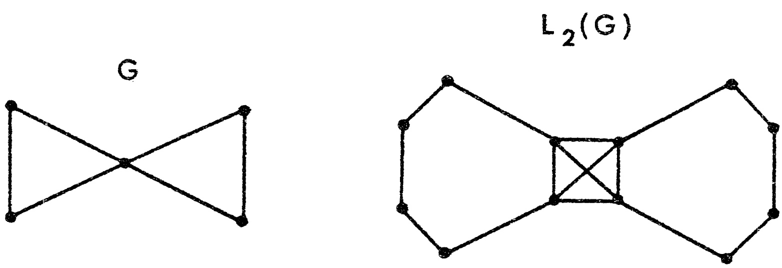

Fig. 5

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That the converse of Proposition 4 does not hold is seen from Figure 5, in which $L_2(G)$ is hamiltonian and G is not. The converse of Proposition 5 is also false, as is seen from Figure 4. Note that $L_2(G)$ may be hamiltonian without G being eulerian (see Figure 1). However, we now find that the corresponding property for $L_3(G)$ is entirely relevant.

PROPOSITION 6. _If_ G _is eulerian, then_ $L_3(G)$ _is hamiltonian, and conversely._

The truth of Propositions 1-5 (and, indeed, the falsity of their converses) is easily seen from the Propositions 7, 8 and 9 which we shall now state. The proof of Proposition 7 is given by Chartrand [1]. We shall prove Propositions 8 and 9, and thereafter Proposition 6.

PROPOSITION 7. $L(G)$ _is eulerian if and only if the degrees of the points of_ G _are all of the same parity._

PROPOSITION 8. $L(G)$ _is hamiltonian if and only if there is a tour in_ G _which includes at least one end-point of each line of_ G.

PROPOSITION 9. $L_2(G)$ _is hamiltonian if and only if there is a spanning tour in_ G.

Proof of Proposition 8. Let us suppose, first, that there is a tour $v_1, x_1, v_2, x_2, \ldots, v_n, x_n, v_1$ in G which includes at least one end-point of each line of G. Divide the lines of G not in this tour into n disjoint sets $S_1, \ldots, S_n$ such that the members of S_i are incident with v_i. If $S_i = \{y_i^1, \ldots, y_i^{r(i)}\}$, then

$$x_n, y_1^1, \ldots, y_1^{r(1)}, x_1, y_2^1, \ldots, y_2^{r(2)}, x_2, \ldots, y_n^1, \ldots, y_n^{r(n)}, x_n$$

is the point-sequence of an hamiltonian cycle of $L(G)$.

To prove the converse, let us assume that $L(G)$ has an hamiltonian cycle with point-sequence $x_1, x_2, \ldots, x_k, x_1$, where $x_1, \ldots, x_k$ are the distinct lines of G. Let v_i be the common end-point of x_i and x_{i+1} in G for $i = 1, \ldots, k-1$ and v_k be the common end-point of x_k and x_1. If $v_1 = \ldots = v_k$, the sequence with sole term v_1 is, in a trivial sense, a tour which includes an end-point of each line of G. Otherwise, let $v_{j(1)}, \ldots, v_{j(p)}$ be the subsequence of $v_1, \ldots, v_k$ consisting of the consecutively distinct points among $v_1, \ldots, v_k$; specifically (i) $v_{j(r)} = v_i \neq v_{j(r+1)}$ for $j(r) \leq i < j(r+1)$ and $r = 1, \ldots, p-1$ and (ii) $v_{j(p)} = v_h \neq v_{j(1)}$ for $h \geq j(p)$ and for $h < j(1)$. Then, since, for $r = 2, \ldots, p$, $v_{j(r-1)} = v_{j(r)-1}$ and $v_{j(r)}$ are distinct points incident with $x_{j(r)}$ in G, they are joined by $x_{j(r)}$; and similarly $v_{j(p)}$ is joined by $x_{j(1)}$ to $v_{j(1)}$. Hence

(1) $$v_{j(p)}, x_{j(1)}, v_{j(1)}, x_{j(2)}, v_{j(2)}, x_{j(3)}, \ldots, x_{j(p)}, v_{j(p)}$$

is a tour in G which includes each v_i and hence includes an end-point of each line of G.

Proof of Proposition 9. Let us suppose, first, that G has a spanning tour $v_1, x_1, v_2, x_2, \ldots, v_n, x_n, v_1$. Then clearly

(2) $$f(v_1, x_n),\ f(v_1, x_1),\ f(v_2, x_1),\ f(v_2, x_2),\ f(v_3, x_2),$$
$$\ldots,\ f(v_n, x_{n-1}),\ f(v_n, x_n),\ f(v_1, x_n)$$

is the point-sequence of a cycle in $L_2(G)$. Since the sequence $v_1, \ldots, v_n$ includes all the points of G (possibly with repetitions), the points of $L_2(G)$ not in (2) may be divided into n disjoint sets $S_1, \ldots, S_n$ such that all elements of S_i have the form $f(v_i, x)$. If, for $i = 2, \ldots, n$, the elements of S_i are inserted

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between $f(v_i, x_{i-1})$ and $f(v_1, x_i)$, and if in addition the elements of S_1 are inserted between $f(v_1, x_n)$ and $f(v_1, x_1)$, then the result is still the point-sequence of a cycle in $L_2(G)$; and this is an hamiltonian cycle since it includes all the points of $L_2(G)$.

To prove the converse, let us suppose that

$$f(v_1, x_1),\ f(v_2, x_2), \ldots,\ f(v_k, x_k),\ f(v_1, x_1)$$

is the point-sequence of an hamiltonian cycle of $L_2(G)$. Let $v_{j(1)}, \ldots, v_{j(p)}$ be the subsequence of $v_1, \ldots, v_k$ defined as in the proof of Proposition 8. Since $v_{j(r)-1} \neq v_{j(r)}$ and $f(v_{j(r)-1}, x_{j(r)-1})$ is adjacent to $f(v_{j(r)}, x_{j(r)})$ in $L_2(G)$, it follows that $x_{j(r)-1} = x_{j(r)}$ and hence that $x_{j(r)}$ joins $v_{j(r)-1} = v_{j(r-1)}$ to $v_{j(r)}$ in G for $r = 2, \ldots, p$; similarly $x_{j(1)}$ joins $v_{j(p)}$ to $v_{j(1)}$. Therefore (1) must be a spanning tour of G.

Proof of Proposition 6. For any line x of G, let $m(x)$ be the middle point of the path W_x. Suppose, first, that G is eulerian. Let $v_1, x_1, v_2, x_2, \ldots, v_n, x_n, v_1$ be an eulerian walk in G. Then clearly

$$f(v_1, x_1),\ m(x_1),\ f(v_2, x_1),\ f(v_2, x_2),\ m(x_2),\ f(v_3, x_2),$$

$$f(v_3, x_3),\ m(x_3),\ f(v_4, x_3),\ \ldots,\ m(x_n),\ f(v_1, x_n),\ f(v_1, x_1)$$

is the point-sequence of an hamiltonian cycle in $L_3(G)$. Conversely, if $L_3(G)$ is hamiltonian, we can clearly select an hamiltonian cycle H of $L_3(G)$ such that the second point in H is $m(x_1)$ for some line x_1 of G. Moreover, if a line x joins the points u and v in G, the terms immediately preceding and following $m(x)$ in the point-sequence of H must clearly be $f(u,x)$, $f(v,x)$ in some order. Therefore the point-sequence of H must be of the form

$$f(u_1,x_1),\ m(x_1),\ f(v_1,x_1),\ f(u_2,x_2),\ m(x_2),\ f(v_2,x_2),$$

$$\ldots,\ f(u_n,x_n),\ m(x_n),\ f(v_n,x_n),\ f(u_1,x_1),$$

where $x_1,\ldots,x_n$ is a list of the lines in G without repetitions and x_i joins u_i, v_i in G. Since $f(v_{i-1},x_{i-1})$, $f(u_i,x_i)$ are successive terms of H, they are adjacent in $L_3(G)$ and therefore $v_{i-1} = u_i$ for $i = 2,\ldots,n$; similarly $v_n = u_1$. Hence $u_1, x_1, u_2, x_2, \ldots, u_n, x_n, u_1$ is an eulerian walk in G.

3. Directed graphs. A digraph D has a finite non-empty set V of points and a subset X of $V \times V$ whose elements are called directed lines, with the convention that $u \neq v$ whenever $(u,v) \in X$. For a comprehensive presentation of the concepts of digraph theory, see [3]. We apply the adjective "directed" to the graphical terms: path, cycle, walk, tour, etc., to indicate that the directions of the directed lines are followed. The line-digraph $L(D)$ of D has X as its set of points, and an element $((u_1,v_1), (u_2,v_2))$ of $X \times X$ is a directed line of $L(D)$ if and only if $v_1 = u_2$. An eulerian digraph has a directed spanning tour containing all its directed lines. An hamiltonian digraph has a directed cycle containing all its points. We call D weakly connected if every two points of D are joined by a (not necessarily directed) walk.

PROPOSITION 10. *A non-trivial weakly connected digraph D is eulerian if and only if its line-digraph $L(D)$ is hamiltonian.*

In fact, it was noted by Kasteleyn [4] that there is a one-to-one correspondence between the eulerian walks of D and the hamiltonian cycles of $L(D)$, so that there is an equal number of each.

4. Problem. Characterize those graphs G for which $L(G)$ is hamiltonian. This class of graphs includes both the eulerian and hamiltonian graphs. (The problems of conveniently characterizing hamiltonian graphs and hamiltonian digraphs appear to be impossibly difficult in the present state of knowledge. We mention, however, that a construction can be given which reduces both of these problems to that of characterizing hamiltonian bicolourable undirected graphs, whose set of points can be coloured with two colours so that no two points of the same colour are adjacent.)

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