ON A THEOREM OF PRIVALOFF

P.S. Bullen

(received January 9, 1967)

1. It is the object of this note to extend to general harmonic structures a theorem due to Privaloff [12] concerning the definition of harmonic functions. The notation is that of [6, 9, 10], where many of the definitions not given here will be found.

The work on this paper was done in part while the author was at the Mathematics Research Centre, United States Army, Madison, Wisconsin.

2. Let X be a locally compact space with a harmonic structure in the sense of Boboc [3], Constantinescu [4] and Cornea [5]. That is to say the following five axioms are satisfied.

I. For all non-empty open subsets U of X there exists a subspace, $H(U)$, of the real continuous functions, such that $U \to H(U)$ is a sheaf. These are the harmonic functions on U.

II. There is a base of regular sets.

III. For all $x \in X$, there exists an h; harmonic at x and such that $h(x) > 0$.

IV. The MP (called MP_o sets in [3]) sets cover X. Hence there is a base of regular MP sets denoted by $\underset{\sim}{V}$; if $V \in \underset{\sim}{V}$ and $\overline{V} \subset U$, U a non-empty open set, V is said to be regular in U. Such sets will always be denoted by V, V', V_1 etc.

V. If A is a non-empty set of functions harmonic on U, directed by $\leq$ and with $\sup A$ finite on a dense set then $\sup A \in H(U)$.

The explanation of the terms in these axioms and of the axioms themselves will be found in the above references.

Let us suppose a further axiom to hold [8].

VI. There exists a locally strongly hypoharmonic function on X, p, that is μ_x^V-summable for all $V \in \underset{\sim}{V}$ and all $x \in V$.

Then [8] a differential operator can be introduced by defining

$$N - \bar{D}F(x) = \limsup_{N(x)} \frac{\bar{\Delta} F(x;V)}{\Delta p(x;V)} ,$$

where

$$\bar{\Delta} F(x;V) = \bar{H}_F^V (x) - F(x) = \int^* F \, d\mu_x^V - F(x) ,$$

$N(x)$ = the filter of sections of a fundamental system of regular MP neighbourhoods of x,

$$\Delta p(x;V) = H_p^V(x) - p(x) .$$

This is the upper derivative and by replacing lim sup by lim inf and $\bar{\Delta}$ by $\underline{\Delta}$, a lower derivative can be defined. If both exist and are equal the common value is called the derivative of F at x, $N - DF(x)$. Remark that $\bar{\Delta} F(x;V)$ and $\underline{\Delta} F(x;V)$ exist, are equal and are finite if F is μ_x^V-summable, in particular if F is hyperharmonic and finite on a dense set, that is if F is superharmonic. If there is no ambiguity the prefix N will be omitted. We write $F \in H^*(U)$ for F hyperharmonic in U.

It is easily seen that we have

THEOREM 1 [8]. If $f \in H^*(X)$ then for any N for which $N - \bar{D}f(x)$ is defined, $N - \bar{D}f(x) \leq 0$.

The result to be proved follows from a general converse of theorem 1. Such a converse has been given in [9] and in [10], but neither is general enough to include Privaloff's theorem in the classical case.

Loosely, the converse says if the derivative of a lower semi-continuous (l.s.c.) function is finite except possibly on a

https://doi.org/10.4153/CMB-1967-032-9 Published online by Cambridge University Press

set where hyperharmonic functions can be infinite, and is there infinite not because f is too large or too small, and if then the derivative is non-positive except possibly on a set where derivatives of hypoharmonic functions can be infinite, then the function is hyperharmonic.

3. A set E is said to be polar if there exists a $u \in H^*(X)$, finite on a dense set, with $E \subset \{u = \infty\}$.

A set Z is called a Z-set [8] if there exists a real non-negative $u \in H^*(U)$ such that $Z \subset \{Du = -\infty\}$. (In [8] u was required to be continuous, but this was for the purpose of defining a Perron integral.).

A function f will be said to be lower smooth at x if $\liminf_{N(x)} \bar{\Delta} f(x;V) \leq 0$. Remark that if f is l.s.c., bounded below and lower smooth then $\liminf_{N(x)} \bar{\Delta} f(x;V) = 0$, and that if f is real and continuous f is lower smooth [1].

4. THEOREM 2. If f is a numerical l.s.c. function on X and if for some N (a) $N - \underline{D}f(x) \leq 0$, $x \in X \sim Z$, Z a Z-set, (b) $N - \underline{D}f(x) < \infty$, $x \in X \sim E$, E polar (c) f is lower smooth on E, then $f \in H^*(X)$. If E is closed (c) can be replaced by (c') $f(x) > -\infty$ for $x \in E$.

Proof. We will assume, with no loss of generality, that $N(x)$ consists of all regular MP sets containing x. It is also immediate that (b) and (c) imply that for all $x \, \underline{\Delta} f(x;V) \neq \infty$, at least for small enough V. Hence we can assume that $f > -\infty$ - in all cases, by (c').

If $f \notin H^*(X)$ then it follows that there exists an x_1, and V_1, $x_1 \in V_1$, and an $h_1 \in C(\overline{V}_1)$ such that

(i) the restriction of h_1 to V_1 is harmonic in V_1,

(ii) for all $z \in V_1^*$, $h_1(z) < f(z)$,

(iii) $h(x_1) > f(x_1)$.

In fact we can take h_1 to be φ on V_1^* and $H_\varphi^{V_1}$ in V_1 for some suitable $\varphi \in C(V_1^*)$. Further by axiom III we can assume

https://doi.org/10.4153/CMB-1967-032-9 Published online by Cambridge University Press

that there exists $h_2 \in C(\overline{V}_1)$, with the restriction of h_2 to V_1 harmonic in V_1 and $h_2 > 0$. Throughout the proof x_1, V_1, h_1, h_2 will be used in this way.

(α) Suppose first that for all x, $\underline{D}f(x) < 0$.

Put $g = f - h_1$ and $g' = \frac{g}{h_2}$. Then (i) g' is l.s.c., (ii) $g' > -\infty$, (iii) $g'(z) > 0$ for all $z \in V_1$, (iv) $g'(x_1) < 0$, (v) $\underline{D}g(x) < 0$ for all $x \in V_1$.

Hence g' assumes a finite negative minimum at some point $x_2 \in V_1$. Consider then a V such that $x_2 \in V \subset \overline{V} \subset V_1$,
$\underline{\Delta}g(x_2;V) = \int_* g'h_2 d\mu_{x_2}^V - g'(x_2)h_2(x_2) \geq g'(x_2)\Delta h_2(x;V) = 0$.

Thus $\underline{D}g(x_2) \geq 0$, which contradicts the above property (v).

(β) Suppose now that $\underline{D}f(x) \leq 0$ for all $x \in X$. Then,

$$\underline{D}(f - \frac{p}{n}) \leq \underline{Df} - \frac{1}{n} < 0 .$$

So, by (α), $f - \frac{p}{n} \in H^*(x)$, for all n. Hence

$$\overline{\Delta}f(x;V) \leq \overline{\Delta}(f - \frac{p}{n})(x;V) + \Delta(\frac{p}{n})(x;V)$$

$$\leq \frac{1}{n}\Delta p(x;V), \text{ for all } n, V, x \in V .$$

Hence $\overline{\Delta}f(x;V) \leq 0$ for all V, $x \in V$, which implies that $f \in H^*(X)$.

(γ) Now let f satisfy the hypotheses of Theorem 2 with $E = \phi$. That is to say $\underline{Df}(x) < \infty$ for all $x \in X$ and $\underline{D}f(x) \leq 0$ f $x \in X \sim Z$, Z a Z-set.

Then let $u \in H^*(X)$, $0 \leq u \leq \infty$, $Z \subset \{Du = -\infty\}$. Put $g = f + \varepsilon u$, $\varepsilon > 0$. Then we can choose ε small enough so that if x_1, V_1, h_1 are as above we still have that for all $z \in V_1^*$, $h_1(z) < g(z)$ and $h_1(x_1) > g(x_1)$. So $g \notin H^*(X)$; but

$$\underline{\Delta}\, g(x;V) \leq \underline{\Delta}\, f(x;V) + \varepsilon\, \Delta\, u(x;V) .$$

https://doi.org/10.4153/CMB-1967-032-9 Published online by Cambridge University Press

So $\underline{D}g(x) \leq 0$ for all $x \in X$; which is a contradiction by (β) .

If then E is closed the argument to this point has shown that $f \in H^*(X \sim E)$ and hence since f is l.s.c. and locally bounded below on X and E is polar, it follows [7] that $f \in H^*(X)$.

(δ) Consider now the general case. Let $u \in H^*(X)$, finite on a dense set, and $E \subset \{u = \infty\}$.

Put $u_n = f + \frac{u}{n}$. Then for all $x \in X \sim Z$, $\underline{D}\, u_n(x) \leq 0$ and so by (γ) , $u_n \in H^*(X)$.

So if $u_o = \lim_{n \to \infty} u_n$, u_o is nearly hyperharmonic on X , and its regularization $\hat{u}_o \in H^*(X)$, [7]. Further, [7] , $\hat{u}_o \geq f$, and

$$\hat{u}_o(x) = \lim_{N(x)} \int^* u_o \, d\mu_x^V = \lim_{N(x)} \int^* f \, d\mu_x^V$$

since μ_x^V is not supported by the points where u is infinite [2]. Hence, since f is lower smooth, $\hat{u}_o = f$, which completes the proof.

COROLLARY 3. If f is a real continuous function on X and if for some N , (a) $N - \underline{D}f(x) \leq 0 \leq N - \overline{D}f(x)$, $x \in X \sim Z$, Z a Z-set, (b) $N - \underline{D}f(x) < \infty$, $N - \overline{D}f(x) > -\infty$, $x \in X \sim E$, E a polar set, then $f \in H(X)$.

Proof. This is immediate from Theorem 2 since as has been remarked real continuous functions are lower smooth.

This corollary is a generalization and an extension of Privaloff's result. In the classical case Z-sets are sets of Lebesgue measure zero and in Privaloff's theorem the polar set E is replaced by a closed set of zero capacity, which is of course polar [6].

COROLLARY 4. If f is l.s.c. and E is a polar set on which f is lower smooth then $\sup_{x \in X} N - \underline{D}f(x) = \sup_{x \in X \sim E} N - \underline{D}f(x)$.

Proof. Let $\Lambda = \sup_{x \in X} \underline{D}f(x)$, $\lambda = \sup_{x \in X \sim E} \underline{D}f(x)$ then since

https://doi.org/10.4153/CMB-1967-032-9 Published online by Cambridge University Press

$\Lambda \geq \lambda$ the result follows if $\lambda = \infty$. To prove equality if $-\infty \leq \lambda < \infty$ it suffices to prove that, for all finite α, $\lambda \leq \alpha$ implies $\Lambda \leq \alpha$. Further it is sufficient to consider the case $\alpha = 0$; since if $\alpha \neq 0$ we can consider $f - \alpha p$. By Theorem 2 if $\lambda \leq 0$, $f \in H^*(X)$ and hence, by Theorem 1, $\Lambda \leq 0$.

This generalized a result due to Denjoy and shows that polar sets are possible E-sets in the theory in [10].

COROLLARY 5. If a numerical function f attains a finite non-positive local minimum at x then $N - \underline{D}f(x) \geq 0$.

Proof. This is just part (α) of the proof of Theorem 2.

Thus the differential operator satisfies what Dynkin [11] calls the minimum principle.

5. If $X = R$ then Theorem 2 is not completely satisfactory since in most examples the only polar set is the empty set. (If $X = R^n$, $n > 1$, then in most examples enumerable sets are polar sets). Thus Theorem 2 does not cover the classical result [13] on convex functions.

However, we have proved [8]:

THEOREM 6. If f is l.s.c. on X and for some N

(a) $N - \underline{D}f(x) \leq 0$, $x \in X \sim Z$, Z a Z-set,

(b) $N - \underline{D}f(x) < \infty$, $X \sim E$, E countable

(c) $\liminf \dfrac{\Delta f(x;V)}{\rho(x;V)} \leq 0$ for all $x \in E$ then $f \in H^*(X)$.

Since $X = R$ the axioms in [8] are satisfied and the above notation, explained in [8], reduces as follows.

$$V = \,]x-h,\ x+k[\,,\quad h,\ k > 0,$$

$$\mu_x^V = \alpha_h \varepsilon_{x-h} + \beta_k \varepsilon_{x+k},\quad \alpha_h,\ \beta_k > 0$$

and ε_z the unit mass at z,

$$\rho(x;V) = \alpha_h h + \beta_k k,$$

$$\Delta(x;V) = \alpha_h f(x-h) + \beta_k f(x+k) - f(x).$$

https://doi.org/10.4153/CMB-1967-032-9 Published online by Cambridge University Press

Then, as explained in [9], this theorem includes the classical result on convex functions.

REFERENCES

1. H. Bauer, Axiomatische Behandlung des Durichletschen Problems für elliptische und parabolische Differentialgleichungen. Math. Annalen, 146 (1962), 1-59.

2. H. Bauer, Propriétés fines des fonctions hyperharmoniques dans une théorie axiomatiques du potentiel. Annales. Inst. Fourier, 15 (1965), 136-154.

3. H. Boboc, C. Constantinescu and A. Cornea, On the Dirichlet problem in the axiomatic theory of harmonic functions. Nagoya Math. J. 23 (1963), 73-96.

4. H. Boboc, C. Constantinescu and A. Cornea, Axiomatic theory of harmonic functions. Non-negative superharmonic functions. Annales. Inst. Fourier, 151 (1965), 283-312.

5. H. Boboc, C. Constantinescu and A. Cornea, Axiomatic theory of harmonic functions. Balayage. Annales Inst. Fourier, 152 (1965), 37-70.

6. M. Brelot, Eléments de la theorie classique du potential. (Paris, 1959).

7. M. Brelot, Lectures on potential theory. (Bombay, 1960).

8. P.S. Bullen, A general Perron integral. Can. J. Math., 17 (1965), 17-30.

9. P.S. Bullen, A general Perron integral II. (to appear in Can. J. Math.).

10. P.S. Bullen, A general Perron integral III. MRC Technical Summary Report, 696 (1966).

11. E.B. Dynkin, Markov processes. (New York, 1965).

12. I.I. Privaloff, Sur la definition d'une fonction harmonique. Comptes Rendus de l'Acad. Sci. U.R.S.S., 31 (1941), 102-103.

13. A. Zygmund, Trigonometrical Series. (Cambridge, 1959).

University of British Columbia

https://doi.org/10.4153/CMB-1967-032-9 Published online by Cambridge University Press