Glasgow Mathematical Journal (2023), **65**, pp. 138–140
doi:10.1017/S0017089522000180

Glasgow Mathematical Journal

RESEARCH ARTICLE

A note on the rational homological dimension of lattices in positive characteristic

Sam Hughes

Mathematical Institute, Andrew Wiles Building, University of Oxford, Oxford OX2 6GG, UK
E-mail: sam.hughes@maths.ox.ac.uk

Received: 6 December 2021; **Revised:** 9 May 2022; **Accepted:** 23 May 2022; **First published online:** 10 June 2022

2020 Mathematics Subject Classification.: *Primary* - 20J06, *Secondary* - 20F67

Abstract

We show via ℓ^2-homology that the rational homological dimension of a lattice in a product of simple simply connected Chevalley groups over global function fields is equal to the rational cohomological dimension and to the dimension of the associated Bruhat–Tits building.

1. Introduction

Let k be the function field of an irreducible projective smooth curve C defined over a finite field $\mathbb{F}_q$. Let S be a finite non-empty set of (closed) points of C. Let $\mathcal{O}_S$ be the ring of rational functions whose poles lie in S. For each $p \in S$, there is a discrete valuation ν_x of k such that $\nu_p(f)$ is the order of vanishing of f at p. The valuation ring $\mathcal{O}_p$ is the ring of functions that do not have a pole at p, that is

$$\mathcal{O}_S = \bigcap_{p \notin S} \mathcal{O}_p.$$

Let $\bar{k}$ denote the algebraic closure of k. Let $\mathbf{G}$ be an affine group scheme defined over $\bar{k}$ such that $\mathbf{G}(\bar{k})$ is almost simple. For each $p \in S$, there is a completion k_p of k and the group $\mathbf{G}(k_p)$ acts on the Bruhat–Tits building X_p. Thus, we may embed $\mathbf{G}(\mathcal{O}_S)$ diagonally into the product $\prod_{p\in S} \mathbf{G}(k_p)$ as an arithmetic lattice.

The *rational cohomological dimension* of a group Γ is defined to be

$$\mathrm{cd}_{\mathbb{Q}}(\Gamma) := \sup\{n \colon H^n(\Gamma;M) \neq 0,\ M \text{ a } \mathbb{Q}\Gamma\text{-module}\},$$

the *rational homological dimension* is defined completely analogously as

$$\mathrm{hd}_{\mathbb{Q}}(\Gamma) := \sup\{n \colon H_n(\Gamma;M) \neq 0,\ M \text{ a } \mathbb{Q}\Gamma\text{-module}\}.$$

In [5], it is shown that $\mathrm{cd}_{\mathbb{Q}}(\mathbf{G}(\mathcal{O}_S)) = \prod_{p\in S} \dim(X_p)$. In light of this, Ian Leary asked the author what is $\mathrm{hd}_{\mathbb{Q}}(\mathbf{G}(\mathcal{O}_S))$?

Theorem A. *Let* $\mathbf{G}$ *be a simple simply connected Chevalley group. Let* k *and* $\mathcal{O}_S$ *be as above. Then*

$$\mathrm{hd}_{\mathbb{Q}}(\mathbf{G}(\mathcal{O}_S)) = \mathrm{cd}_{\mathbb{Q}}(\mathbf{G}(\mathcal{O}_S)) = \prod_{p\in S} \dim(X_p).$$

More generally, we obtain the following.

Corollary B. *Let* Γ *be a lattice in a product of simple simply connected Chevalley groups over global function fields with associated Bruhat–Tits building* X. *Then* $\mathrm{hd}_{\mathbb{Q}}(\Gamma) = \mathrm{cd}_{\mathbb{Q}}(\Gamma) = \dim(X)$.

The author expects these results are well-known; however, they do not appear in the literature so we take the opportunity to record them here.

2. ℓ^2–homology and measure equivalence

Let Γ be a group. Both Γ and the complex group algebra $\mathbb{C}\Gamma$ act by left multiplication on the Hilbert space $\ell^2\Gamma$ of square-summable sequences. The *group von Neumann algebra* $\mathcal{N}\Gamma$ is the ring of Γ-equivariant bounded operators on $\ell^2 G$. The non-zero divisors of $\mathcal{N}G$ form an Ore set and the Ore localization of $\mathcal{N}\Gamma$ can be identified with the *ring of affiliated operators* $\mathcal{U}\Gamma$.

There are inclusions $\mathbb{Q}\Gamma \subseteq \mathcal{N}\Gamma \subseteq \ell^2\Gamma \subseteq \mathcal{U}\Gamma$, and it is also known that $\mathcal{U}\Gamma$ is a self-injective ring which is flat over $\mathcal{N}\Gamma$. For more details concerning these constructions, we refer the reader to [12] and especially to Theorem 8.22 of Section 8.2.3 therein. The *von Neumann dimension* and the basic properties we need can be found in [12, Section 8.3].

The ℓ^2-Betti numbers of a group Γ, denoted $b_i^{(2)}(\Gamma)$, are then defined to be the von-Neumann dimensions of the homology groups $H_i(\Gamma; \mathcal{U}\Gamma)$. The following lemma is a triviality.

Lemma 2.1. *Let* Γ *be a discrete group and suppose that* $b_i^{(2)}(\Gamma) > 0$. *Then the homology group* $H_i(\Gamma; \mathcal{U}\Gamma)$ *is non-trivial.*

Two countable groups Γ and Λ are said to be *measure equivalent* if there exist commuting, measure-preserving, free actions of Γ and Λ on some infinite Lebesgue measure space (Ω, m), such that the action of each of the groups Γ and Λ admits a finite measure fundamental domain. The key examples of measure equivalent groups are lattices in the same locally compact group [6]. The relevance of this for us is the following deep theorem of Gaboriau.

Theorem 2.2. (Gaboriau's Theorem [4]) *Suppose a discrete group* Γ *is measure equivalent to a discrete group* Λ. *Then* $b_p(\Gamma) = 0$ *if and only if* $b_p(\Lambda) = 0$.

3. Proofs

Proof of Theorem A. We first note that the group $\Gamma := \mathbf{G}(\mathcal{O}_S)$ is measure equivalent to the product $\Lambda := \prod_{p\in S} \mathbf{G}(\mathbb{F}_q[t_p])$ for some suitably chosen $t_p \in \mathcal{O}_p$. By [13, Theorem 1.6] (see also [2,3,1]), the group $\mathbf{G}(\mathbb{F}_q[t_p])$ has one non-vanishing ℓ^2-Betti number in dimension $\dim(X_p)$. Hence, by the Künneth formula Λ has one non-vanishing ℓ^2-Betti number in dimension $d = \prod_{p\in S} \dim(X_p)$. Thus, by Gaboriau's Theorem, the group Γ has exactly one non-vanishing ℓ^2-Betti number in dimension d. It follows from Lemma 2.1 that $\mathrm{hd}_{\mathbb{Q}}(\Gamma) \geq d$. The reverse inequality follows from the fact that Γ acts properly on the d-dimensional space $\prod_{p\in S} \dim(X_p)$.

Proof of Corollary B. The proof of the corollary is entirely analogous. First, we split $\mathbf{G}$ into a product of simple groups $\prod_{i=1}^n \mathbf{G}_i$ corresponding to the decomposition of the Bruhat–Tits building $X = \prod_{i=1}^n X_i$. Let Λ_i be a lattice in $\mathbf{G}_i$ and let $\Lambda = \prod_{i=1}^n \Lambda_i$. Each Λ_i has a non-vanishing ℓ^2-Betti number in dimension $\dim(X_i)$. In particular, Λ has a non-vanishing ℓ^2-Betti number in dimension $\dim(X) = \prod_{i=1}^n \dim(X_i)$. By Gaboriau's Theorem, Γ also has non-vanishing ℓ^2-Betti number in dimension $\dim(X)$. It follows from Lemma 2.1 that $\mathrm{hd}_{\mathbb{Q}}(\Gamma) \geq d$. The reverse inequality follows from the fact that Γ acts properly on the d-dimensional space $\prod_{p\in S} X_p$.

Remark 3.1. *A similar argument can be applied to lattices in products of simple simply connected algebraic groups over locally compact p-adic fields. One obtains the analogous result for such a lattice* Γ *that* $\mathrm{cd}_{\mathbb{Q}}(\Gamma) = \mathrm{hd}_{\mathbb{Q}}(\Gamma) = \dim(X)$, *where X is the associated Bruhat–Tits building.*

Acknowledgements. The author would like to thank his PhD supervisor Professor Ian Leary for his guidance, support, and suggesting of the question. This note contains material from the author's PhD thesis [7] and was originally part of [8], but was split off into a number of companion papers [9,10] (see also [11]) at the request of the referee. This work was supported by the Engineering and Physical Sciences Research Council grant number 2127970. The author would like to thank the referee for a number of helpful comments.

Competing interests. The author declares none.

References

[1] M. W. Davis et al., Weighted L2-cohomology of Coxeter groups, *Geom. Topol.* **11** (2007), 47–138. ISSN: 1465-3060. doi: 10.2140/gt.2007.11.47.
[2] J. Dymara, L2-cohomology of buildings with fundamental class, Proc. Amer. Math. Soc. **132**(6) (2004), 1839–1843. ISSN: 0002-9939. doi: 10.1090/S0002-9939-03-07234-4.
[3] J. Dymara, Thin buildings, *Geom. Topol.* **10** (2006), 667–694. ISSN: 1465-3060. doi: 10.2140/gt.2006.10.667.
[4] D. Gaboriau, Invariants l2 de relations d'Équivalence et de groupes, *Publ. Math. Inst. Hautes Études Sci.* **95** (2002), 93–150. ISSN: 0073-8301. doi: 10.1007/s102400200002.
[5] G. Gandini, Bounding the homological finiteness length, *Bull. London Math. Soc.* **44**(6) (2012), 1209–1214. ISSN: 0024-6093. doi: 10.1112/blms/bds047.
[6] M. Gromov, Asymptotic invariants of infinite groups, in *Geometric group theory, Vol. 2 (Sussex, 1991)*, London Mathematical Society Lecture Note *Series*, vol. **182** (Cambridge University Press, Cambridge, 1993), 1–295.
[7] S. Hughes, Equivariant cohomology, lattices, and trees, PhD Thesis (School of Mathematical Sciences, University of Southampton, 2021).
[8] S. Hughes, Graphs and complexes of lattices (2021). arXiv:2104.13728 [math.GR].
[9] S. Hughes, Lattices in a product of trees, hierarchically hyperbolic groups, and virtual torsion-freeness, *Bull. London Math*. *Soc*. doi: 10.1112/blms.12637
[10] S. Hughes, Irreducible lattices fibring over the circle (2022). arXiv:2201.06525 [math.GR].
[11] S. Hughes and M. Valiunas, Commensurating HNN-extensions: hierarchical hyperbolicity and biautomaticity (2021). arXiv: 2203.11996 [math.GR].
[12] W. Lück, *L^2-invariants: theory and applications to geometry and K-theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3^{rd} Series. A Series of Modern Surveys in Mathematics], vol. 44 (Springer-Verlag, Berlin, 2002), xvi+595. iSBN: 3-540-43566-2. doi: 10.1007/978-3-662-04687-6.
[13] H. D. Petersen, R. Sauer and A. Thom, L^2-Betti numbers of totally disconnected groups and their approximation by Betti numbers of lattices, *J. Topol.* **11**(1) (2018), 257–282. ISSN: 1753–8416. doi: 10.1112/topo.12056.