Forum of Mathematics, Sigma (2020), **Vol. 8**:e62, 1–6
doi:10.1017/fms.2020.61

RESEARCH ARTICLE

Dynamics of plane partitions: Proof of the Cameron–Fon-Der-Flaass conjecture

Rebecca Patrias[1] and Oliver Pechenik[2]

[1]Department of Mathematics, University of St. Thomas, St. Paul, MN 55105, USA; E-mail: patr0028@stthomas.edu.
[2]Department of Combinatorics & Optimization, University of Waterloo, Waterloo, ON N2L 3G1, Canada;
E-mail: oliver.pechenik@uwaterloo.ca.

Received: 4 May 2020; **Revised:** 28 October 2020; **Accepted:** 11 November 2020

2020 Mathematics Subject Classification: *Primary* – 05E18; *Secondary* – 05A17

Abstract
One of the oldest outstanding problems in dynamical algebraic combinatorics is the following conjecture of P. Cameron and D. Fon-Der-Flaass (1995): consider a plane partition P in an $a \times b \times c$ box B. Let $\Psi(P)$ denote the smallest plane partition containing the minimal elements of $\mathsf{B} - P$. Then if $p = a + b + c - 1$ is prime, Cameron and Fon-Der-Flaass conjectured that the cardinality of the Ψ-orbit of P is always a multiple of p.

This conjecture was established for $p \gg 0$ by Cameron and Fon-Der-Flaass (1995) and for slightly smaller values of p in work of K. Dilks, J. Striker and the second author (2017). Our main theorem specializes to prove this conjecture in full generality.

Contents

1. Introduction

The relatively young field of *dynamical algebraic combinatorics* studies dynamical properties of actions on various fundamental objects of algebraic combinatorics. For example, alternating sign matrices, plane partitions, root systems and Young tableaux all carry combinatorially natural cyclic group actions. In dynamical algebraic combinatorics, we are interested in establishing features of the resulting orbit structures, such as *cyclic sieving phenomena* [RSW04], *homomesies* [PR15], *periodicities* and *resonance phenomena* [DPS17]. For an excellent survey of the area, see [Str17].

One of the most studied actions in dynamical algebraic combinatorics is called *rowmotion*. Rowmotion can be defined as an action on the order ideals of any finite poset P. Interesting dynamical properties appear when P is chosen to be a poset of significance in algebraic combinatorics. While much of the literature on dynamical algebraic combinatorics dates from the past 15 or so years, rowmotion has older roots; it first appeared in 1974 through independent work of P. Duchet [Duc74] (in a special case) and of A. Brouwer and A. Schrijver [BS74] (in full generality).

One of the oldest open problems in dynamical algebraic combinatorics has been a 1995 conjecture of P. Cameron and D. Fon-Der-Flaass [CFDF95] on the periodicity of rowmotion for plane partitions. The main goal of this paper is to prove their conjecture, which we now recall.

Fix positive integers $a, b, c \in \mathbb{Z}^+$ and consider plane partitions sitting inside a rectangular $a \times b \times c$ box. We identify this box with the poset $\mathsf{B}_{a,b,c} = \mathbf{a} \times \mathbf{b} \times \mathbf{c}$ that is the product of three chains, and identify plane partitions in this box with order ideals of the poset $\mathsf{B}_{a,b,c}$.

We write $J(\mathsf{P})$ for the set of all order ideals of a poset P. Given $I \in J(\mathsf{P})$, define $\Psi(I)$ to be the order ideal generated by the minimal elements of the complementary order filter $\mathsf{P} - I$. Following [SW12], we refer to the operator Ψ as *rowmotion*. It is straightforward to see that the action of Ψ is reversible, so it permutes the elements of $J(\mathsf{P})$ and partitions them into disjoint orbits. For a general poset P, these orbits tend to be large and without discernible structure. However, for special posets P, intricate structure has been discovered (for various such results, see, e.g., [AST13, BS74, CFDF95, MP18, Pan09, PR15, RS13, SW12, Vor19]).

Cameron and Fon-Der-Flaass [CFDF95] made the following periodicity conjecture for rowmotion on the poset $\mathsf{B}_{a,b,c}$:

Conjecture 1.1 ([CFDF95]). *Suppose $p = a + b + c - 1$ is prime. Then the cardinality of every Ψ-orbit of $J(\mathsf{B}_{a,b,c})$ is a multiple of p.*

Remark 1.2. Conjecture 1.1 proposes a special kind of *resonance* in the sense of [DPS17]. That is, while the Ψ-orbit cardinalities remain unknown, they all ‘resonate with the frequency p,’ being all of the form hp for some positive integers h. It would be very interesting to understand the values h that appear. Experimentally, there appears to be a strong bias toward odd values of h. We currently have no explanation for this phenomenon, nor do we have good upper bounds on the values h.

Our main result is the following, which implies Conjecture 1.1:

Theorem 1.3. *Let k be the cardinality of any Ψ-orbit of $J(\mathsf{B}_{a,b,c})$. Then*

$$\gcd(k, a + b + c - 1) > 1.$$

Previous work has succeeded in establishing Conjecture 1.1 only for very small and very large values of c. The case $c = 1$ was established earlier by Brouwer and Schrijver [BS74] and the case $c = 2$ by Cameron and Fon-Der-Flaass [CFDF95]. (Indeed, in these ‘small c’ cases the size of every Ψ-orbit is exactly p.) Cameron and Fon-Der-Flaass [CFDF95] also established the ‘large c’ case $c > ab - a - b + 1$. This bound was later improved to

$$c > \frac{2ab - 2}{3} - a - b + 2$$

in [DPS17, Theorem 4.13].

Our superficially short proof of Theorem 1.3 and Conjecture 1.1 is uniform and does not rely on any of these previous partial results. Nonetheless, we are heavily indebted to previous work that was not available when Cameron and Fon-Der-Flaass first made their conjecture. Explicitly, our proof calls upon some of the main results of [DPS17] and [Pec17]. In a deeper sense, our proof builds as well on technology and theorems developed previously in various other papers [Pec14, SW12, TY09, TY11].

More specifically, in Section 2 we use the results of [DPS17] to translate Theorem 1.3 into an equivalent statement about the combinatorics of K-*promotion* on *increasing tableaux*. K-promotion was first studied in [Pec14] as an outgrowth of the combinatorics of K-theoretic Schubert calculus for Grassmannians introduced in [TY09], and has since been studied in several purely combinatorial contexts. In Section 3 we then prove this translated conjecture, relying on the main theorem of [Pec17].

https://doi.org/10.1017/fms.2020.61 Published online by Cambridge University Press

2. Reformulation in terms of increasing tableaux

Our first step in proving Theorem 1.3 is to use the results of [DPS17] to translate it into an equivalent statement regarding different combinatorics. First, we recall the definitions of increasing tableaux and the K-promotion operator on them.

We write $a \times b$ to denote the grid of boxes with a rows and b columns. Equivalently, this is the Young diagram of the partition with a parts all of size b. Index the boxes of $a \times b$ as in a matrix, so the box $(1, 2)$ is the box in the second column from the left in the top row. For a box b in $a \times b$, we write $\mathsf{b}^{\rightarrow}$ for the box immediately right of b, $\mathsf{b}^{\downarrow}$ for the box immediately below b, etc. A *short ribbon* in $a \times b$ is an edge-connected subset of boxes with at most two in any row or column.

An *increasing tableau* of shape $a \times b$ is a filling T of the boxes of $a \times b$ with positive integers, so that rows strictly increase from left to right and columns strictly increase from top to bottom. That is, for every box b, we have $T(\mathsf{b}) < T(\mathsf{b}^{\rightarrow})$ and $T(\mathsf{b}) < T(\mathsf{b}^{\downarrow})$. We write $\mathrm{Inc}(a \times b)$ for the set of all increasing tableaux of shape $a \times b$ and $\mathrm{Inc}^q(a \times b)$ for the finite subset with entries at most q. Note that in an increasing tableau, if we look at the set of boxes containing either i or $i + 1$, the edge-connected components of this set are all short ribbons.

Example 2.1. An increasing tableau of shape 3×6 is

$$T = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 5 & 6 & 10 \\ \hline 2 & 4 & 5 & 8 & 9 & 11 \\ \hline 6 & 7 & 9 & 10 & 13 & 17 \\ \hline \end{array} \in \mathrm{Inc}^{17}(3 \times 6).$$

Note that not every number from 1 to 17 need appear. Note also that, for example, the boxes labeled 4 and 5 make up two short ribbons, while the boxes labeled 1 and 2 make up a single short ribbon.

Following [BS16], we say that $T \in \mathrm{Inc}(a \times b)$ is *minimal* if we have

- $T(1, 1) = 1$,
- $T(\mathsf{b}^{\rightarrow}) = T(\mathsf{b}) + 1$ (for all b not in the rightmost column) and
- $T(\mathsf{b}^{\downarrow}) = T(\mathsf{b}) + 1$ (for all b not in the bottom row).

Note that there is a unique minimal tableau $M_{a\times b}$ of each shape $a \times b$, and that $M_{a\times b}$ is the unique element of $\mathrm{Inc}^{a+b-1}(a \times b)$. Moreover, $\mathrm{Inc}^q(a \times b)$ is empty if $q < a + b - 1$.

We now recall the definition of *K-promotion* on increasing tableaux. Let $T \in \mathrm{Inc}^q(a \times b)$. Consider the short ribbons consisting of the boxes labeled 1 and 2. Say a short ribbon is *trivial* if it consists of a single box and *nontrivial* otherwise. For each trivial short ribbon we do nothing, while for each nontrivial short ribbon we swap the labels 1 and 2. The result is generally not an increasing tableau, but nonetheless consider the short ribbons in it consisting of the boxes labeled 1 and 3 and repeat this process, successively swapping the pairs of labels $(1, 4), (1, 5), \ldots, (1, q)$ in nontrivial short ribbons. Note that if the box in position $(1, 1)$ originally had label 1, then label 1 finally appears only in position (a, b). To finish, decrement the label in each box by 1, and replace any resulting 0 label by q. The result is now an increasing tableau in $\mathrm{Inc}^q(a\times b)$, the K-promotion of T. See Example 2.3 for an example of this process. We will abuse notation by also denoting the K-promotion of T by $\Psi(T)$, as there can be no confusion with rowmotion of plane partitions. We write $\Psi^{\bullet}(T)$ to denote the Ψ-orbit of the increasing tableau T.

Remark 2.2. Increasing tableaux are a special case of the more classically studied *semistandard tableaux*, and K-promotion shares features with the *promotion* for semistandard tableaux of M.-P. Schützenberger [Sch72]; however, promotion of semistandard tableaux does not preserve the subset of increasing tableaux, and K-promotion does not coincide with promotion.

Example 2.3. Starting with the tableau $T \in \mathrm{Inc}^9(4 \times 4)$ shown here, we illustrate the process of computing its K-promotion $\Psi(T)$. At each step, trivial short ribbons are shown in light grey and nontrivial short ribbons are shown in darker grey.

https://doi.org/10.1017/fms.2020.61 Published online by Cambridge University Press

$T =$

1	2	4	5
2	3	5	6
4	5	7	8
5	6	8	9

$\rightarrow$

2	1	4	5
1	3	5	6
4	5	7	8
5	6	8	9

$\rightarrow$

2	3	4	5
3	1	5	6
4	5	7	8
5	6	8	9

$\rightarrow$

2	3	4	5
3	1	5	6
4	5	7	8
5	6	8	9

$\rightarrow$

2	3	4	5
3	5	1	6
4	1	7	8
5	6	8	9

$\rightarrow$

2	3	4	5
3	5	6	1
4	6	7	8
5	1	8	9

$\rightarrow$

2	3	4	5
3	5	6	1
4	6	7	8
5	1	8	9

$\rightarrow$

2	3	4	5
3	5	6	8
4	6	7	1
5	8	1	9

$\rightarrow$

2	3	4	5
3	5	6	8
4	6	7	9
5	8	9	1

$\rightarrow$

1	2	3	4
2	4	5	7
3	5	6	8
4	7	8	9

$= \Psi(T)$

By [DPS17, Theorem 4.4], there is a Ψ-equivariant bijection between the sets $J(\mathsf{B}_{a,b,c})$ and $\mathrm{Inc}^{a+b+c-1}(a \times b)$. Hence, to prove Theorem 1.3 and Conjecture 1.1, it is sufficient to establish the following:

Theorem 2.4. *Let $q > a + b - 1$ and suppose the Ψ-orbit of $T \in \mathrm{Inc}^q(a \times b)$ has cardinality k. Then* $\gcd(k, q) > 1$.

Remark 2.5. The hypothesis $q > a + b - 1$ in Theorem 2.4 is necessary merely to exclude the minimal tableau $M_{a\times b}$, corresponding under the Ψ-equivariant bijection of [DPS17] to the empty plane partition in the degenerate $a \times b \times 0$ box. Obviously, these objects have Ψ-orbits of size 1.

3. Proof of Theorem 2.4

Let $q \geq a + b - 1$ and fix $T \in \mathrm{Inc}^q(a \times b)$. Suppose $|\Psi^\bullet(T)| = k$ and $\gcd(k, q) = 1$. We aim to show that T is minimal, so $q = a + b - 1$.

The *frame* of the shape $a \times b$ is the set $\mathsf{Frame}(a \times b)$ consisting of those boxes in the first or last column, or first or last row, of $a \times b$. The frame $\mathsf{Frame}(U)$ of the tableau $U \in \mathrm{Inc}^q(a \times b)$ is the restriction of the filling U to $\mathsf{Frame}(a \times b)$.

Example 3.1. For T as in Example 2.3, the frame consists of the boxes shaded in light grey here:

1	2	4	5
2	3	5	6
4	5	7	8
5	6	8	9

Consider the cyclic group $C_k = \langle \psi \rangle$ of order k. Define an action of C_k on $\Psi^\bullet(T)$ by $\psi \cdot U = \Psi(U)$ for all $U \in \Psi^\bullet(T)$. Since k and q are relatively prime, the group element ψ^q also generates C_k. Hence, every $U \in \Psi^\bullet(T)$ is of the form $\Psi^{mq}(T)$ for some positive integer $m \in \mathbb{Z}^+$.

By [Pec17, Theorem 2], we have $\mathsf{Frame}(U) = \mathsf{Frame}(\Psi^q(U))$ for all tableaux $U \in \mathrm{Inc}^q(a\times b)$. Hence, by the observation of the previous paragraph, we have $\mathsf{Frame}(U) = \mathsf{Frame}(T)$ for every $U \in \Psi^\bullet(T)$. In particular, $\mathsf{Frame}(T) = \mathsf{Frame}(\Psi(T))$.

The condition $\mathsf{Frame}(T) = \mathsf{Frame}(\Psi(T))$ turns out to be very strict. Indeed, Proposition 3.2 implies that T is therefore a minimal tableau and $q = a + b - 1$, completing the proof of Theorem 2.4. $\square$

https://doi.org/10.1017/fms.2020.61 Published online by Cambridge University Press

Proposition 3.2. *Suppose* $V \in \mathrm{Inc}^{\ell}(a \times b)$ *satisfies* $\mathsf{Frame}(V) = \mathsf{Frame}(\Psi(V))$. *Then* V *is minimal and* $\ell = a + b - 1$.

Before proving this proposition, we need a few more definitions. Let $V \in \mathrm{Inc}^{\ell}(a \times b)$. Following [DPS17], we define the *flow path* of V to be the set of pairs $\{\mathsf{b}, \mathsf{b}'\}$ of adjacent boxes of V such that b and b' are at some point part of the same nontrivial short ribbon during the application of K-promotion to V. We define the *stream-bed* of V to be the union of the flow path – that is, the set of all boxes b appearing in any pair $\{\mathsf{b}, \mathsf{b}'\}$ of the flow path of V. (Warning: In [Pec14], the term 'flow path' was used to refer to what we here call a 'stream-bed.') Observe that if $\mathsf{b} \neq (1, 1)$ is in the stream-bed of V, then either $\{\mathsf{b}^{\leftarrow}, \mathsf{b}\}$ or $\{\mathsf{b}^{\uparrow}, \mathsf{b}\}$ is in the flow path of V. Similarly observe that if $\mathsf{b} \neq (a, b)$ is in the stream-bed of V, then either $\{\mathsf{b}, \mathsf{b}^{\rightarrow}\}$ or $\{\mathsf{b}, \mathsf{b}^{\downarrow}\}$ is in the flow path of V.

Example 3.3. Let T be as in Example 2.3. Then its stream-bed is the union of all the dark-grey short ribbons in all the tableaux illustrated there.

1	2	4	5
2	3	5	6
4	5	7	8
5	6	8	9

Proof of Proposition 3.2. We have $V(1, 1) = 1$, for otherwise we would have $\Psi(V)(1, 1) = V(1, 1) - 1$, contradicting $\mathsf{Frame}(V) = \mathsf{Frame}(\Psi(V))$. For any tableau $W \in \mathrm{Inc}^{\ell}(a \times b)$ with $W(1, 1) = 1$, we have $\Psi(W)(a, b) = \ell$. Hence, by $\mathsf{Frame}(V) = \mathsf{Frame}(\Psi(V))$, we also have $V(a, b) = \ell$.

Consider $\mathsf{b} \in \mathsf{Frame}(a \times b)$. If b is not in the stream-bed of V, then $\Psi(V)(\mathsf{b}) = V(\mathsf{b}) - 1$, contradicting $\mathsf{Frame}(V) = \mathsf{Frame}(\Psi(V))$. Hence, every box of $\mathsf{Frame}(a \times b)$ must be in the stream-bed of V.

Consider $\{\mathsf{b}, \mathsf{b}^{\rightarrow}\}$ in the top row of V. Since $\mathsf{b}^{\rightarrow}$ is in the stream-bed of V, the pair $\{\mathsf{b}, \mathsf{b}^{\rightarrow}\}$ must be in the flow path of V. Hence $\Psi(V)(\mathsf{b}) = V(\mathsf{b}^{\rightarrow}) - 1$. But by assumption, $\Psi(V)(\mathsf{b}) = V(\mathsf{b})$, so we have $V(\mathsf{b}^{\rightarrow}) = V(\mathsf{b}) + 1$. Similarly, we have $V(\mathsf{b}^{\downarrow}) = V(\mathsf{b}) + 1$ for b in the leftmost column of V.

Consider $\{\mathsf{b}, \mathsf{b}^{\rightarrow}\}$ in the bottom row of V. Since b is in the stream-bed of V, the pair $\{\mathsf{b}, \mathsf{b}^{\rightarrow}\}$ must be in the flow path of V. Thus again we have $V(\mathsf{b}^{\rightarrow}) = V(\mathsf{b}) + 1$. Similarly, we have $V(\mathsf{b}^{\downarrow}) = V(\mathsf{b}) + 1$ for b in the rightmost column of V.

Therefore, the entries of V increase consecutively around $\mathsf{Frame}(a \times b)$ from upper left to lower right. In particular, the largest entry of V must be $a + b - 1$. But we have already determined that this largest entry is ℓ in position (a, b). Hence, $\ell = a + b - 1$, and V is the minimal tableau $M_{a \times b}$. □

Acknowledgements. The second author was partially supported by a Mathematical Sciences Postdoctoral Research Fellowship (#1703696) from the National Science Foundation.

The authors are grateful to Hugh Thomas and the anonymous referees for comments on earlier drafts of this paper.

Conflict of Interest: None.

References

[AST13] D. Armstrong, C. Stump and H. Thomas, 'A uniform bijection between nonnesting and noncrossing partitions', *Trans. Amer. Math. Soc.* **365**(8) (2013), 4121–4151.

[BS16] A. S. Buch and M. J. Samuel, 'K-theory of minuscule varieties', *J. Reine Angew. Math.* **719** (2016), 133–171.

[BS74] A. E. Brouwer and A. Schrijver, *On the Period of an Operator, Defined on Antichains*, Mathematisch Centrum Afdeling Zuivere Wiskunde ZW, 24/74 (Mathematisch Centrum, Amsterdam, 1974).

[CFDF95] P. J. Cameron and D. G. Fon-Der-Flaass, 'Orbits of antichains revisited', *European J. Combin.* **16**(6) (1995), 545–554.

[DPS17] K. Dilks, O. Pechenik and J. Striker, 'Resonance in orbits of plane partitions and increasing tableaux', *J. Combin. Theory Ser. A* **148** (2017), 244–274.

[Duc74] P. Duchet, 'Sur les hypergraphes invariantes', *Discrete Math.* **8** (1974), 269–280.

[MP18] H. Mandel and O. Pechenik, 'Orbits of plane partitions of exceptional Lie type', *European J. Combin.* **74** (2018), 90–109.

[Pan09] D. I. Panyushev, 'On orbits of antichains of positive roots', *European J. Combin.* **30**(2) (2009), 586–594.

[Pec14] O. Pechenik, 'Cyclic sieving of increasing tableaux and small Schröder paths', *J. Combin. Theory Ser. A* **125** (2014), 357–378.

[Pec17] O. Pechenik, 'Promotion of increasing tableaux: Frames and homomesies', *Electron. J. Combin.* **24**(3) (2017), 3.50.

[PR15] J. Propp and T. Roby, 'Homomesy in products of two chains', *Electron. J. Combin.* **22**(3) (2015), 3.4.

[RS13] D. B. Rush and X. Shi, 'On orbits of order ideals of minuscule posets', *J. Algebraic Combin.* **37**(3) (2013), 545–569.

[RSW04] V. Reiner, D. Stanton and D. White, 'The cyclic sieving phenomenon', *J. Combin. Theory Ser. A* **108**(1) (2004), 17–50.

[Sch72] M. P. Schützenberger, 'Promotion des morphismes d'ensembles ordonnés', *Discrete Math.* **2** (1972), 73–94.

[Str17] J. Striker, 'Dynamical algebraic combinatorics: Promotion, rowmotion, and resonance', *Notices Amer. Math. Soc.* **64**(6) (2017), 543–549.

[SW12] J. Striker and N. Williams, 'Promotion and rowmotion', *European J. Combin.* **33**(8) (2012), 1919–1942.

[TY09] H. Thomas and A. Yong, 'A jeu de taquin theory for increasing tableaux, with applications to K-theoretic Schubert calculus', *Algebra Number Theory* **3**(2) (2009), 121–148.

[TY11] H. Thomas and A. Yong, 'Longest increasing subsequences, Plancherel-type measure and the Hecke insertion algorithm', *Adv. in Appl. Math.* **46**(1-4) (2011), 610–642.

[Vor19] C. Vorland, 'Homomesy in products of three chains and multidimensional recombination', *Electron. J. Combin.* **26**(4) (2019), 4.30.

https://doi.org/10.1017/fms.2020.61 Published online by Cambridge University Press