HOMOTOPY OF COMPACT SYMMETRIC SPACES

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Introduction. In recent years a new approach to the study of compact symmetric spaces has been taken by Nagano and Chen [**10**]. This approach assigned to each pair of antipodal points on a closed geodesic a pair of totally geodesic submanifolds. In this paper we will show how these totally geodesic submanifolds can be used in conjunction with a theorem of Bott to compute homotopy in compact symmetric spaces. Some of the results are already known (see [**1**], [**5**], [**11**] for example) but we include them here for completeness and to illustrate this unified approach. We also exhibit a connection between the second homotopy group of a compact symmetric space and the multiplicity of the highest root. Using this in conjunction with a theorem of J. H. Cheng [**6**] we obtain a topological characterization of quaternionic symmetric spaces with antiquaternionic involutive isometry. The author would like to thank Prof T. Nagano for all his help and his detailed descriptions of the totally geodesic submanifolds mentioned above.

Preliminaries.

Definition. A Riemannian manifold M is called a *symmetric space*, if for each point q of M, there exists an involutive isometry s_q of M such that q is an isolated fixed point of s_q. We call s_q the *symmetry* at q.

If G is the closure in the compact open topology of the group of symmetries generated by $\{s_q \mid q \in M\}$, then it is known that G is transitive on M, (provided M is connected, which we will assume hereafter) and hence the typical isotropy subgroup K, say at o, is compact and $M = G/K$. We will assume throughout that G is semisimple and that M (and therefore G) is compact. We will denote the Lie algebra of G by $\mathbf{g}$ and the involution $\mathrm{ad}(s_o)(g) = s_o g s_o$ by $\mathrm{ad}(s_o)$. We will use the same notation for the induced involution on $\mathbf{g}$. Since G is semisimple, the Killing form $B_{\mathbf{g}}$ is a negative definite bilinear form on $\mathbf{g}$ invariant under $\mathrm{ad}\, G$.

The Cartan decomposition of $\mathbf{g}$ with respect to $\mathrm{ad}(s_o)$ is given by $\mathbf{g} = \mathbf{k} + \mathbf{m}$ where $\mathbf{k}$ and $\mathbf{m}$ denote the eigenspaces of plus and minus one respectively. Since $\mathbf{k}$ is the Lie algebra of K, we identify $\mathbf{m}$ with the tangent space to M at o. The following inclusions are well known.

$$[\mathbf{m}, \mathbf{m}] \subset \mathbf{k}, \qquad [\mathbf{k}, \mathbf{m}] \subset \mathbf{m}, \qquad [\mathbf{k}, \mathbf{k}] \subset \mathbf{k}.$$

We will denote by $<,>$ the unique Riemannian metric on M, which is invariant under G and coincides on the tangent space T_oM to M at o with $-B_{\mathbf{g}}$. Recall that on a symmetric space M the exponential map $\mathrm{Exp}: T_oM \to M$ is given by $\mathrm{Exp}\, X = (\exp X)(o)$, where exp is the exponential map of G, sometimes just written as e^X, for $X \in \mathbf{m}$. We will denote by $\mathbf{h}$ a maximal abelian subalgebra of $\mathbf{m}$ and by A its image under $\mathrm{Exp}: T_oM \to M$, that is A is a maximal torus through o in M. The dimension of such a torus is then by definition the rank of M denoted by $r(M)$. Using the fact that $\{\mathrm{ad}(H)^2 \mid H \in \mathbf{h}\}$ is a commutative system of semisimple operators stabilizing $\mathbf{m}$ and $\mathbf{k}$ we get the following well known result [**7**, Chapter 7].

THEOREM A. (1) *We have the following orthogonal root space decompositions*

$$\mathbf{m} = \mathbf{h} + \sum_{\alpha \in R(M)} \mathbf{m}_\alpha, \qquad \mathbf{k} = \mathbf{k}_0 + \sum_{\alpha \in R(M)} \mathbf{k}_\alpha.$$

(2) *For each $\alpha \in R(M)$ we can choose bases* $\{X_\alpha\}$ and $\{Y_\alpha\}$ *for $\mathbf{m}_\alpha$ and $\mathbf{k}_\alpha$ respectively such that*

(a)

$$[H, X_\alpha] = \alpha(H)Y_\alpha \quad \textit{and} \quad [H, Y_\alpha] = -\alpha(H)X_\alpha, \qquad \textit{for all } H \in \mathbf{h}$$

(b)

$$\mathrm{ad}(\exp H)X_\alpha = \cos \alpha(H)X_\alpha + \sin \alpha(H)Y_\alpha \quad \textit{and}$$

$$\mathrm{ad}(\exp H)Y_\alpha = \cos \alpha(H)Y_\alpha - \sin \alpha(H)X_\alpha, \qquad \textit{for all } H \in \mathbf{h}.$$

DEFINITION. The linear forms $\alpha : \mathbf{h} \to \mathbb{R}$ are called the *roots* of M with respect to $\mathbf{h}$ and $\mu(\alpha) = \dim \mathbf{m}_\alpha = \dim \mathbf{k}_\alpha$ is called the *multiplicity* of α.

We can order the roots in a standard manner and this enables us to define positive roots and simple roots. We will denote the simple roots by $\alpha_1, \ldots, \alpha_r$ where $r = r(M)$ and we will use the symbol Σ to denote the set of simple roots. Before proceeding we recall a few more facts. For $X \in \mathbf{g}$ we may consider X as a vector field on M, which we will also denote by X. Its value $X(p)$ at $p \in M$ is the initial tangent to the curve $(\exp tX)(p)$. Since G acts on M as a group of isometries, it therefore carries geodesics to geodesics and hence X restricted to a geodesic is a Jacobi field along that geodesic.

For each smoothly closed geodesic c through o, we consider the antipodal point p of o on c. Denote by $M^+(p)$ the orbit $K_{(1)}(p)$ of the identity component of K through p, then $M^+(p) = F(s_o, M)(p)$ is a symmetric space and is called a *polar set* of M. Note that s_o fixes the point p and therefore $s_o \circ s_p = s_p \circ s_o$, so that $\mathrm{ad}(s_p)$ stabilizes $\mathbf{k}$ and $\mathbf{m}$ giving us a Cartan decomposition $\mathbf{g} = (\mathbf{k}^+ + \mathbf{m}^+) + (\mathbf{k}^- + \mathbf{m}^-)$ at p. $M^+(p) = K/K^+$ and the tangent space to $M^+(p)$ at p can be identified with $\mathbf{k}^-$, that is $T_pM^+(p) = \{Y(p) \mid Y \in k^-\}$. The normal space to $T_pM^+(p)$ is the tangent space to another connected totally geodesic submanifold denoted by $M^-(p)$. Now $M^-(p) = F(s_p \circ s_o, M)(p) = F(s_p \circ s_o, M)(o)$ and $T_oM^-(p)$ can be identified with $\mathbf{m}^-$ as can $T_pM^-(p)$. We also have that $M^-(p) = G^-/K^+$ where G^- is the connected subgroup of G given by the Lie subalgebra $\mathbf{k}^+ + \mathbf{m}^-$. We define two lattices as follows $\Gamma = \{H \in \mathbf{h} \mid \exp H \in K\}$ and $\Gamma^0 = \sum \mathbb{Z}\alpha^\vee$ where the sum is taken over all roots and $\alpha^\vee = 2\pi A_\alpha / \langle A_\alpha, A_\alpha \rangle$, where A_α is the unique vector in $\mathbf{h}$ such that $\langle H, A_\alpha \rangle = \alpha(H)$ for all $H \in \mathbf{h}$.

DEFINITION. The *index* of a geodesic γ from p to q is defined to be the number of conjugate points of p counted with their multiplicities, in the open geodesic segment from p to q. We will denote the space of shortest geodesics from p to q by the symbol Ω^d.

THEOREM (R. Bott). *If Ω^d is a topological manifold and if every non shortest geodesic from p to q has index greater than or equal to λ_0, then*

$$\Pi_{i+1}(M) \cong \Pi_i(\Omega^d), \qquad \text{for } i < \lambda_0 - 1.$$

DEFINITION. A polar set $M^+(p)$ is called a *pole* of the symmetric space M if and only if $M^+(p) = \{p\}$. This is equivalent to the condition that the symmetries s_p and s_o agree on M.

Now $s_o \circ s_q = s_q \circ s_o$ if and only if either $s_o(q) = q$ or q is the midpoint of a geodesic from o to a pole o' of M.

DEFINITION. Let o' be a pole of M, then the *centrosome*, denoted by $C(o, o')$ is defined as follows $C(o, o') = \{y \in M \mid s_o \circ s_y = s_y \circ s_o, s_o(y) \neq y\}$.

Note if we project to the symmetric space $M/(o, o')$ (obtained from M by identifying o and o') or M^* the symmetric space characterized by the property that every space locally isometric to M is a covering manifold of M, then $C(o, o')$ projects to a polar set of this space. Thus finding the new polar set in this space (which is starred in the list of Nagano and Chen [**10**]) gives us a method of finding the space of shortest geodesics from o to o'. The procedure will therefore be as follows. We choose a polar set $M^+(p)$ furthest away from o in order to make the index of non shortest geodesics to p as large as possible. All shortest geodesics from o to p will lie in $M^-(p)$ since they are perpendicular to $M^+(p)$. By definition of $M^-(p)$ we have that $s_o \circ s_p$ is the identity on $M^-(p)$ and therefore p is a pole of $M^-(p)$. The space of shortest geodesics from o to p can then be calculated by applying the centrosome argument as explained above to the space $M^-(p)$. In order to carry out the computations we will need to know all the polar sets for a given space, these have all been calculated by Nagano and Chen [**10**]. In order to find the polar set furthest from o we will use [**3**, Proposition 2.2], which tells us the initial tangent to the shortest geodesic to p. If $p = \operatorname{Exp} X$ then the initial tangent to any other geodesic from o to p is of the form $X + H$ with H in Γ. We note that in order to calculate higher homotopy we can work on the universal covering space of M and we may assume that $H \in \Gamma^0 = \sum \mathbb{Z}\alpha^\vee$. Since a lot of the calculations are long but not difficult, we will not include them all here.

1. Spaces of classical type.

***R(M)* of type A_r.** We first consider the spaces $AI(2m) = SU(2m)/SO(2m)$, $A_{2m-1} = SU(2m)$ and $AII(2m) = SU(4m)/SP(2m)$ all of which have the root system A_{2m-1}. All these spaces have a pole p which is the furthest polar set from o. The shortest geodesic to p has initial tangent given by the weight $\pi\omega_m = \pi/2(\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_m - \varepsilon_{m+1} - \ldots - \varepsilon_{2m})$. In the case of the group $SU(2m)$ all tangent vectors in the discussion should be doubled. The roots for A_{2m-1} are $\varepsilon_i - \varepsilon_j (i \neq j, 1 \leqslant i, j \leqslant 2m)$ and therefore $(\varepsilon_i - \varepsilon_j)^\vee = \pi(\varepsilon_i - \varepsilon_j)$. We have therefore that the initial tangents to the non shortest geodesics from o to p have the form $\pi/2(n_1\varepsilon_1 + n_2\varepsilon_2 + \ldots - n_{m+1}\varepsilon_{m+1} - \ldots - n_{2m}\varepsilon_{2m})$ where the n_i's are odd integers. The initial tangent to a typical non shortest geodesic for which the index λ_0 is minimal has $n_{m-1} = 3$ and all other $n_i = 1$. For such a geodesic we get $m + 1$ conjugate points before reaching p (not counting multiplicity), since each of the positive roots $\varepsilon_{m-1} - \varepsilon_m, \ldots, \varepsilon_{m-1} - \varepsilon_{2m}$ evaluate on $\pi/2(\varepsilon_1 + \ldots + 3\varepsilon_{m-1} - \ldots - \varepsilon_{2m})$ to give a value greater than π and therefore the Jacobi fields Y_α (which behave like the sine function, see [**3**]) for $\alpha = \varepsilon_{m-1} - \varepsilon_j, j > m - 1$ when restricted to the non shortest geodesic vanish before p. Since the roots all have the same length, they all have the same multiplicity for a given space. The multiplicities μ are as follows, $\mu = 1$ for $AI(2m)$, $\mu = 2$ for A_{2m-1}, and $\mu = 4$ for $AII(2m)$. We have therefore that $\lambda_0 = m + 1$ for $AI(2m)$, $\lambda_0 = 2m + 2$ for A_{2m-1}, and $\lambda_0 = 4m + 4$ for $AII(2m)$. Using the centrosome argument we see by consulting the list of Nagano and Chen that the corresponding spaces of shortest

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geodesics are as follows. $\Omega^d = G_m(\mathbb{C}^{2m})$ for A_{2m-1}, $\Omega^d = G_m(\mathbb{R}^{2m})$ for $AI(2m)$, and $\Omega^d = G_m(\mathbb{H}^{2m})$ for $AII(2m)$. We therefore have the following proposition.

PROPOSITION 1.1

$$\Pi_{i+1}SU(2m) \cong \Pi_i G_m(\mathbb{C}^{2m}), \qquad \textit{for } 1 \leq i \leq 2m,$$
$$\Pi_{i+1}SU(2m)/SO(2m) \cong \Pi_i G_m(\mathbb{R}^{2m}), \qquad \textit{for } 1 \leq i \leq m-1,$$
$$\Pi_{i+1}SU(4m)/Sp(2m) \cong \Pi_i G_m(\mathbb{H}^{2m}), \qquad \textit{for } 1 \leq i \leq 4m+2.$$

The above result for $SU(2m)$ was first obtained by Bott by choosing I_{2m} and $-I_{2m} \in SU(2m)$ as the points p and q in his theorem. Note that $-I_{2m}$ is in fact the pole of $SU(2m)$. The same argument as above also works for the cases $SU(2m+1)$, $AI(2m+1)$, and $AII(2m+1)$. The corresponding weight vector in this case is given as follows:

$$\omega_m = \pi(\varepsilon_1 + \ldots + \varepsilon_m - (m/2m+1)(\varepsilon_{m+1} + \ldots + \varepsilon_{2m+1}))$$

and the Ω^d's are $G_m(\mathbb{C}^{2m+1})$ for A_{2m}, $G_m(\mathbb{R}^{2m+1})$ for $AI(2m+1)$, and $G_m(\mathbb{H}^{2m+1})$ for $AII(2m+1)$.

***R(M)* of type *D_r*.** Here we are considering the spaces $SO(2m)$ and $G_m(\mathbb{R}^{2m})$ which have the root system D_m. Again these spaces have a pole, and it is the polar set furthest from o. The initial tangent to the geodesic going to the pole is given by the weight $\pi\omega_m = \pi/2(\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_m)$. The roots are $\pm\varepsilon_i \pm \varepsilon_j$ and therefore the corresponding $\alpha^\vee$'s are $\pi(\pm\varepsilon_i \pm \varepsilon_j)$. The index for non shortest geodesics is minimized by those having initial tangent of the form $\pi/2(3\varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \ldots + \varepsilon_m)$. This gives us $m-1$ conjugate points (not counting multiplicity) coming from the positive roots $\varepsilon_1 - \varepsilon_j, 2 \leq j \leq m$. Now applying the centrosome argument the spaces of shortest geodesics turn out to be as follows. For $SO(2m)$, $\Omega^d = DIII(m) \cup DIII(m)$, where the union means two disjoint copies of $DIII(m) = SO(2m)/U(m)$. For the space $G_m(\mathbb{R}^{2m})$, $\Omega^d = SO(m)$. Since all the roots have the same length the root spaces all have the same multiplicities μ, which are as follows: $\mu = 1$ for $G_m(\mathbb{R}^{2m})$ and as is always the case for a group, $\mu = 2$ for $SO(2m)$. We therefore have the following proposition.

PROPOSITION 1.2.

$$\Pi_{i+1}SO(2m) \cong \Pi_i DIII(m) \cup DIII(M), \qquad \textit{for } 1 \leq i \leq 2m-4,$$
$$\Pi_{i+1}G_m(\mathbb{R}^{2m}) \cong \Pi_i SO(m), \qquad \textit{for } 1 \leq i \leq m-3.$$

***R(M)* of type *C_r* or *BC_r*.** In this case we are considering the spaces $C_m = Sp(m)$, $CI(m) = Sp(m)/Su(m)$, and $DIII(m)$. Note that $DIII(m)$ has the root system C_m if m is even and BC_m if m is odd. All these spaces have a pole and the initial tangent to a shortest geodesic going to a pole is given the vector $\pi/2\omega_m = \pi/2(\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_m)$. The roots for C_m are $\pm\varepsilon_i \pm \varepsilon_j$, $1 \leq i < j \leq m$, and $\pm 2\varepsilon_i$, $1 \leq i \leq m$. Adding elements of the lattice Γ^0 to this vector we see that the initial tangents to non shortest geodesics are of the form $\pi/2(n_1\varepsilon_1 + \ldots + n_m\varepsilon_m)$ where the n_i's are odd integers not all equal to plus or minus one, this being the case for the shortest ones. The minimal λ_0 is attained when one $n_i = \pm 3$ and the rest are ± 1. The number of conjugate points (not counting multiplicity) is therefore $m - 1 + 2$, since the Jacobi field Y_α will have vanished twice before it gets to

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the pole when $\alpha = 2\varepsilon_i$. In the case of the root system BC_m the number of conjugate points will be $m+2$, since ε_i is also a root. The multiplicities μ are then as follows. For $Sp(m)$ all roots have $\mu = 2$ and so $\lambda_0 = 2m+2$. For $Sp(m)/U(m)$ all roots have $\mu = 1$, therefore $\lambda_0 = m+1$. In the case of $DIII(m)$ with m even all roots have $\mu = 4$ except those of the same length as the highest root which have $\mu = 1$, and therefore $\lambda_0 = 4m+2$. When m is odd the shorter roots have $\mu = 4$, and the longer roots have $\mu = 1$, and hence $\lambda_0 = 4m-2$. By reading off the appropriate centrosomes we get the following spaces of shortest geodesics from o to p. For $Sp(m), \Omega^d = Sp(m)/U(m)$, for $DIII(m), \Omega^d = UII(m) = U(2m)/SP(m)$, and for $CI(m), \Omega^d = UI(m) = U(m)/O(m)$. We have now proved the following proposition.

Proposition 1.3.

$$\begin{aligned} \Pi_{i+1}Sp(m) &\cong \Pi_i Sp(m)/U(m), && \textit{for } 1 \leqslant i \leqslant 2m, \\ \Pi_{i+1}SO(2m)/U(m) &\cong \Pi_i U(2m)/Sp(2m), && \textit{for } 1 \leqslant i \leqslant 4m, i \textit{ even}, \\ & && \textit{for } 1 \leqslant i \leqslant 4m-4, i \textit{ odd}, \\ \Pi_i Sp(m)/U(m) &\cong \Pi_i U(m)/O(m), && \textit{for } 1 \leqslant i \leqslant m-1. \end{aligned}$$

2. Spaces of exceptional type. In computing the homotopy groups for these spaces we will usually choose the antipodal point on what I will call the Helgason sphere $S(\tilde{\alpha})$ (see **[8]**), as the second point in Bott's theorem.

$M = FII$: We choose p to be the antipodal point on the Helgason sphere $S(\tilde{\alpha}) = S^8$. Since $\mu(\tilde{\alpha}) = 7$ where $\tilde{\alpha} = 2\varepsilon_1$, and since the other root ε_1 has multiplicity $\mu(\varepsilon_1) = 8$ we get the following values for λ_0, and Ω^d, $\lambda_0 = 15$ and $\Omega^d = S^7$. We therefore have the following proposition.

Proposition 2.1.

$$\Pi_{i+1}FII \cong \Pi_i S^7, \qquad \textit{for } 1 \leqslant i \leqslant 13.$$

$M = EIV$: Again we use the antipodal point on the Helgason sphere $S(\tilde{\alpha}) = S^9$. The roots are $\varepsilon_1 - \varepsilon_2$, $\varepsilon_2 - \varepsilon_3$, and $\varepsilon_1 - \varepsilon_3$ which is the highest root. They all have multiplicity $\mu = 8$. The shortest geodesics to the pole of this S^9 all lie in the $M^-(p) = T \,.\, S^9$, where T denotes a circle and the dot denotes the space $T \times S^9$ with the pairs of points (x, y) and (x', y') identified. Here x' is the pole of x in T, and y' is the pole of y in S^9. We know from the Borel-Siebenthal construction for $M^-(p)$ (see **[10]**), that the geodesic segment in T to the pole has initial tangent $\pi/2(2\varepsilon_1 - \varepsilon_2 - \varepsilon_3)$ on which the highest root and $\varepsilon_1 - \varepsilon_3$ evaluate as $3\pi/2$. We therefore have that the shortest geodesics to p lie in the S^9 and $\Omega^d = S^8$. It is a simple matter to show that $\lambda_0 = 16$, and we now have the following proposition.

Proposition 2.2.

$$\Pi_{i+1}EIV \cong \Pi_i S^8, \qquad \textit{for } 1 \leqslant i \leqslant 14.$$

$M = EIII$: Here again we argue as above. The initial tangent to the shortest geodesic going to the antipodal point on the Helgason sphere is given by $\pi/2\varepsilon_1$. Since the root system is BC_2 the multiplicities are as follows: $\mu(2\varepsilon_i) = 1$, $\mu(\varepsilon_i \pm \varepsilon_j) = 6$, and $\mu(\varepsilon_i) = 8$. The minimal index λ_0 is attained for non shortest geodesics with initial tangents of the form $\pi(\pm\varepsilon_1/2 - \varepsilon_2)$ and $\lambda_0 = 7$. Since $S(\tilde{\alpha}) = S(2\varepsilon_1) = S^2$ we now have proved the following proposition.

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PROPOSITION 2.3.

$$\Pi_{i+1}EIII \cong \Pi_i S^1, \qquad \textit{for } 1 \leq i \leq 5.$$

$M = EV$: The space EV has a pole and it is the furthest polar set from o. The initial tangent to a shortest geodesic to this point is given by $\pi\omega_7 = \pi(\varepsilon_6 + \frac{1}{2}(\varepsilon_8 - \varepsilon_7))$. Since the root system is E_7 all roots have multiplicity $\mu = 1$. The minimal index for a non shortest geodesic to the pole is attained for geodesics with initial tangents of the form $\pi(\varepsilon_6 + \frac{1}{2}(\varepsilon_8 - \varepsilon_2) \pm \varepsilon_1 \pm \varepsilon_2)$ or $\pi(\varepsilon_1 - \varepsilon_2 - \ldots - \varepsilon_6)$, and the index for such geodesics is $\lambda_0 = 7$. The centrosome argument gives the space of shortest geodesics as $\Omega^d = AII(4)/\mathbb{Z}_2$ where $AII(2)/\mathbb{Z}_2$ denotes a symmetric space with $AII(4)$ as its double cover. We therefore have the following proposition.

PROPOSITION 2.4.

$$\Pi_{i+1}EV \cong \Pi_i AII(4)/\mathbb{Z}_2, \qquad \textit{for } 1 \leq i \leq 5.$$

$M = EVII$: Here again we will choose as our second point the antipodal point on the Helgason sphere. The initial tangent to a shortest geodesic going to this point is given by $\pi/2\varepsilon_1$. The root system of this space is C_3, the minimal index λ_0 is attained by geodesics with initial tangent of the form $\pi(\varepsilon_i/2 - \varepsilon_2)$ and is equal to nine, giving us the following proposition.

PROPOSITION 2.5.

$$\Pi_{i+1}EVII \cong \Pi_i S^1, \qquad \textit{for } 1 \leq i \leq 7.$$

$M = EVIII$: Here we choose the point p on the polar set $M^+(p)$ furthest away from o. The polar set in question is the symmetric space $G_8(\mathbb{R}^{16})^\#$ which is the space with $G_8^0(\mathbb{R}^{16})$ as a double covering and is not $G_8(\mathbb{R}^{16})$. The initial tangent to a shortest geodesic to p is given by ε_8 and the minimal λ_0 is attained for geodesics with initial tangents of the form $\pi/2(\varepsilon_8 + \varepsilon_1 - \varepsilon_2 - \ldots - \varepsilon_6 + \varepsilon_7)$ for which the index is $\lambda_0 = 7$. The centrosome argument then gives that the space of shortest geodesics is $\Omega^d = G_4(\mathbb{R}^{12})$ giving us the following result.

PROPOSITION 2.6.

$$\Pi_{i+1}EVIII \cong \Pi_i G_4(\mathbb{R}^{12}), \qquad \textit{for } 1 \leq i \leq 5.$$

$M = F4, FI, EII, EVI$, *and* EIX: For all these spaces the root system is F_4. The $M^-(p)$ corresponding to the polar set furthest from o in the above spaces are Spin(9), $G_4^0(\mathbb{R}^9)$, $G_4^0(\mathbb{R}^{10})$, $G_4^0(\mathbb{R}^{12})$, $G_4^0(\mathbb{R}^{16})$ respectively. The initial tangent to the shortest geodesic to p is ω_4 of the root system F_4. A typical non shortest geodesic to p for which we obtain a minimal λ_0 has initial tangent $\pi(\varepsilon_1 + \varepsilon_2 - \varepsilon_3)$ for which λ_0 is $10, 5, 7, 11$, and 19 for the respective spaces. The centrosome argument now gives the space of shortest geodesics Ω^d as $G_2^0(\mathbb{R}^9)$, $S^3 \,.\, S^4$, $S^3 \,.\, S^5$, $S^3 \,.\, S^7$, and $S^3 \,.\, S^{11}$ for the respective spaces. The dot denotes the product of the two spaces with the pairs of points (x, y) and (x', y') identified, where x' and y' are the poles of x and y respectively. We therefore have the following result.

PROPOSITION 2.7.

$$\begin{aligned} \Pi_{i+1}F_4 &\cong \Pi_i G_2^0(\mathbb{R}^9), && \text{for } 1 \leq i \leq 8,\\ \Pi_{i+1}FI &\cong \Pi_i S^3 . S^4, && \text{for } 1 \leq i \leq 3,\\ \Pi_{i+1}EII &\cong \Pi_i S^3 . S^5, && \text{for } 1 \leq i \leq 5,\\ \Pi_{i+1}EVI &\cong \Pi_i S^3 . S^7, && \text{for } 1 \leq i \leq 9,\\ \Pi_{i+1}EIX &\cong \Pi_i S^3 . S^{11}, && \text{for } 1 \leq i \leq 17. \end{aligned}$$

Now from these results and the fact that the second homotopy group of a Lie group is trivial, and using the homotopy sequence for a fibration in the cases of EI and G_2, we conclude the following.

THEOREM 2.8.

$$\mu(\tilde{\alpha}) = 1 \qquad \text{if and only if } \Pi_2 M \neq 0.$$

Note that when $\mu(\tilde{\alpha}) = 1$, the Helgason sphere has dimension two and gives a non trivial element in the second homotopy group. From our computations for the spaces EIV and FII we have that again the Helgason sphere gives a non trivial element in the appropriate dimension of homotopy. There would appear to be an interesting connection between this fact and the results of Burstall, Rawnsley and Salamon **[4]**. M. Takeuchi **[11]** has also computed Π_2 for all compact symmetric spaces.

3. In **[6]** Cheng constructed a one to one correspondence between quaternionic symmetric spaces with anti-quaternionic involution and simple five step graded Lie algebras, for the necessary background see **[6]**, **[12]** and **[9]**. Theorem 2.8 now gives an equivalent classification in terms of the second homotopy group of an associated compact symmetric space. In order to state the theorem we must quote some facts (see **[6]**).

Let $\mathfrak{l} = \mathfrak{l}_{-2} \oplus \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1 \oplus \mathfrak{l}_2$ be a simple five step graded Lie algebra. Let L and L_u denote the adjoint groups whose Lie algebras are $\mathfrak{l}$ and the compact dual of $\mathfrak{l}$ respectively. Let L^c denote the adjoint group of $\mathfrak{l}^c$, the complexification of $\mathfrak{l}$. There exists $z \in \mathfrak{l}$ such that $\operatorname{ad} z = p1$ on $\mathfrak{l}_p, p \in \{-2, -1, 0, 1, 2\}$ and there exists a maximal compact subgroup G of L such that $\mathfrak{l} = \mathfrak{g} \oplus \mathfrak{g}^{\text{perp}}$ where $\mathfrak{g}^{\text{perp}}$ denotes the orthogonal complement of $\mathfrak{g}$, and $\mathfrak{g}^{\text{perp}}$ contains a maximal abelian subalgebra containing z. We now get the following theorem by combining Theorem 2.8 and the results of Cheng.

THEOREM 2.9. *The following statements are equivalent.*

(i) $\mathfrak{l}$ *is a simple five step graded Lie algebra.*

(ii) *There exists a quaternionic symmetric space with anti-quaternionic involution, whose associated twistor space is the complex flag manifold* L^c/P^c, *where* P *is a parabolic subgroup of* L.

(iii) *The multiplicity of the highest root of the symmetric space* L_u/G *is equal to one.*

(iv) $\Pi_2(L_u/G) \neq \{0\}$.

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