J. Aust. Math. Soc. **80** (2006), 367–373

OMITTED RAYS AND WEDGES OF FRACTIONAL CAUCHY TRANSFORMS

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(Received 16 April 2004; revised 15 February 2005)

Communicated by P. Fenton

Abstract

For $\alpha > 0$ let $\mathscr{F}_\alpha$ denote the set of functions which can be expressed

$$f(z) = \int_{|\zeta|=1} \frac{1}{(1-\bar{\zeta}z)^\alpha}\, d\mu(\zeta) \quad \text{for } |z| < 1,$$

where μ is a complex-valued Borel measure on the unit circle. We show that if f is an analytic function in $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and there are two nonparallel rays in $\mathbb{C}\backslash f(\Delta)$ which do not meet, then $f \in \mathscr{F}_\alpha$ where $\alpha\pi$ denotes the largest of the two angles determined by the rays. Also if the range of a function analytic in Δ is contained in an angular wedge of opening $\alpha\pi$ and $1 < \alpha < 2$, then $f \in \mathscr{F}_\alpha$.

2000 *Mathematics subject classification*: primary 30E20; secondary 30H05.

1. Introduction

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and $\Lambda = \{z \in \mathbb{C} : |z| = 1\}$, and let $\mathscr{M}$ denote the set of complex-valued Borel measures on Λ. For $\alpha > 0$, let $\mathscr{F}_\alpha$ denote the set of functions $f : \Delta \to \mathbb{C}$ for which there exists $\mu \in \mathscr{M}$ such that

$$(1) \qquad f(z) = \int_\Lambda \frac{1}{(1-\bar{\zeta}z)^\alpha}\, d\mu(\zeta) \quad \text{for } |z| < 1.$$

The power function in (1) denotes the principal branch. Each function given by (1) is analytic in Δ. For $f \in \mathscr{F}_\alpha$, let $\|f\|_{\mathscr{F}_\alpha} = \inf \|\mu\|$, where $\|\mu\|$ denotes the total variation of μ, and μ varies over all measures in $\mathscr{M}$ for which (1) holds. This defines a norm on $\mathscr{F}_\alpha$, and $\mathscr{F}_\alpha$ is a Banach space with respect to this norm. The family $\mathscr{F}_1$

https://doi.org/10.1017/S1446788700014075 Published online by Cambridge University Press

was first studied by Havin [2]. The general class $\mathscr{F}_\alpha$, where $\alpha > 0$, was introduced in [5] and has been studied extensively. In [6] a survey is given about these so-called fractional Cauchy transforms.

Several conditions on an analytic function f are known to be sufficient to imply $f \in \mathscr{F}_\alpha$. Most of these conditions are analytic. Here we are concerned with geometric conditions. The Riesz-Herglotz formula provides information of this type. It implies that if $f : \Delta \to \mathbb{C}$ is analytic and $f(\Delta)$ is contained in a half-plane, then $f \in \mathscr{F}_1$. Another result of this kind was obtained by Bourdon and Cima in [1], namely, if $f : \Delta \to \mathbb{C}$ is analytic and there are two oppositely directed rays in $\mathbb{C}\backslash f(\Delta)$, then $f \in \mathscr{F}_1$.

This paper contains generalizations of the two results described above. We show that if $f(\Delta)$ is contained in an angular wedge with opening $\alpha\pi$ and $1 < \alpha < 2$, then $f \in \mathscr{F}_\alpha$. Also if there are two nonparallel rays in $\mathbb{C}\backslash f(\Delta)$ which do not meet and the angles at infinity between these two rays are $\alpha\pi$ and $\beta\pi$, then $f \in \mathscr{F}_\gamma$, where $\gamma = \max\{\alpha, \beta\}$.

If $f(\Delta)$ is contained in an angular wedge of opening less than π, then $f \in \mathscr{F}_1$, but f need not belong to $\mathscr{F}_\alpha$ for any α, $0 < \alpha < 1$. This holds more generally for bounded analytic functions. To see this, let

$$f(z) = \sum_{n=1}^{\infty} \frac{z^{2^n}}{n^2}, \qquad |z| < 1.$$

The function f is analytic and bounded in Δ. However $f \notin \mathscr{F}_\alpha$ when $0 < \alpha < 1$. This is because the Taylor coefficients of f do not satisfy the condition $a_n = O(n^{\alpha-1})$, which is necessary for membership in $\mathscr{F}_\alpha$.

Finally we mention that if $f : \Delta \to \mathbb{C}$ is analytic and $\mathbb{C}\backslash f(\Delta)$ contains a ray, then $f \in \mathscr{F}_2$ [5, Theorem 5].

2. Preliminaries

This section contains lemmas which will be used to prove the main results. The first two lemmas are in [5, Lemma 1]. Lemma 2.3 is in [3, Theorem 2]. Lemma 2.4 is known but we give a proof.

LEMMA 2.1. *For every* $\alpha > 0$, $f \in \mathscr{F}_\alpha$ *if and only if* $f' \in \mathscr{F}_{\alpha+1}$.

LEMMA 2.2. *If* $0 < \alpha < \beta$, *then* $\mathscr{F}_\alpha \subset \mathscr{F}_\beta$.

LEMMA 2.3. *If* $\alpha \geq 1$, $f \in \mathscr{F}_\alpha$ *and the function* $\varphi : \Delta \to \Delta$ *is analytic, then the composition* $f \circ \varphi \in \mathscr{F}_\alpha$.

LEMMA 2.4. *Suppose that $f : \Delta \to \mathbb{C}$ is analytic and f' belongs to Hardy space H^1. Then $f \in \mathscr{F}_\alpha$ for every $\alpha > 0$.*

PROOF. Suppose that $f' \in H^1$ and let $g = f'$. Let $\zeta = e^{i\theta}$. Then

$$G(\zeta) \equiv \lim_{r \to 1-} g(r\zeta)$$

exists for almost all θ in $[-\pi, \pi]$ and $G(e^{i\theta}) \in L^1([-\pi, \pi])$. Also g is represented by the Cauchy formula

(2)
$$g(z) = \frac{1}{2\pi i} \int_\Lambda \frac{G(\zeta)}{\zeta - z} d\zeta, \quad |z| < 1.$$

Equation (2) yields (1), where $d\mu(\zeta) = (G(\zeta)/2\pi i \zeta) d\zeta$ and hence $g = f' \in \mathscr{F}_1$. By Lemma 2.2, $f' \in \mathscr{F}_\alpha$ for $\alpha > 1$. Lemma 2.1 implies that $f \in \mathscr{F}_\alpha$ for all $\alpha > 0$. □

LEMMA 2.5. *Suppose that the function g is analytic in a neighbourhood of Δ. Let N be a positive integer and suppose that $|\zeta_n| = 1$, $\alpha_n > 0$ for $n = 1, 2, \ldots, N$, and $\zeta_n \neq \zeta_m$ for $n \neq m$. Let*

(3)
$$f(z) = \frac{g(z)}{\prod_{n=1}^{N} (z - \zeta_n)^{\alpha_n}}, \quad |z| < 1.$$

Then $f \in \mathscr{F}_\alpha$, where $\alpha = \max\{\alpha_n : 1 \leq n \leq N\}$.

PROOF. We give the proof for the case $N = 2$. A similar argument can be given for other values of N.

Suppose that $|\zeta| = |\sigma| = 1$, $\zeta \neq \sigma$, $\beta > 0$, and $\gamma > 0$. Suppose that the function g is analytic in a neighborhood of Δ and let

(4)
$$f(z) = \frac{g(z)}{(z - \zeta)^\beta (z - \sigma)^\gamma}, \quad |z| < 1.$$

We shall show that $f \in \mathscr{F}_\alpha$, where $\alpha = \max\{\beta, \gamma\}$.

The function $z \mapsto g(z)/(z - \sigma)^\gamma$ is analytic at $z = \zeta$, and hence

$$\frac{g(z)}{(z - \sigma)^\gamma} = \sum_{m=0}^{\infty} a_m (z - \zeta)^m,$$

for z in some neighbourhood of ζ. Let p be the least integer such that $p \geq \beta$ and let $s = p - \beta$. Then

(5)
$$f(z) = \sum_{m=0}^{p-1} \frac{a_m}{(z - \zeta)^{\beta - m}} + (z - \zeta)^s h(z),$$

where the function h is analytic in some neighbourhood of ζ. Suppose that β is not an integer. Then

$$\frac{d}{dz}\left[(z-\zeta)^s h(z)\right] = (z-\zeta)^s h'(z) + s(z-\zeta)^{s-1}h(z).$$

Since $(z-\zeta)^s$ is bounded in $\overline{\Delta} \setminus \{\zeta\}$, this implies that there is a positive constant A such that

$$\left|\frac{d}{dz}\left[(z-\zeta)^s h(z)\right]\right| \le A|z-\zeta|^{s-1}, \tag{6}$$

for $z \in \overline{\Delta}$, z near ζ, and $z \neq \zeta$. Likewise if γ is not an integer, q is the least integer such that $q \ge \gamma$ and $t = q - \gamma$, then

$$f(z) = \sum_{m=0}^{q-1} \frac{b_m}{(z-\sigma)^{\gamma-m}} + (z-\sigma)^t k(z), \tag{7}$$

where k is a function analytic in some neighbourhood of σ and b_m $(m = 0, 1, \ldots, q-1)$ are suitable constants. Thus

$$\left|\frac{d}{dz}\left[(z-\sigma)^t k(z)\right]\right| \le B|z-\sigma|^{t-1}, \tag{8}$$

for $z \in \overline{\Delta}$, z near σ, and $z \neq \sigma$, where B is a positive constant.

For $z \in \overline{\Delta} \setminus \{\zeta, \sigma\}$, let

$$r(z) = f(z) - \sum_{m=0}^{p-1} \frac{a_m}{(z-\zeta)^{\beta-m}} - \sum_{m=0}^{q-1} \frac{b_m}{(z-\sigma)^{\gamma-m}}. \tag{9}$$

The relations (5), (6) and (9) imply that there is a constant C such that

$$|r'(z)| \le C|z-\zeta|^{s-1}, \tag{10}$$

for $z \in \overline{\Delta}$, z near ζ, and $z \neq \zeta$. Likewise (7)–(9) imply that

$$|r'(z)| \le D|z-\sigma|^{t-1}, \tag{11}$$

for $z \in \overline{\Delta}$, z near σ, and $z \neq \sigma$, where D is a positive constant.

The function $z \mapsto (z-\tau)^u$ belongs to H^1 when $|\tau| = 1$ and $u > -1$. Hence the inequalities (10) and (11) and the fact that r' is analytic in $\overline{\Delta} \setminus \{\zeta, \sigma\}$ imply that $r' \in H^1$. This proves that $r' \in H^1$ when β and γ are not integers. A similar argument shows that $r' \in H^1$ when only one of the numbers β and γ is not an integer. If both

https://doi.org/10.1017/S1446788700014075 Published online by Cambridge University Press

β and γ are integers then $r = 0$. Therefore, in general, $r' \in H^1$. Lemma 2.4 yields $r \in \mathscr{F}_\delta$ for every $\delta > 0$.

Equation (9) gives

$$f = f_1 + f_2 + r, \tag{12}$$

where

$$f_1(z) = \sum_{m=0}^{p-1} \frac{a_m}{(z-\zeta)^{\beta-m}} \quad \text{and} \quad f_2(z) = \sum_{m=0}^{q-1} \frac{b_m}{(z-\sigma)^{\gamma-m}}.$$

The function $z \mapsto 1/(z-\zeta)^\delta$ belongs to $\mathscr{F}_\beta$ when $0 < \delta \le \beta$ and hence $f_1 \in \mathscr{F}_\beta$. Likewise $f_1 \in \mathscr{F}_\gamma$. Lemma 2.2 yields $f_1 \in \mathscr{F}_\alpha$ and $f_2 \in \mathscr{F}_\alpha$. Since $r \in \mathscr{F}_\alpha$, (12) implies that $f \in \mathscr{F}_\alpha$. □

Lemma 2.5 contrasts with the following result obtained in [5, Lemma 1].

THEOREM 2.6. *If $f \in \mathscr{F}_\alpha$ and $g \in \mathscr{F}_\beta$ then $f \cdot g \in \mathscr{F}_{\alpha+\beta}$.*

Since the function g in Lemma 2.5 is analytic in $\overline{\Delta}$, g is a multiplier of $\mathscr{F}_\delta$ for every $\delta > 0$ [4, Theorem 3.5]. This fact and Theorem 2.6 imply that the function f in (3) belongs to $\mathscr{F}_{\alpha'}$, where $\alpha' = \sum_{n=1}^{N} \alpha_n$. Lemma 2.5 is clearly an improvement of this result. The assumption that $\zeta_n \ne \zeta_m$ for $n \ne m$ is critical in Lemma 2.5. To see how this is reflected in our argument, suppose that the numbers ζ_n ($n = 1, 2, \ldots, N$) are distinct, $\zeta_2 \to \zeta_1$, and the numbers α_n are fixed. Suppose that $\alpha = \max\{\alpha_1, \alpha_2, \ldots, \alpha_N\}$. Then the norm $\|f\|_{\mathscr{F}_\alpha}$ of the corresponding function in (3) goes to infinity as $\zeta_2 \to \zeta_1$.

3. The main results

Let f be analytic in Δ. In this section we give two geometric conditions on $f(\Delta)$ sufficient to imply that $f \in \mathscr{F}_\alpha$.

THEOREM 3.1. *Suppose that the function $f : \Delta \to \mathbb{C}$ is analytic and let $\Phi = \mathbb{C} \setminus f(\Delta)$.*

(a) *Suppose that Φ contains two nonparallel rays. Let $\alpha\pi$ and $\beta\pi$ denote the angles at ∞ between these two rays, where $\alpha \ge \beta$. If $\alpha < 2$, then $f \in \mathscr{F}_\alpha$.*

(b) *If Φ contains a ray then $f \in \mathscr{F}_2$.*

PROOF. First assume that Φ contains two nonparallel rays. Since $\alpha + \beta = 2$, the assumptions imply that $1 < \alpha < 2$. We may assume that the rays do not intersect.

Let F denote a conformal mapping of Δ onto the complement of the two rays. The Schwarz-Christoffel formula gives

$$F(z) = b \int_0^z \frac{(w-\zeta_1)(w-\zeta_2)}{(w-\zeta_3)^{\alpha+1}(w-\zeta_4)^{3-\alpha}}\, dw + c, \tag{13}$$

where ζ_1, ζ_2, ζ_3, and ζ_4 are distinct points on Λ, and b and c are suitable complex numbers. Hence

$$F'(z) = \frac{g(z)}{(w-\zeta_3)^{\alpha+1}(w-\zeta_4)^{3-\alpha}},$$

where g is a quadratic polynomial. Since $3-\alpha < \alpha+1$, Lemma 2.5 yields $F' \in \mathscr{F}_{\alpha+1}$. Hence Lemma 2.1 implies $F \in \mathscr{F}_\alpha$.

Since $f(\Delta) \subset F(\Delta)$ and F is univalent, the function $\varphi = F^{-1} \circ f$ is analytic in Δ and maps Δ into Δ. Since $F \in \mathscr{F}_\alpha$ and $\alpha > 1$, Lemma 2.3 yields $f = F \circ \varphi \in \mathscr{F}_\alpha$. This proves the first assertion.

The second assertion can be proved in a similar way. The conformal mapping of Δ onto the complement of a ray has the form $F(z) = P(z)/(z-\zeta)^2$, where P is a quadratic polynomial and $|\zeta| = 1$. This yields $F \in \mathscr{F}_2$ and hence Lemma 2.3 yields $f \in \mathscr{F}_2$. □

THEOREM 3.2. *Suppose that f is analytic in Δ. If $f(\Delta)$ is contained in an angular wedge of opening $\alpha\pi$ and $1 < \alpha < 2$, then $f \in \mathscr{F}_\alpha$.*

PROOF. The function $z \mapsto [(1+z)/(1-z)]^\alpha$ maps Δ one-to-one onto the wedge $\{w : |\arg w| < \alpha\pi/2\}$. Hence there are complex numbers b and c such that the function defined by $F(z) = b[(1+z)/(1-z)]^\alpha + c$ maps Δ one-to-one onto the angular wedge containing $f(\Delta)$. The function $z \mapsto 1/(1-z)^\alpha$ belongs to $\mathscr{F}_\alpha$. Let $h(z) = (1+z)^\alpha$. Since $\alpha > 1$, h' is bounded. Thus $h' \in H^1$ and it follows that h is a multiplier of $\mathscr{F}_\delta$ for every $\delta > 0$ [4, Theorem 3.5]. Therefore $F \in \mathscr{F}_\alpha$. Since $f(\Delta) \subset F(\Delta)$ and F is univalent, we have $f = F \circ \varphi$, where the function $\varphi : \Delta \to \Delta$ is analytic. Since $F \in \mathscr{F}_\alpha$ and $\alpha > 1$, Lemma 2.3 gives $f \in \mathscr{F}_\alpha$. □

References

[1] P. Bourdon and J. A. Cima, 'On integrals of Cauchy-Stieltjes type', *Houston J. Math.* **14** (1988), 465–474.

[2] V. P. Havin, 'On analytic functions representable by an integral of Cauchy-Stieltjes type', *Vestnik Leningrad Univ. Ser. Mat. Meh. Astronom.* **13** (1958), 66–79 (in Russian).

[3] R. A. Hibschweiler and T. H. MacGregor, 'Closure properties of families of Cauchy-Stieltjes transforms', *Proc. Amer. Math. Soc.* **105** (1989), 615–621.

[4] ———, 'Multipliers of families of Cauchy-Stieltjes transforms', *Trans. Amer. Math. Soc.* **331** (1992), 377–394.

[5] T. H. MacGregor, 'Analytic and univalent functions with integral representations involving complex measures', *Indiana Univ. Math. J.* **36** (1987), 109–130.

[6] ———, 'Fractional Cauchy transforms', *J. Comput. Appl. Math.* **105** (1999), 93–108.

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