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Reconstruction problems of convex bodies from surface area measures and lightness functions

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Abstract. First, we build a computational procedure to reconstruct the convex body from a pre-given surface area measure. Nontrivially, we prove the convergence of this procedure. Then, the sufficient and necessary conditions of a Sobolev binary function to be a lightness function of a convex body are investigated. Finally, we present a computational procedure to compute the curvature function from a pre-given lightness function, and then we reconstruct the convex body from this curvature function by using the first procedure. The convergence is also proved. The main computations in both procedures are simply about the spherical harmonics.

1 Introduction

Reconstruction problems of convex bodies (compact convex sets with nonempty interiors in $\mathbb{R}^n$) for some known geometric functions and measures (e.g., area measures, curvatures, brightness function, and lightness functions in this paper) have caught great attention in convex geometric analysis. For instance, a convex body K can be uniquely determined, up to translations, by its *surface area measure* $S(K,\cdot)$, which is a spherical Borel measure defined by

$$S(K,\eta) = \mathcal{H}^{n-1}\big(\nu_K^{-1}(\eta)\big), \qquad \forall \text{ Borel set } \eta \subset \mathbb{S}^{n-1},$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure, and $\nu_K : \partial K \to \mathbb{S}^{n-1}$ is the Gauss map defined $\mathcal{H}^{n-1}$-almost everywhere on the boundary ∂K. Investigation about whether a spherical Borel measure μ is the surface area measure of a convex body K is the classical *Minkowski problem*. Its complete solution goes back to the work of Aleksandrov and Fenchel-Jessen (see [1, 19, 28] for references). Around 1937, Aleksandrov also showed that a centrally symmetric convex body K is uniquely determined, up to translations, by its *brightness function*

$$b_K(v) = \frac{1}{2}\int_{\mathbb{S}^{n-1}} |u\cdot v|\,dS(K,u).$$

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Independently, Campi [8] and Bourgain and Lindenstrauss [6] proved a stability version of this result.

1.1 Reconstruction problems for a geometric function (or measure)

In view of much of previous work, we roughly summarize the reconstruction problems into the following three types:

(I) Study whether a geometric function (or measure) can uniquely and stably determine a convex body.

(II) Given a function (or measure) Ξ, study both the necessary and sufficient conditions of Ξ so that there is a convex body K whose geometric function (or measure) is the pre-given Ξ.

(III) Suppose the geometric function (or measure) of a convex body K is given. Try to build a computational procedure to reconstruct the related convex body; prove the convergence; and design an algorithm based on the computational procedure, and then implement it and test it.

Aleksandrov's projection theorem and its stability versions belong to the first topic (I). The *spherical harmonic* is the powerful tool therein. This technique can also solve problem (II) for brightness function (see, e.g., Lemma 4.4). Gardner and Milanfar [13, 14] first studied the type (III) problem for brightness function. In fact, they provided algorithms to reconstruct the origin-symmetric convex body from its given brightness function. Before that, they gave an algorithm to reconstruct the polytope from the pre-given discrete measure. Gardner, Kiderlen, Milanfar, and Peyman [12] studied the convergence of related algorithms, which belongs to the topic (III). Bianchi, Gardner, and Kiderlen [2] studied problem (III) for covariograms of convex bodies. For studies about the reconstruction problems for other geometric functions (or measures), see, e.g., [20, 21, 23, 26].

The Minkowski problems for geometric measures always involve type (I) and type (II). Except for the surface area measure, Aleksandrov, Fenchel, and Jessen also introduced the *mixed area measures*, and related problems are called the *Christoffel–Minkowski problem*. For these geometric measures, type (I) problems are completely solved, but type (II) problems are not (see [17] for a regular case). While the *curvature measures* were introduced by Federer about half a century ago, its dual counterpart was introduced quite recently by Huang, Lutwak, Yang, and Zhang [19, 24]. The studies of the reconstruction problems of type (II) for the new geometric measures have caught great attention, in both convex geometry and partial differential equations (see, e.g., [4, 3, 10, 22, 30]). However, the problem of type (I) for the dual curvature measure is only solved in one special case on the plane [5]. It seems that the problem of type (III) is only considered for surface area measure. For instance, see [14, 23] for discrete case in $\mathbb{R}^n$, and see [20] for smooth case in $\mathbb{R}^3$. In [14], the authors reconstructed polytopes from pre-given discrete measures, and they proved the convergence of their algorithm; in [20], by using the spherical harmonic, the authors gave an effective numerical algorithm for smooth curvature functions in $\mathbb{R}^3$, but they did not rigorously prove the convergence.

Given a convex body K in $\mathbb{R}^n$, a *lightness function* [9] is a binary function that gives the total reflected light resulting from illumination by a light source at infinity in the direction w that is visible when looking in the direction v. An interesting model appeared in [9] is the *Lambertian lightness function*, which means that it follows *Lambert's cosine law*, and it is formally defined by

$$L_K(v,w) = \int_{\mathbb{S}^{n-1}} (v\cdot u)_-(w\cdot u)_+ dS(K,u), \tag{1.1}$$

where $(t)_+$ and $(t)_-$ denote, respectively, the positive part and negative part of a real number t. In view of the general description of lightness function, the authors [9] also introduced a more general lightness function

$$Q_K(v,w) = \int_{\mathbb{S}^{n-1}} (v\cdot u)_- f_w(u) dS(K,u), \tag{1.2}$$

where $f_w : \mathbb{S}^{n-1} \to [0,\infty)$ is positive and continuous in the open half-space $w^+ = \{x : x\cdot w > 0\}$ and vanishes in its complement. This assumption is to fulfill that a watcher from the viewing direction w can only see a visible part of ∂K. The authors [9] gave exhaustive studies about the first topic (I), for these lightness functions.

The *aim* of this is to study (the theoretical part of) problem (III) for surface area measure, and problems (II) and (III) for lightness functions. We explicitly restate them as follows:

- (III) Given the surface area measure $S(K,\cdot)$, try to build a computational procedure to reconstruct the convex body, and prove the convergence.
- (II) What are the sufficient and necessary conditions of a binary function $G : \mathbb{S}^{n-1}\times\mathbb{S}^{n-1} \to \mathbb{R}$ so that there is a convex body K whose lightness function Q_K is the pre-given G?
- (III) Given a lightness function Q_K, try to build a computational procedure to reconstruct the convex body, and prove the convergence.

The *spherical harmonics* will be an important tool in this paper. They are the eigenvectors of spherical Laplace–Beltrami operator $\Delta_S : C^\infty(\mathbb{S}^{n-1}) \to C^\infty(\mathbb{S}^{n-1})$. For more detailed introduction, see Section 2.5. We also refer to [6, 15, 27–29] as good general references.

In Section 3, we give a computational procedure to reconstruct the convex body from a pre-given surface area measure μ on $\mathbb{S}^{n-1}$. The main computations in the procedures will be the integrals of spherical harmonics and the mixed discriminants (which will be introduced in Section 2.3) of their Hessians. First, we consider a sequence of convex optimization problems (P_N), and we prove in Theorem 3.5 that the solutions of (P_N) will converge to a dilation of the convex body whose surface area measure is the pre-given μ. Noticing that (P_N) has infinite many constraints, it is very difficult to solve (P_N) directly. Therefore, we construct the subproblem $(P_{N,k})$, which is an approximation problem of (P_N). Nontrivially, we prove in Theorem 3.3 that $(P_{N,k})$ are also convex optimization problems when k is sufficiently large, but with only finitely many constraints. Since problem $(P_{N,k})$ are convex optimization problems with only finite many constraints, they can be efficiently solved by many existing optimization package, and we do not build the explicit algorithm. Finally, by combining Theorem 3.3 with Theorem 3.5, we conclude that the solution of $(P_{N,k})$ (after normalization) converges (in Hausdorff metric) to the convex body K whose surface area measure is μ.

In Section 4.2, using the language of spherical harmonics, we give both the necessary and sufficient conditions of a binary function to be a lightness function, under the Sobolev setting. We use $H_k(\mathbb{S}^{n-1})$ to denote the Sobolev space of order k, and the detailed introduction can be seen in Section 2.4. We take the Lambertian lightness function $L_K(v, w)$ as an example, since it has more physical explanations. A trivial attempt is that if K is smooth and symmetric,

$$L_K(v, -v) = \frac{1}{2} \int_{\mathbb{S}^{n-1}} |v \cdot u|^2 dS(K, u),$$

which is the L^2 *cosine transform* of $S(K, \cdot)$. However, it is well known that the L^p cosine transform is not surjective from $C_e^\infty(\mathbb{S}^{n-1})$ (even functions in $C^\infty(\mathbb{S}^{n-1})$) into $C_e^\infty(\mathbb{S}^{n-1})$ when p is even.

In view of this, the interesting but also reasonable thing in Section 4 is that the convex body K can be determined by the *even expansion* of the binary function L_K without symmetric assumption, which is different from the brightness function. Moreover, the representations in the main theorems naturally lead to an algorithm to compute the curvature function of a convex body K. Our proof relies on some results that we slightly develop in Section 4.1, including the eigenvalues and the reverse of the *partial cosine transform*

$$\mathcal{C}_- f(v) = \int_{\mathbb{S}^{n-1}} (v \cdot u)_- f(u) du$$

in the Sobolev setting. We note that the Sobolev setting and the tool of spherical harmonics are very suitable and convenient to build algorithms to reconstruct convex bodies from known surface area measures or lightness functions.

In Section 5, we use the main results in Section 4.2 to construct a computational procedure to compute the surface area measure from a pre-given lightness function. Then, using the Minkowski procedure (Mink-Pro) in Section 3, we are able to reconstruct the convex body K (without the symmetric assumption). The convergence of the procedure is also shown.

The main computations in the two procedures of Sections 3 and 5 are merely about the spherical harmonics $Y_{m,j}$'s, including the mixed discriminants of their Hessians, and the scalar products of them with given function (or measure). These computations are independent of the indices N, k of the problem $(P_{N,k})$, and hence can be computed in advance and stored. In other words, the two procedures are robust and easily implementable.

2 Preliminaries

In this section, we collect some definitions, notations, and basic facts about convex bodies, the Minkowski problem, Sobolev space on $\mathbb{S}^{n-1}$, and the spherical harmonics. We refer to [11, 16, 18, 28, 29] as good general references.

2.1 Convex bodies

Let K be a nonempty convex compact set. Its support function $h_K : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$h_K(u) = \max\{x \cdot u : x \in K\}.$$

Sometimes, we also view it as a spherical function by restricting it to $\mathbb{S}^{n-1}$. The support function is always sublinear. Conversely, a sublinear function uniquely determines a nonempty convex compact set.

The *Hausdorff distance* between two nonempty compact convex sets K_1, K_2 is defined by

$$d_H(K_1, K_2) = \|h_{K_1} - h_{K_2}\|_\infty.$$

Given a function $f \in C(\mathbb{S}^{n-1})$, denote $W[f]$ to be the Wulff shape determined by f:

$$W[f] = \left\{x \in \mathbb{R}^n : x \cdot u \le f(u), \forall u \in \mathbb{S}^{n-1}\right\}.$$

If $W[f]$ has interiors, and $f_i \in C(\mathbb{S}^{n-1})$, $i = 1, \dots,$ is a sequence of continuous functions that converges to f uniformly, then $W[f_i] \to W[f]$ in the Hausdorff metric.

We say the surface area measure $S(K, \cdot)$ has a density, if it is absolutely continuous with respect to the spherical Lebesgue measure. In this case, we denote the Radon–Nikodym derivative by

$$F_K(u) = \frac{dS(K,u)}{du},$$

and we call F_K the *curvature function*.

Given a function $h \in C^2(\mathbb{S}^{n-1})$, denote D^2h by

$$D^2h = \nabla^2 h + hI,$$

where $\nabla^2 h$ is the Hessian of h with respect to the standard metric on $\mathbb{S}^{n-1}$, and I denotes the identity map. If one extend the h to a positive-homogeneous function on $\mathbb{R}^n$, then $D^2h(u)$ is exactly the restriction of its Hessian in $\mathbb{R}^n$ to the tangent space of $\mathbb{S}^{n-1}$ at u.

In fact, if ∂K is C^2 and has positive Gauss curvature everywhere, then $S(K, \cdot)$ has a density,

$$F_K(u) = \frac{dS(K,u)}{du} = \det(D^2 h_K),$$

which is the reciprocal Gauss curvature at $\nu_K^{-1}(u) \in \partial K$.

2.2 Minkowski problem and the Monge–Ampère equation

Solution to the classical Minkowski problem: Suppose μ is a finite Borel measure on satisfying:

(1) μ is not concentrated in a half-sphere;

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(2) the centroid of μ is the origin:

$$\int_{\mathbb{S}^{n-1}} u d\mu(u) = 0.$$

Then, there is a unique (up to translations) convex body K in $\mathbb{R}^n$ so that $\mu(\cdot) = S(K, \cdot)$. Moreover, the convex body K is homothetic to the solution of the maximization problem

$$\max V(Q), \quad \text{subject to that } Q \text{ is a convex body and } \int_{\mathbb{S}^{n-1}} h_Q d\mu = 1.$$

If the measure μ has a density function $g : \mathbb{S}^{n-1} \to \mathbb{R}$ with respect to the Lebesgue measure of the unit sphere $\mathbb{S}^{n-1}$, the Minkowski problem is equivalent to the study of solution to the following *Monge–Ampère equation* on $\mathbb{S}^{n-1}$:

$$\det(D^2 h) = \det(\nabla^2 h + hI) = g. \tag{2.1}$$

Let $f \in L^1(\mathbb{S}^{n-1})$ be nonnegative, satisfying $\int_{\mathbb{S}^{n-1}} f > 0$ and

$$\int_{\mathbb{S}^{n-1}} u f(u) du = 0.$$

Then, by the solution of the Minkowski problem, there is always a convex body K such that

$$dS(K, u) = f(u) du.$$

2.3 Mixed discriminant and mixed volume

Let $A_1, \dots, A_m$ be self-adjoint linear transforms from a k-dimensional Hilbert V to itself. Then, the following is a k-homogeneous polynomial of $t_1, \dots, t_m \in \mathbb{R}$:

$$\det\left(t_1 A_1 + \cdots t_m A_m\right) = \sum_{i_1=1}^{m} \dots \sum_{i_k=1}^{m} \mathrm{D}(A_{i_1}, \dots, A_{i_k}) t_{i_1} \dots t_{i_k}.$$

Its coefficients $\mathrm{D}(A_{i_1}, \dots, A_{i_k})$, which are symmetric functions of $A_{i_1}, \dots, A_{i_k}$, are called the *mixed discriminants* of the linear transforms.

If we choose an orthonormal basis in V, these A_i can be represented by symmetric $k \times k$ matrices, and we still denote them by A_i. Denote the matrices $A_r = \left(a_{ij}^r\right)_{i,j=1}^k$, $r = 1, \dots, k$, and the mixed discriminant $\mathrm{D}(A_1, \dots, A_k)$ can be computed by

$$\mathrm{D}\left(A_1, \dots, A_k\right) = \frac{1}{k!} \sum_{\sigma \in P_k} \begin{vmatrix} a_{11}^{\sigma(1)} & \cdots & a_{1k}^{\sigma(k)} \\ \vdots & & \vdots \\ a_{k1}^{\sigma(1)} & \cdots & a_{kk}^{\sigma(k)} \end{vmatrix}, \tag{2.2}$$

where P_k is the group of all permutations of the set $\{1, \dots, k\}$. Clearly, $\mathrm{D}(A_1, \dots, A_k)$ is linear in each argument.

Let $K_1, \ldots, K_m$ be convex bodies, and let $\lambda_1, \ldots, \lambda_m \geq 0$. Then, the volume of $\lambda_1 K_1 + \cdots \lambda_m K_m$ is an nth homogeneous polynomial of $\lambda_1, \ldots, \lambda_m$,

$$V(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1,\ldots,i_n=1} \mathrm{V}(K_{i_1}, \ldots, K_{i_n})\lambda_{i_1} \cdots \lambda_{i_n}.$$

The coefficients $\mathrm{V}(K_{i_1}, \ldots, K_{i_n})$, which will be called the *mixed volumes*, are nonnegative, symmetric in the indices, and dependent only on $K_{i_1}, \ldots, K_{i_n}$.

If ∂K is C^2 and has positive Gauss curvature everywhere, then

$$V(K) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_K(u)\det(D^2 h_K)(u)du.$$

In addition, if $h_K = t_1 h_1 + \cdots t_m h_m$, for some $h_i \in C^2$ and $t_i \in \mathbb{R}$, we have

(2.3)
$$V(K) = \sum_{i_1=1}^{m} \cdots \sum_{i_n=1}^{m} \Big(\frac{1}{n}\int_{\mathbb{S}^{n-1}} h_{i_n}(u)\mathrm{D}(D^2 h_{i_1}, \ldots, D^2 h_{i_{n-1}})(u)du\Big) t_{i_1} \ldots t_{i_n}.$$

Here, $\mathrm{D}(D^2 h_{i_1}, \ldots, D^2 h_{i_{n-1}})(u)$ should be understood as the mixed discriminant of linear transforms restricted on $u^\perp$.

Especially, notice that the spherical harmonics $(Y_{m,j})$ belong to $C^\infty(\mathbb{S}^{n-1})$. When we are considering the numerical algorithm in Section 3, we represent the support function in finite-dimensional function space generated by the spherical harmonics in each step N, and (2.3) will be useful. Moreover, (2.3) also suggests the relation between mixed discriminant and mixed volume, which goes back to Hilbert–Minkowski–Aleksandrov (see [1] for reference).

2.4 Sobolev space H_k

Let $k \geq 0$ be an integer. We denote $H_k(\mathbb{S}^{n-1})$ to be the *Sobolev space*, which is the completion of $C^\infty(\mathbb{S}^{n-1})$ with respect to the norm

$$\|f\|_{H_k} = \sum_{m=0}^{k} \Big(\int_{\mathbb{S}^{n-1}} |\nabla^m f|^2 du\Big)^{1/2}, \quad f \in C^\infty(\mathbb{S}^{n-1}),$$

where $\nabla^m f$ denotes the mth covariant derivative of f. For a function $g \in H_k(\mathbb{S}^{n-1})$ and $m \leq k$, $\nabla^m g \in L^2(\mathbb{S}^{n-1})$ denotes its *weak covariant derivative*, which can be defined as the limit of $\nabla^m f_k$ in L^2, where $f_k \in C^\infty(\mathbb{S}^{n-1})$ $(k \in \mathbb{N})$ is an arbitrary Cauchy sequence with respect to the norm $\|\cdot\|_{H_k}$. See [18] for reference.

Let $\Delta_S : C^\infty(\mathbb{S}^{n-1}) \to C^\infty(\mathbb{S}^{n-1})$ be the Laplace operator on $\mathbb{S}^{n-1}$, which is self-adjoint. Extend it to $\Delta_S : H_{k+2}(\mathbb{S}^{n-1}) \to H_k(\mathbb{S}^{n-1})$. Then, for $f, g \in H_{k+2}(\mathbb{S}^{n-1})$, it satisfies

$$(\Delta_S f, g) = (f, \Delta_S g).$$

Note that $(\cdot,\cdot)$ always denotes the scalar product in $L^2(\mathbb{S}^{n-1})$ in this paper.

2.5 Spherical harmonics

The *spherical harmonics* are the eigenvectors (eigenfunctions) of spherical Laplace-Beltrami operator $\Delta_S : C^\infty(\mathbb{S}^{n-1}) \to C^\infty(\mathbb{S}^{n-1})$. It is well known that these eigenvectors can be chosen to be the restriction onto $\mathbb{S}^{n-1}$ of the harmonic polynomials (with respect to an orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$) on $\mathbb{R}^n \backslash \{0\}$. The eigenvalues of Δ_S are

$$\lambda_m = -m(m+n-2), \quad m = 0, 1, \ldots.$$

With regard to λ_m, the related eigenspace E_m consists of all the harmonic polynomials on $\mathbb{R}^n \backslash \{0\}$ of degree m. The dimension of E_m is

$$N_m = \frac{(2m+n-2)\cdot(n+m-3)!}{(n-2)!\cdot m!}.$$

The compact operator theory in functional analysis tells that one can choose the eigenvectors to be a Hilbert basis of $L^2(\mathbb{S}^{n-1})$. More precisely, we can choose $(Y_{m,j})$ such that the eigenspace of λ_m is

$$E_m = \operatorname{span}\left\{Y_{m,1}, \ldots, Y_{m,N_m}\right\}$$

and

$$\left(Y_{m,j}, Y_{m,k}\right) = \int_{\mathbb{S}^{n-1}} Y_{m,j}(u) Y_{m,k}(u) du = \delta_j^k.$$

For distinct m, m', we automatically have $\left(Y_{m,j}, Y_{m',j'}\right) = 0$. Especially, when $m = 1$, the eigenfunctions $\{Y_{1,j}\}_{j=1}^n$ are chosen to be

$$Y_{1,j}(u) = \frac{1}{\sqrt{\omega_n}} e_j \cdot u, \qquad j = 1, \ldots, n,$$

so that $(Y_{1,k}, Y_{1,j}) = \delta_j^k$. From this, it is easy to obtain the following formula that describes the barycenter of an integrable function g:

$$\begin{aligned} v \cdot \left(\frac{1}{\omega_n} \int_{\mathbb{S}^{n-1}} u g(u) du\right) &= \sum_{j=1}^n \left[\int_{\mathbb{S}^{n-1}} \left(\frac{e_j}{\sqrt{\omega_n}} \cdot u\right) g(u) du\right]\left(\frac{e_j}{\sqrt{\omega_n}} \cdot v\right) \\ &= \sum_{j=1}^n (g, Y_{1,j}) Y_{1,j}(v). \end{aligned} \tag{2.4}$$

Since $(Y_{m,j})$ is a Hilbert basis, for each $g \in L^2(\mathbb{S}^{n-1})$, we have

$$g = \sum_{m=0}^{\infty} \sum_{j=1}^{N_m} \left(g, Y_{m,j}\right) Y_{m,j}.$$

3 Reconstruction from a pre-given surface area measure

For a given finite Borel measure μ that is not concentrated in any hemisphere and satisfies

$$\int_{\mathbb{S}^{n-1}} v d\mu(v) = 0,$$

we shall give a computational procedure to reconstruct the convex body whose surface area measure is μ, and its convergence will also be proved.

For our convenience, only in this section, we relabel the spherical harmonics $\{Y_{m,j} : m = 0, 1, \dots, j = 1, \dots, N_m\} \backslash \{Y_{1,1}, \dots, Y_{1,n}\}$ by $\{Y_m\}_{m=0}^{\infty}$. That is to say, in $\{Y_i\}_{i=0}^{\infty}$, there are no spherical harmonics of degree 1. As a result, if $h_Q = \sum_{i=0}^{N} x_i Y_i$ is a support function of a convex body Q. Then, by (2.4),

$$\int_{\mathbb{S}^{n-1}} v h_Q(v) dv = 0.$$

So, one convenience is to assign a specific translation of Q, since the surface area measure $S(Q, \cdot)$ is translation invariant.

Denote

(3.1)

$$\mathrm{V}_{i_1,\dots,i_n} = \frac{1}{n} \int_{\mathbb{S}^{n-1}} Y_{i_n}(v)\, \mathrm{D}(D^2 Y_{i_1}, \dots, D^2 Y_{i_{n-1}})(v)\, dv, \quad i_1, \dots, i_n = 0, 1, \dots, N,$$

and

$$\mathrm{V}_i = \int_{\mathbb{S}^{n-1}} Y_i(v) d\mu(v), \quad i = 0, 1, \dots, N. \tag{3.2}$$

Here, D is the mixed discriminant that can be computed by (2.2).

For a point $(x_0, \dots, x_N) \in \mathbb{R}^{N+1}$, let V be the volume function

$$V(x_0, \dots, x_N) = \sum_{i_1=0}^{N} \dots \sum_{i_n=0}^{N} \mathrm{V}_{i_1,\dots,i_n} \cdot x_{i_1} \dots x_{i_n}.$$

3.1 Optimization problem $(\overline{P}_N)$

Originally, our aim in Step N is to solve the following optimization problem (P_N):

$$\max \ V(x_0, \dots, x_N),$$

$$\text{subject to} \quad (x_0, \dots, x_N) \in \overline{\mathcal{C}}_N,$$

where $\mathcal{C}_N$ is a convex set (will be proved later) defined by

$$\overline{\mathcal{C}}_N = \left\{ (x_0, \dots, x_N) \in \mathbb{R}^{N+1} : D^2 \left(\sum_{m=0}^{N} x_m Y_m \right)(u) \geq 0, \ \forall u \in \mathbb{S}^{n-1}, \sum_{m=0}^{N} \mathrm{V}_m \cdot x_m = 1 \right\}.$$

Solve the optimization problem $(\overline{P}_N)$ and normalize the solution.

However, the problem (P_N) cannot be solved directly by computer, since the requirement $(x_0, \dots, x_N) \in \overline{\mathcal{C}}_N$ has actually infinitely many inequality constraints, although it can be shown to be a convex optimization problem. Therefore, we consider the following optimization problem $(P_{N,k})$, for $N, k \in \mathbb{N}$. We will show in Theorem 3.3 that the subproblem $(P_{N,k})$ is also a convex optimization problem with only finitely many constraints when k is large, and it follows from Theorems 3.3 and 3.5 that its solution will converge to the solution of the Minkowski problem as $N, k \to \infty$.

Now, we put our computational procedure *Mink-Pro* and the subproblem $(P_{N,k})$ as follows.

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For each $k \in \mathbb{N}$, we assign a kth partition $T_k = \{O_1^k, \dots, O_{2^k}^k\}$ of $\mathbb{S}^{n-1}$ such that

$$O_i^k \subset \mathbb{S}^{n-1}, \quad \mathcal{H}^{n-1}(O_i^k) = \frac{n\omega_n}{2^k}, \quad \bigcup_{i=1}^{2^k} O_i^k = \mathbb{S}^{n-1}, \quad \text{and} \quad T_{k+1} \subset T_k. \tag{3.3}$$

Extract one point from each O_i^k to obtain a set $U_k \subset \mathbb{S}^{n-1}$ such that U_k contains 2^k elements and

$$U_{k+1} \subset U_k. \tag{3.4}$$

Define

$$\mathcal{C}_N = \left\{(x_0, \dots, x_N) \in \mathbb{R}^{N+1} : D^2\left(\sum_{i=0}^{N} x_i Y_i\right)(u) \ge \frac{I}{N}, \ \forall u \in \mathbb{S}^{n-1}, \text{ and } \sum_{m=0}^{N} \mathrm{V}_m \cdot x_m = 1\right\},$$

and

(3.5)

$$\mathcal{C}_{N,k} = \left\{(x_0, \dots, x_N) \in \mathbb{R}^{N+1} : D^2\left(\sum_{i=0}^{N} x_i Y_i\right)(u) \ge \frac{I}{N}, \ \forall u \in U_k, \text{ and } \sum_{m=0}^{N} \mathrm{V}_m \cdot x_m = 1\right\}.$$

Clearly, $\mathcal{C}_{N,k+1} \subset \mathcal{C}_{N,k}$, and

$$\bigcap_{k=1}^{\infty} \mathcal{C}_{N,k} = \mathcal{C}_N.$$

Consider the modified *optimization problem* (P_N^*):

$$\max \ V(x_0, \dots, x_N),$$

$$\text{subject to} \quad (x_0, \dots, x_N) \in \mathcal{C}_N.$$

Denote its solution by $(x_0^*, \dots, x_N^*)$, and define

$$h_N = \left(\sum_{i=0}^{N} x_i^* Y_i\right) V(x_0^*, \dots, x_N^*)^{-1/(n-1)}. \tag{3.6}$$

3.2 Minkowski Procedure

- **Input:** Natural numbers n, N, k, the general mixed volumes $\mathrm{V}_{i_1,\dots,i_n}$ and V_i for the spherical harmonics $Y_0, \dots, Y_N$ defined by (3.1) and (3.2), the finite set U_k satisfying (3.4) and (3.3).
- **Task:** Compute the function $h_{N,k}$.
- **Action:**
  1. Solve the following convex optimization problem $(P_{N,k})$:

$$\sup \ V(x_0, \dots, x_N),$$

$$\text{subject to} \quad (x_0, \dots, x_N) \in \mathcal{C}_{N,k}.$$

  Denote the solution of $(P_{N,k})$ by $(x_0^k, \dots, x_N^k)$.

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2. Compute

$$h_{N,k} = \left(\sum_{i=0}^{N} x_i^k Y_i\right) V(x_0^k, \dots, x_N^k)^{-1/(n-1)}.$$

Remark 3.1 1. We will show in Theorem 3.3 that $(P_{N,k})$ and (P_N) are convex optimization problems, for sufficiently large k. Thus, the maximizers $(x_0^k, \dots, x_0^k)$ and $(x_0^*, \dots, x_0^*)$ (theoretically) exist and are unique.

Especially, the approach to find the solution of $(P_{N,k})$ is computationally tractable, since $(P_{N,k})$ has only finitely many constraints. To write the detailed algorithm for the approach, see [7, 25] for references.

2. We will prove in Theorem 3.3(6) that the convex body, determined by $h_{N,k}$,

$$K_{N,k} = \left\{z : z \cdot u \le h_{N,k}(u), \quad \forall u \in \mathbb{S}^{n-1}\right\}$$

converges to K_N as $k \to \infty$. In addition, we will prove in Theorem 3.5 that $K_N \to K$, when $N \to \infty$, where K_N is the convex body determined by h_N, and $S(K, \cdot) = \mu$.

3. Recalling equation (2.3), $V(x_0, \dots, x_N)$ is actually the volume of the convex body Q whose support function is

$$h_Q = \sum_{i=0}^{N} x_i Y_i,$$

for $(x_0, \dots, x_N) \in \mathcal{C}_N$. The quantity $\sum_{i=0}^{N} \mathrm{V}_i \cdot x_i$ is $\int h_Q d\mu$. This explains that Action 1 coincides with the optimization problem in solving the Minkowski problem. However, they are not completely the same, and we have to prove the convergence.

4. Although our procedure can reconstruct the body from a general measure, in practice, the smooth functions in the form

$$F(u) = \sum_{i=0}^{k_0} z_i Y_i(u)$$

are more tractable. Since the 1-degree spherical harmonics $Y_{1,1}, \dots, Y_{1,n}$ are removed from Y_i's, such a function F always satisfies $\int_{\mathbb{S}^{n-1}} uF(u)du = 0$.

Lemma 3.1 *Suppose K is a convex body such that*

$$\int_{\mathbb{S}^{n-1}} h_K(u) d\mu(u) = 1$$

and

$$\int_{\mathbb{S}^{n-1}} u h_K(u) du = 0. \tag{3.7}$$

Then, there is a constant c_1 that only depends on μ, such that $\max\{|x| : x \in K\} \le c_1$.

Proof Since μ is not concentrated in any hemisphere, the function

$$v \mapsto \int_{S^{n-1}} (u \cdot v)_+ d\mu(u)$$

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is strictly positive on S^{n-1}, and since the function is continuous on S^{n-1}, which is compact, there exists $c_0 > 0$ such that

$$\int_{S^{n-1}} (u \cdot v)_+ d\mu(u) \geq c_0 > 0 \qquad \text{for all } v \in S^{n-1}.$$

Denote $R_0 = \max\{|x| : x \in \bar{K}_N\}$. Then, there exists $v_0 \in \mathbb{S}^{n-1}$ such that $R_0 \cdot v_0 \in K$. By (3.7), we must have $o \in K$. Thus, we obtain

$$R_0 \cdot c_0 \leq \int_{\mathbb{S}^{n-1}} R_0 (v_0 \cdot u)_+ d\mu(u) \leq \int_{\mathbb{S}^{n-1}} h_K(u) d\mu(u) = 1,$$

and hence $R_0 \leq 1/c_0$. Denote $c_1 = 1/c_0$, and we complete the proof. ∎

Lemma 3.2 (1) *The sets* $\overline{\mathcal{C}}_N, \mathcal{C}_N, \mathcal{C}_{N,k} \subset \mathbb{R}^{N+1}$ *are nonempty, closed, and convex.*
(2) $\overline{\mathcal{C}}_N$ *is bounded.*

Proof (1) Since Y_0 is a constant, $(c, 0, \dots, 0) \in \mathcal{C}_N$ for some sufficiently large constant c. Since $\mathcal{C}_N \subset \overline{\mathcal{C}}_N$ and $\mathcal{C}_N \subset \mathcal{C}_{N,k}$, they are all nonempty. Since $D^2\Big(\sum_{i=0}^N x_i Y_i\Big)(u)$ is continuous in $(x_0, \dots, x_N)$, $\overline{\mathcal{C}}_N$, $\mathcal{C}_N$, and $\mathcal{C}_{N,k}$ are closed.

If, at some $u \in \mathbb{S}^{n-1}$,

$$D^2\left(\sum_{i=0}^N x_i Y_i\right)(u) \quad \text{and} \quad D^2\left(\sum_{i=0}^N y_i Y_i\right)(u)$$

are positive semidefinite, then

$$D^2\left(\sum_{i=0}^N \Big((1-\lambda)x_i + \lambda y_i\Big) Y_i\right)(u)$$

is also positive semidefinite at this u, $\forall \lambda \in [0,1]$. It follows immediately that $\overline{\mathcal{C}}_N$, $\mathcal{C}_N$, and $\mathcal{C}_{N,k}$ are convex.

(2) By Lemma 3.1, for any $(x_0, \dots, x_N) \in \mathcal{C}_N$, we have

$$|h_Q(u)| = \left|\sum_{m=0}^N x_m Y_m(u)\right| \leq c_1, \quad \forall u \in \mathbb{S}^{n-1},$$

where c_1 is a constant that only depends on μ. Then,

$$|x_m| = |(h_Q, Y_m)| = \left|\int_{\mathbb{S}^{n-1}} h_Q(u) Y_m(u) du\right| \leq c_1 \|Y_m\|_1.$$

Since $Y_m \in C^\infty(\mathbb{S}^{n-1})$ are fixed functions, we deduce that $\overline{\mathcal{C}}_N$ is bounded. ∎

Theorem 3.3 (1) $V(x_0, \dots, x_N)$ *is strictly positive on* $\mathcal{C}_N$.
(2) *The function* $V^{1/n}$ *is strictly concave on* $\mathcal{C}_N$.
(3) $\mathcal{C}_{N,k} \to \mathcal{C}_N$ *in the Hausdorff metric; and* $\mathcal{C}_N \subset \operatorname{int} \overline{\mathcal{C}}_N$.
(4) $(V)^{1/n}$ *is strictly concave on* $\mathcal{C}_{N,k}$, *for sufficiently large* k.
(5) $(x_0^k, \dots, x_N^k) \to (x_0^*, \dots, x_N^*)$ *as* $k \to \infty$.
(6) *The convex body* $K_{N,k}$ *determined by* $h_{N,k}$ *converges to the convex body* K_N *determined by* h_N *(defined in (3.6)) as* $k \to \infty$.

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Proof (1) We will use the fact that a smooth support function is either the support function of a convex body or the support function of a fixed point. By our construction, $\sum_{i=0}^{N} x_i Y_i$ is always smooth, and $\{Y_i\}_{i=0}^{N}$ does not contain the spherical harmonics of degree 1. Thus, for $(x_0, \dots, x_N) \in \mathcal{C}_N$, it cannot be the support function of a fixed point. Thus, V is always positive.

(2) Since $(x_0, \dots, x_N) \in \mathcal{C}_N$,

$$\sum_{i=0}^{N} x_i Y_i(u)$$

is a support function of some convex compact set, which we denote by $K(x_0, \dots, x_N)$. Now,

$$V(x_0, \dots, x_N) = V\big(K(x_0, \dots, x_N)\big)$$

is its volume. Notice also that, for $(y_0, \dots, y_N) \in \mathcal{C}_N$ and $\lambda \in [0,1]$,

$$(1-\lambda)K(x_0, \dots, x_N) + \lambda K(y_0, \dots, y_N) = K\big((1-\lambda)x_0 + \lambda y_0, \dots, (1-\lambda)x_N + \lambda y_N\big).$$

Thus, statement (2) follows immediately from the Brunn–Minkowski inequality, statement (1), and the fact that

$$\int_{\mathbb{S}^{n-1}} u h_{K(x_0,\dots,x_N)} du = \int_{\mathbb{S}^{n-1}} u h_{K(y_0,\dots,y_N)}(u) du = 0.$$

(3) Since $\mathcal{C}_{N,k}$ is convex and closed for each $k \in \mathbb{N}$,

$$\bigcap_{k=1}^{\infty} \mathcal{C}_{N,k} = \mathcal{C}_N,$$

and $\mathcal{C}_N$ is compact, statement (3) follows immediately.

(4) Since $\mathcal{C}_{N,k} \to \mathcal{C}_N \subset \operatorname{int}\overline{\mathcal{C}}_N$, and $\overline{\mathcal{C}}_N$ is compact, V is still positive on $\mathcal{C}_{N,k}$ when k is sufficiently large. Moreover, $\mathcal{C}_{N,k}$ will also be compact. Recall that

$$V(x_0, \dots, x_N) = \sum_{i_1=0}^{N} \dots \sum_{i_n=0}^{N} \mathrm{V}_{i_1,\dots,i_n} \cdot x_{i_1} \dots x_{i_n}$$

is a polynomial of degree n, where $\mathrm{V}_{i_1,\dots,i_n}$ are fixed constants. Thus, $V(\cdot) C^{\infty}(\mathbb{R}^{N+1})$. By the construction of $\mathcal{C}_{N,k}$, when k is sufficiently large, there is a constant $\theta > 0$, such that the Hessian

$$D^2(V^{1/n})(x_0, \dots, x_n) \geq \theta I_{N+1}$$

everywhere on $\mathcal{C}_{N,k}$.

(5) As a consequence of (3), for sufficiently large k, $\mathcal{C}_{N,k}$ is also compact. Thus, any subsequence of $(x_0^k, \dots, x_N^k)$ has a subsubsequence, which is also denoted by $(x_0^k, \dots, x_N^k)$, converging to a point $(\bar{x}_0, \dots, \bar{x}_N) \in \bigcap_k \mathcal{C}_{N,k} = \mathcal{C}_N$. It suffices to show that $(\bar{x}_0, \dots, \bar{x}_N) = (x_0^*, \dots, x_N^*)$.

Since $(x_0^*, \dots, x_N^*) \in \mathcal{C}_{N,k}$, for each $k \in \mathbb{N}$, and $(x_0^k, \dots, x_N^k)$ is the maximizer of $(P_{N,k})$ for sufficiently large k, we have

$$V(x_0^k, \dots, x_N^k) \geq V(x_0^*, \dots, x_N^*). \tag{3.8}$$

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Taking limits in (3.8), we obtain

$$V(\bar{x}_0, \ldots, \bar{x}_N) \geq V(x_0^*, \ldots, x_N^*).$$

Since $(\bar{x}_0, \ldots, \bar{x}_N) \in \mathcal{C}_N$, and $(x_0^*, \ldots, x_N^*)$ is the unique maximizer of (P_N^*), we must have $(\bar{x}_0, \ldots, \bar{x}_N) = (x_0^*, \ldots, x_N^*)$. This completes the proof.

(6) By (5), we have

$$\sum_{i=0}^{N} x_i^k Y_i \to \sum_{i=0}^{N} x_i^* Y_i$$

uniformly on $\mathbb{S}^{n-1}$. Recall the fact collected in Section 2.1: $f_i \to f$ uniformly and that $W[f]$ has interior imply that $W[f_i] \to W[f]$ in the Hausdorff metric. Now, (6) follows immediately from this property, the continuity of volume, and the definitions of h_N and $h_{N,k}$. ∎

Lemma 3.4 [29, Theorems 4 and 5] *Suppose $F \in C^\infty(\mathbb{S}^{n-1})$ satisfying $\int F(u)udu = 0$. Denote*

$$F_N(u) = \sum_{m=0}^{N} (F, Y_m)(u).$$

Then:

(1) *F_N converges to F uniformly on $\mathbb{S}^{n-1}$;*
(2) *$D^2 F_N$ converges to $D^2 F$ uniformly on $\mathbb{S}^{n-1}$.*

Remark 3.2 This Lemma follows immediately from the estimates in [29, Theorems 4 and 5], which suggest the relationship between the max norm and the L^2-norm of Fourier expansion into spherical harmonics. The assumption $\int F(u)udu = 0$ is because our $\{Y_i\}_{i=0}^{\infty}$ does not contain the spherical harmonics $Y_{1,1}, \ldots, Y_{1,n}$. Except for this, the only difference is that the D^α in [29, Theorem 4(b)] is slightly different from ours.

Let $f \in C^\infty(\mathbb{S}^{n-1})$. Denote $\bar{D}^2 f(x/|x|)$ to be the Euclidean Hessian, and $D^2 f$ to be our notation in Section 2.1. Then, restricting to $x^\perp$,

$$\bar{D}^2 f(x/|x|) + f(x/|x|)I = D^2 f(x/|x|),$$

and $\bar{D}^2 f(x/|x|) = 0$ on the line $\mathbb{R}x$.

Theorem 3.5 *Denote the solution of (P_N) by $(x_0^*, \ldots, x_N^*)$, and*

$$h_N = \left(\sum_{m=0}^{N} x_m Y_m \right) V(x_0^*, x_1^*, \ldots, x_N^*)^{-1/(n-1)}.$$

Let K_N be the convex body determined by h_N,

$$K_N = \{x : x \cdot u \leq h_N(u), \ \forall u \in \mathbb{S}^{n-1}\}.$$

Then, K_N converges to K in the Hausdorff metric, as $N \to \infty$. Here, K is the unique (theoretical) solution of the Minkowski problem

$$S(K, \cdot) = \mu(\cdot)$$

such that

$$\int_{\mathbb{S}^{n-1}} u h_K(u) du = 0.$$

Proof Denote

$$h_{\bar{K}_N} = \sum_{m=0}^{N} x_m^* Y_m,$$

and denote $\bar{K} = \lambda K$ ($\lambda > 0$) to be the dilation of K such that $\int_{\mathbb{S}^{n-1}} h_{\bar{K}} d\mu = 1$. To prove this theorem, it suffices to prove that $\bar{K}_N \to \bar{K}$.

Step 1. Since $\{Y_m\}_{m=0}^{\infty}$ does not contain the $\{Y_{1,j}\}$'s, by equation (2.4), we always have

$$\int_{\mathbb{S}^{n-1}} u h_{\bar{K}_N}(u) du = 0. \tag{3.9}$$

Step 2. It follows immediately from Lemma 3.1 that the sequence $\{\bar{K}_N\}_{N=1}^{\infty}$ is bounded.

Step 3. By Step 2 and Blaschke's selection theorem, any subsequence of $\{\bar{K}_N\}_{N=1}^{\infty}$ has a subsubsequence $\bar{K}_{N_i}$ converging to a convex body K^* in the Hausdorff metric. In this step, we will show that K^* satisfies

$$V(K^*) = \sup\left\{V(Q) : \int_{\mathbb{S}^{n-1}} h_Q(u) d\mu(u) = 1,\ Q \text{ is a convex body}\right\}, \tag{3.10}$$

and

$$\int_{\mathbb{S}^{n-1}} h_{K^*}(u) d\mu(u) = 1. \tag{3.11}$$

The second equation follows immediately from the convergence. Let us consider (4.10).

- *Step 3.1.* Suppose Q is a smooth convex body that has positive Gauss curvature, and $\int_{\mathbb{S}^{n-1}} h_Q d\mu = 1$. We will show that $V(K^*) \ge V(Q)$. By the assumption, $h_Q \in C^{\infty}(\mathbb{S}^{n-1})$, and

  $$\det(D^2 h_Q)(u) > 0, \quad \forall u \in \mathbb{S}^{n-1}.$$

  Denote

  $$h_{Q_N} = \sum_{m=0}^{N} (h_Q, Y_m) Y_m.$$

  Then, by Lemma 3.4(1), h_{Q_N} converges to h_Q uniformly. In addition, by Lemma 3.4(2), for sufficiently large N, we still have

  $$\det(D^2 h_{Q_N})(u) > 0, \quad \forall u \in \mathbb{S}^{n-1},$$

  and hence h_{Q_N} is also support function. Therefore, if we denote

  $$\bar{Q}_N = \frac{1}{\int_{\mathbb{S}^{n-1}} h_{Q_N} d\mu} Q_N,$$

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by the construction of $\bar{K}_N$, we have $V(\bar{K}_N) \geq V(\bar{Q}_N)$. Since $\bar{K}_N \to K^*$, $\bar{Q}_N \to Q$, we get

$$V(K^*) \geq V(Q).$$

- *Step 3.2.* For any convex body L, we can choose a sequence of smooth convex bodies Q_m that have positive curvatures, and such that $Q_m \to L$. By the approximation argument, we complete the proof of (4.10).

By the solution of the Minkowski problem, the convex body satisfying (4.10) and (3.11) is unique up to translations. Recalling the fact that

$$\int_{\mathbb{S}^{n-1}} u h_{\bar{K}}(u)du = \int_{\mathbb{S}^{n-1}} u h_{K^*}(u)du = 0,$$

we have $\bar{K} = K^*$. Since we have proved that any subsequence of $\{\bar{K}_N\}$ has a convergent subsequence converging to $\bar{K}$, we have actually completed the proof of this theorem. ■

4 Reconstruction problems (II) for the lightness function

In this section, we consider the general lightness function Q_K, which was defined by

$$Q_K(v,w) = \int_{\mathbb{S}^{n-1}} (v \cdot u)_- f_w(u) dS(K,u),$$

where f_w is positive and continuous in the open set $w^+ \cap \mathbb{S}^{n-1}$ and vanishes in its complement. Before this, we need to develop some applications about the Funk–Hecke theorem.

4.1 Further applications of the Funk–Hecke theorem

The following is known as the *Funk–Hecke theorem*.

Lemma 4.1 [28] *Let $f \in L_2[-1,1]$ be a function satisfying*

$$\int_{-1}^{1} |f(t)|(1-t^2)^{\frac{n-3}{2}} < \infty.$$

If $Y_m \in \mathscr{F}^m$ is a spherical harmonic of degree m, then

$$\int_{\mathbb{S}^{n-1}} f(u \cdot v) Y_m(u) du = \lambda_m(f) Y_m(v),$$

for $v \in \mathbb{S}^{n-1}$, where

$$\lambda_m(f) = (n-1)\omega_{n-1}[C_m^{\nu}(1)]^{-1} \int_{-1}^{1} f(t) C_m^{\nu}(t)(1-t^2)^{\frac{n-3}{2}} dt,$$

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where

$$C_m^\nu(t) = a_m^\nu \left(1-t^2\right)^{-\nu+1/2}\left(\frac{d}{dt}\right)^m \left(1-t^2\right)^{m+\nu-1/2},$$

$$a_m^\nu = C_m^\nu(1)\left(-\frac{1}{2}\right)^m \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(m+\frac{n-1}{2}\right)}, \quad \text{and} \quad C_m^\nu(1) = \frac{\Gamma(m+n-2)}{\Gamma(n-2)\Gamma(m+1)}.$$

Lemma 4.2 [28] *For the function $f(t) = |t|$, the eigenvalue $\lambda_m(f)$ in the Funk–Hecke theorem is*

$$\lambda_m(|t|) = \frac{(-1)^{\frac{m-2}{2}}\pi^{\frac{n-1}{2}}\Gamma(m-1)}{2^{m-2}\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m+n+1}{2}\right)},$$

for even m, and $\lambda_m(|t|) = 0$ for odd m. Furthermore, $|\lambda_m(|t|)^{-1}| = O\left(m^{(n+2)/2}\right)$ for even $m \to \infty$.

Since the lightness functions are defined in the form of partial cosine transform

$$\mathcal{C}_- g(v) = \int_{\mathbb{S}^{n-1}} (v\cdot u)_- g(u)du,$$

we require the following Lemma to compute its eigenvalues.

Lemma 4.3 *Let $f(t) = t_-$, and denote β_m to be the $\lambda_m(f)$ in the Funk–Hecke theorem. Then,*

$$\beta_m = \frac{(-1)^{\frac{m-2}{2}}\pi^{\frac{n-1}{2}}\Gamma(m-1)}{2^{m-1}\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{m+n+1}{2}\right)},$$

for even m, $\beta_m = 0$ if m is odd and $m \geq 3$, and

$$\beta_1 = -\frac{\omega_n}{2} \neq 0.$$

Applying Lemma 4.1, one can directly compute the coefficients β_m, but it will be complicated when $m \geq 3$ is odd. However, we would like to provide the following simpler method, to understand the geometric fact that the partial cosine transform $\mathcal{C}_- g$ also defines a support function of a zonoid (which may not be symmetric with respect to the origin).

Proof Since $t_- = 1/2(|t| - t)$, we have

$$\begin{aligned}\int_{\mathbb{S}^{n-1}}(v\cdot u)_- g(u)du &= \frac{1}{2}\int_{\mathbb{S}^{n-1}}|v\cdot u|g(u)du - \frac{1}{2}\int_{\mathbb{S}^{n-1}}(v\cdot u)g(u)du \\ &= \frac{1}{2}\int_{\mathbb{S}^{n-1}}|v\cdot u|g(u)du + v\cdot\left(-\frac{1}{2}\int_{\mathbb{S}^{n-1}}ug(u)du\right).\end{aligned} \tag{4.1}$$

Denote $a_{m,j} = (g, Y_{m,j})$. By the above equation and Lemmas 4.1 and 4.2, we have

$$\int_{\mathbb{S}^{n-1}}(v\cdot u)_- g(u)du = \frac{1}{2}\sum_{m \text{ even}}\sum_{j=1}^{N_m}\lambda_m(|t|)a_{m,j}Y_{m,j}(v) - \frac{1}{2}\int_{\mathbb{S}^{n-1}}(v\cdot u)g(u)du. \tag{4.2}$$

Using Lemma 4.1 again, for the function $(v \cdot u)_-$, we know that

$$\int_{\mathbb{S}^{n-1}} (v \cdot u)_- g(u) du = \sum_{m=0}^{\infty} \sum_{j=1}^{N_m} \beta_m a_{m,j} Y_{m,j}(v). \tag{4.3}$$

By (2.4), we have

$$-\frac{1}{2} \int_{\mathbb{S}^{n-1}} (v \cdot u) g(u) du = -\frac{\omega_n}{2} \cdot \sum_{j=1}^{n} a_{1,j} Y_{1,j}(v).$$

By this and Lemma 4.2, comparing (4.2) with (4.3), we get the desired result. ∎

Lemma 4.4 (Theorem 3.5.4 [28]) *If ρ is an even signed measure on $\mathbb{S}^{n-1}$ with*

$$\int_{\mathbb{S}^{n-1}} |(u \cdot v)| d\rho(u) = 0 \quad \forall\, v \in \mathbb{S}^{n-1},$$

then $\rho = 0$.

If G is an even real function on $\mathbb{S}^{n-1}$ of differentiability class C^k, where $k \geq n+2$ is even, then there exist an even continuous function g on $\mathbb{S}^{n-1}$ such that

$$G(v) = \int_{\mathbb{S}^{n-1}} |(u \cdot v)| g(u) du, \tag{4.4}$$

for $v \in \mathbb{S}^{n-1}$.

The above Lemma and its proof can be found in [28]. Since we want to work with the nonsmooth convex bodies, we slightly improve this result to the functions in Sobolev spaces. Precisely, we consider the functions in $H_k(\mathbb{S}^{n-1})$. Although the approach is standard, we did not find the result as follows. So, for completeness, we give a proof here.

Lemma 4.5 *Let $k \geq n+1$ be an even integer. If $G \in H_k(\mathbb{S}^{n-1})$ is an even function, then there exists an even function $g \in L^2(\mathbb{S}^{n-1})$ such that*

$$G(v) = \int_{\mathbb{S}^{n-1}} |(u \cdot v)| g(u) du, \tag{4.5}$$

for all $v \in \mathbb{S}^{n-1}$. Especially, g can be formulated by the convergent function series

$$\sum_{m \text{ even}} \sum_{j=1}^{N_m} \lambda_m^{-1}(|t|)(G, Y_{m,j}) Y_{m,j}, \tag{4.6}$$

where $\lambda_m^{-1}(|t|)$ is provided by the Funk–Hecke theorem.

Proof Denote $k = 2l$. Since $G \in H_k(\mathbb{S}^{n-1})$, we have

$$G(v) = \sum_{m=0}^{\infty} \sum_{j=1}^{N_m} (G, Y_{m,j}) Y_{m,j}(v).$$

We aim to prove that the series

$$\sum_{m \text{ even}} \lambda_m(|t|)^{-1} \left(\sum_{j=1}^{N_m} (G, Y_{m,j}) Y_{m,j} \right) \tag{4.7}$$

converges in $L^2(\mathbb{S}^{n-1})$.

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Recall that

$$\Delta_S Y_{m,j} = -m(m+n-2)Y_{m,j}.$$

Since Δ_S is self-adjoint, we have

$$(G, Y_{m,j}) = \left[\frac{-1}{m(m+n-2)}\right]^l (\Delta_S^l G, Y_{m,j}).$$

By the Cauchy–Schwarz inequality, we have

$$|(G, Y_{m,j})| \le \left[\frac{1}{m(m+n-2)}\right]^l \cdot \|G\|_{H_k}.$$

As a result, the right-hand side of (4.7) satisfies

$$\left\|\lambda_m(|t|)^{-1}\left(\sum_{j=1}^{N_m}(G, Y_{m,j})Y_{m,j}\right)\right\|_2^2 \le \|G\|_{H_k}^2 \left[\frac{1}{m(m+n-2)}\right]^{2l} \cdot \lambda_m(|t|)^{-2} \cdot N_m.$$

Since $G \in H_k$, it suffices to prove that

$$[m(m+n-2)]^{-2l} \cdot \lambda_m(|t|)^{-2} \cdot N_m = O(m^\alpha), \qquad \text{for some } \alpha < -1. \tag{4.8}$$

Here,

$$\lambda_m(|t|)^{-1} = O\left(m^{\frac{n+2}{2}}\right), \quad \text{and} \quad N_m = \frac{(2m+n-2)\cdot(n+m-3)!}{(n-2)!\cdot m!} = O\left(m^{\frac{n-2}{2}}\right).$$

If $l > n/2 + 1/4$, the inequality (4.8) holds for

$$\alpha = -4l + 2n < -1.$$

Therefore, when $k = 2l \ge n+1$, we have the desired result. ∎

Lemmas 4.3 and 4.5 immediately lead to the following.

Corollary 4.6 *Let $k \ge n+1$ be an even integer, and let $G \in H_k(\mathbb{S}^{n-1})$. Suppose $g \in L^2(\mathbb{S}^{n-1})$ satisfies*

$$\int_{\mathbb{S}^{n-1}} (v \cdot u)_- g(u)du = G(v).$$

Then,

$$g(u) + g(-u) = 2\sum_{m \text{ even}} \sum_{j=1}^{N_m} \beta_m^{-1}(G, Y_{m,j})Y_{m,j}$$

and

$$\int_{\mathbb{S}^{n-1}} ug(u)du = -\frac{2}{\omega_n}\int_{\mathbb{S}^{n-1}} uG(u)du.$$

Moreover, $(G, Y_{m,j}) = 0$, for each odd integer $m \ge 3$.

Proof Denote $\tilde{g}(u) = g(u) + g(-u)$, $a_{m,j} = (g, Y_{m,j})$, and $b_{m,j} = (\tilde{g}, Y_{m,j})$. When m is even, $Y_{m,j}$ is even, and thus $b_{m,j} = 2a_{m,j}$. On one hand, by Lemmas 4.1 and 4.2, we have

$$G(v) = \int_{\mathbb{S}^{n-1}} (v \cdot u)_- g(u) du = \sum_{m \text{ even}} \sum_{j=1}^{N_m} \beta_m a_{m,j} Y_{m,j}(v) - \frac{\omega_n}{2} \sum_{j=1}^{n} a_{1,j} Y_{1,j}(v).$$

On the other hand,

$$G(v) = \sum_{m=0}^{\infty} \sum_{j=1}^{N_m} (G, Y_{m,j}) Y_{m,j}(v).$$

Comparing the coefficients, we have the following. For each odd integer $m \geq 3$, we have $(G, Y_{m,j}) = 0$. For $m = 1$, we have $(G, Y_{1,j}) = -(\omega_n/2) a_{1,j}$. Applying (2.4) for g and G, respectively, we get

$$-\frac{\omega_n}{2} \int_{\mathbb{S}^{n-1}} u g(u) du = \int_{\mathbb{S}^{n-1}} u G(u) du.$$

For even m, we have

$$(G, Y_{m,j}) = \frac{\beta_m}{2} b_{m,j}.$$

Since $\tilde{g}$ is an even function, we get

$$\tilde{g}(u) = g(u) + g(-u) = 2 \sum_{m \text{ even}} \sum_{j=1}^{N_m} \beta_m^{-1} (G, Y_{m,j}) Y_{m,j}.$$

The convergence of the above function series follows from Lemma 4.5 and the fact that $2\beta_m = \lambda_m(|t|)$. ∎

4.2 The necessary and sufficient conditions of being a lightness function

Given a binary function $G : \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \to \mathbb{R}$ so that $G(\cdot, w)$ is Sobolev for a fixed $w \in \mathbb{S}^{n-1}$, we shall use the notation

$$(G(\cdot, w), f) = \int_{\mathbb{S}^{n-1}} G(v, w) f(v) dv.$$

Lemma 4.7 *Let K be a convex body, and let $k \geq n+1$ be an even integer. If, for each $i \in \{1, \dots, n\}$, its lightness functions $Q_K(\cdot, e_i), Q_K(\cdot, -e_i) \in H_k(\mathbb{S}^{n-1})$, then the surface area measure $S(K, \cdot)$ has a density in $L^2(\mathbb{S}^{n-1})$.*

Proof Denote

$$G_{K,i}(v) = Q_K(v, e_i) + Q_K(-v, e_i) + Q_K(v, -e_i) + Q_K(-v, -e_i),$$

which is an even function in $H_k(\mathbb{S}^{n-1})$. On one hand,

$$G_{K,i}(v) = \int_{\mathbb{S}^{n-1}} |u \cdot v| \left(f_{e_i}(u) + f_{-e_i}(u) \right) dS(K, u).$$

On the other hand, by Lemma 4.5, there is an even function $F_{K,i} \in L^2(\mathbb{S}^{n-1})$ such that

$$G_{K,i}(v) = \int_{\mathbb{S}^{n-1}} |u \cdot v| F_{K,i}(u) du.$$

Since the cosine transform is injective (Lemma 4.4), we have

$$(f_{e_i}(u) + f_{-e_i}(u))\, dS(K,u) = F_{K,i}(u) du.$$

Note that, for each $u \in \mathbb{S}^{n-1}$, there exists an $i \in \{1, \dots, n\}$ such that $|u_i| \geq 1/\sqrt{n}$. Since $|\tilde{f}_{e_i}|$ is positive and continuous in $\{u : |u_i| \geq 1/\sqrt{n}\}$, it has a strict positive lower bound. Thus,

$$dS(K,u) = \frac{F_{K,i}(u)}{f_{e_i}(u) + f_{-e_i}(u)} du$$

has a density in $L^2(\mathbb{S}^{n-1})$, where the function $F_{K,i}/(f_{e_i} + f_{-e_i})$ must be independent of i. ■

Theorem 4.8 *Let K be a convex body, and let $k \geq n+1$ be an even integer. Suppose $Q_K(\cdot, w) \in H_k(\mathbb{S}^{n-1})$, for any $w \in \mathbb{S}^{n-1}$. Then:*

(1) *The curvature function F_K satisfies*

$$\frac{1}{2} f_w(u) F_K(u) = 1_{w^+}(u) \cdot \sum_{m \text{ even}} \sum_{j=1}^{N_m} \beta_m^{-1} \big(Q_K(\cdot, w), Y_{m,j}\big) Y_{m,j}(u).$$

(2) *For each odd integer $m \geq 3$, and for each fixed $w \in \mathbb{S}^{n-1}$, the function $Q_K(\cdot, w) : \mathbb{S}^{n-1} \to \mathbb{R}$ satisfies*

$$\big(Q_K(\cdot, w), Y_{m,j}\big) = 0, \quad \forall j = 1, \dots, N_m.$$

(3) *Moreover, F_K satisfies*

$$\int_{\mathbb{S}^{n-1}} v Q_K(v,w) dv = -\frac{\omega_n}{2} \int_{\mathbb{S}^{n-1}} u f_w F_K(u) du, \quad \forall i = 1, \dots, n.$$

Proof By Lemma 4.7, $S(K, \cdot)$ has an $L^2(\mathbb{S}^{n-1})$ density $F_K : \mathbb{S}^{n-1} \to [0, \infty)$. Substitute the function $f_w F_K$ into Corollary 4.6, and observe that

$$Q_K(v,w) = \int_{\mathbb{S}^{n-1}} (v \cdot u)_- f_w(u) F_K(u) du.$$

Then, we immediately obtain (2) and (3), and

$$\frac{1}{2}\left(f_w(u) F_K(u) + f_w(-u) F_K(-u)\right) = \sum_{m \text{ even}} \sum_{j=1}^{N_m} \beta_m^{-1} \big(Q_K(\cdot, w), Y_{m,j}\big) Y_{m,j}(u).$$

Since $f_w(u)$ is concentrated on w^+, $f_w(-u)F_K(-u)$ is just a symmetric copy of $f_w(u)F_K(u)$ in w^-. Thus,

$$\frac{1}{2} f_w(u) F_K(u) = 1_{w^+}(u) \cdot \sum_{m \text{ even}} \sum_{j=1}^{N_m} \beta_m^{-1} \big(Q_K(\cdot, w), Y_{m,j}\big) Y_{m,j}(u).$$

This explains (1). ■

Theorem 4.9 *Let $G : \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \to \mathbb{R}$ be a binary function, and let $k \geq n+1$ be an even integer. Suppose $G(\cdot, w) \in H_k(\mathbb{S}^{n-1})$, for any fixed $w \in \mathbb{S}^{n-1}$. In addition, suppose G satisfies the following conditions:*

(1) *The following function*

$$F(u) = \begin{cases} \sum_{m \text{ even}} \sum_{k=1}^{N_m} \beta_m^{-1}\big(G(\cdot, w), Y_{m,k}\big) \frac{Y_{m,k}(u)}{f_w(u)}, & u \cdot w > 0 \\ \sum_{m \text{ even}} \sum_{k=1}^{N_m} \beta_m^{-1}\big(G(\cdot, -w), Y_{m,k}\big) \frac{Y_{m,k}(u)}{f_{-w}(u)}, & u \cdot w < 0 \end{cases} \tag{4.9}$$

is nonnegative and independent of $w \in \mathbb{S}^{n-1}$.

(2) *For each odd integer $m \geq 3$, and for each fixed $w \in \mathbb{S}^{n-1}$, the function $G(\cdot, w) : \mathbb{S}^{n-1} \to \mathbb{R}$ satisfies*

$$\big(G(\cdot, w), Y_{m,j}\big) = 0, \quad \forall k = 1, \ldots, N_m.$$

(3) *The function F in (1) satisfies*

$$\int_{\mathbb{S}^{n-1}} uF(u)du = 0.$$

(4) *G and the function F in (1) satisfy*

$$\frac{2}{\omega_n} \int_{\mathbb{S}^{n-1}} vG(v, w)dv = \int_{\mathbb{S}^{n-1}} uf_w(u)F(u)du.$$

Then, there is a convex body K such that

$$G = Q_K,$$

and K is unique up to translations.

Condition (1) means that for any $w_1, w_2 \in \mathbb{S}^{n-1}$, the related two function series are equivalent almost everywhere.

Proof By Lemma 4.5, for each fixed $w \in \mathbb{S}^{n-1}$, the function series

$$\sum_{m \text{ even}} \sum_{j=1}^{N_m} \beta_m^{-1}\big(G(\cdot, w), Y_{m,j}\big) Y_{m,j}(u) \in L^2(\mathbb{S}^{n-1}). \tag{4.10}$$

Let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathbb{R}^n$, and denote

$$\Omega_i = \left\{u \in \mathbb{S}^{n-1} : |u \cdot e_i| \geq 1/\sqrt{n}\right\}.$$

By (4.9) and (4.10), we infer that $F \in L^2(\Omega_i)$, $i = 1, \ldots, n$. Since $\{\Omega_i\}_{i=1}^n$ covers $\mathbb{S}^{n-1}$, we see that $F \in L^2(\mathbb{S}^{n-1})$. Since $\mathbb{S}^{n-1}$ is compact, F is also in $L^1(\mathbb{S}^{n-1})$. By conditions (1) and (3), F is independent of w and is of centroid 0. Then, there is a unique (up to translations) convex body K that solves the classical Minkowski problem

$$dS(K, u) = F(u)du.$$

Notice that

$$\frac{1}{2}\left(f_w(u)F(u)+f_w(-u)F(-u)\right)=\sum_{m\text{ even}}\sum_{j=1}^{N_m}\beta_m^{-1}\big(Q_K(\cdot,w),Y_{m,j}\big)Y_{m,j}(u).$$

For this K, by Corollary 4.6, Lemma 4.3, condition (4), equation (2.4), and finally by condition (2), we have

$$\begin{aligned}
Q_K(v,w) &= \int_{\mathbb{S}^{n-1}}(v\cdot u)_- f_w(u)F(u)du\\
&= \frac{1}{2}\int_{\mathbb{S}^{n-1}}(v\cdot u)_-\frac{1}{2}\left(f_w(u)F(u)+f_w(-u)F(-u)\right)du\\
&\quad -\frac{1}{2}\int_{\mathbb{S}^{n-1}}(u\cdot v)f_w(u)F(u)du\\
&= \int_{\mathbb{S}^{n-1}}(v\cdot u)_-\sum_{m\text{ even}}\sum_{j=1}^{N_m}\beta_m^{-1}\big(G(\cdot,w),Y_{m,j}\big)Y_{m,j}(u)\\
&\quad +\frac{1}{\omega_n}\int_{\mathbb{S}^{n-1}}(u\cdot v)G(u,w)du\\
&= \sum_{m\text{ even}}\sum_{j=1}^{N_m}\big(G(\cdot,w),Y_{m,j}\big)Y_{m,j}(v)+\frac{1}{\omega_n}\int_{\mathbb{S}^{n-1}}(u\cdot v)G(u,w)du\\
&= \sum_{m\text{ even}}\sum_{j=1}^{N_m}\big(G(\cdot,w),Y_{m,j}\big)Y_{m,j}(v)+\sum_{j=1}^{N_1}\big(G(\cdot,w),Y_{1,j}\big)Y_{1,j}(v)\\
&= \sum_{m=0}^{\infty}\sum_{j=1}^{N_m}\big(G(\cdot,w),Y_{m,j}\big)Y_{m,j}(v)\\
&= G(v,w).
\end{aligned}$$

That the convex body K is unique follows from Theorem 4.8 and the uniqueness of the Minkowski problem. ■

5 Reconstruction from the pre-given lightness function

In this section, for a pre-given lightness function $Q_K(v,w)$ together with a function f_w as in (1.2), we give the following numerical procedure to reconstruct K.

Light-Pro

- **Input:** Natural numbers n, M, N, k, the spherical harmonics $Y_{m,j}$, where $m = 1,\dots,M$, and $j = 1,\dots,N_m$.
- **Task:** Compute the function $h_{M,N,k}$.
- **Action:**
  1. Fix a $w\in\mathbb{S}^{n-1}$. Compute

$$A_{m,j}=\beta_m^{-1}(Q(\cdot,w),Y_{m,j})$$

and

$$B_{m,j}=\beta_m^{-1}(Q(\cdot,-w),Y_{m,j}).$$

2. Compute

$$F_M(u)=\begin{cases}\sum_{\substack{m=0\\ m \text{ even}}}^{M}\sum_{k=1}^{N_m} A_{m,j}\frac{Y_{m,k}(u)}{f_w(u)}, & u\cdot w>0,\\[2ex] \sum_{\substack{m=0\\ m \text{ even}}}^{M}\sum_{k=1}^{N_m} B_{m,j}\frac{Y_{m,k}(u)}{f_{-w}(u)}, & u\cdot w<0.\end{cases} \tag{5.1}$$

3. Compute

$$\tilde{F}_M(u)=F_M(u)-\frac{1}{\omega_n}\int_{\mathbb{S}^{n-1}}(u\cdot v)F_M(v)dv. \tag{5.2}$$

4. If $\tilde{F}_M \geq 0$, then apply the *Mink-Pro* to find the solution of the $(P_{N,k})$ problem for

$$d\mu=\tilde{F}_M du,$$

and denote the resulting function by $h_{M,N,k}$.

Remark 5.1 1. On one hand, it follows from a direct computation that the $\tilde{F}_M$ defined in (5.2) satisfies

$$\int_{\mathbb{S}^{n-1}} u\tilde{F}_M(u)du=0.$$

On the other hand, since f_w is positive and continuous on $w^+\cap\mathbb{S}^{n-1}$, by Lemma 3.4 (1) and (5.1), we see that $F_M\to F_K$ uniformly. By this and

$$\int_{\mathbb{S}^{n-1}} uF_K(u)du=0,$$

we also have $\tilde{F}_M\to F_K$ uniformly, too.

2. If ∂K is smooth and has positive curvature, we can prove in Theorem 5.1 that $K_{M,N,k}\to K$ as $M,N,k\to\infty$, where $K_{M,N,k}$ is the convex body determined by $h_{M,N,k}$.

Theorem 5.1 *If ∂K is smooth and has positive curvature, then $K_{M,N,k}\to K$ in Hausdorff metric as $M,N,k\to\infty$.*

Proof By Remark 5.1(1), $\tilde{F}_M\to F_K$ uniformly. Since F_K is smooth and has positive curvature, when M is sufficiently large, we also have $\tilde{F}_M>0$. In addition, for any $g\in C(\mathbb{S}^{n-1})$,

$$\left|\int_{\mathbb{S}^{n-1}} g(u)\tilde{F}_M(u)du-\int_{\mathbb{S}^{n-1}} g(u)F_K(u)du\right|\to 0, \quad \text{as } M\to\infty.$$

That is to say, $\tilde{F}_M(u)du\to F_K(u)du$ weakly. Let K_M be the convex body whose curvature function is $\tilde{F}_M$. Since K_M, K are convex bodies such that

$$\int_{\mathbb{S}^{n-1}} uh_M(u)du=\int_{\mathbb{S}^{n-1}} uh_K(u)du=0,$$

$K_M\to K$ in the Hausdorff metric. This is a classical result that can be proved by using Blaschke's selection theorem (see [28] for reference). The assertion follows from this and Theorem 3.3. ■

When $f_w = 1_{w^+ \cap \mathbb{S}^{n-1}}$, the characteristic function of the open hemisphere, Q_K becomes the *partial brightness function* (which is a binary function rather than b_K)

$$R_K(v, w) = \int_{w^+} (v \cdot u)_- dS(K, u).$$

Recalling that the brightness function b_K can only determine an origin-symmetric convex body, for the partial brightness function, we are able to construct the convex body without symmetric assumption. It is reasonable because the change of viewing and illumination directions provides more information.

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References

[1] A. D. Alexandrov, *A. D. Alexandrov selected works. Part II: Intrinsic geometry of convex surfaces*, Chapman & Hall/CRC, Boca Raton, FL, 2006, Edited by S. S. Kutateladze, Translated from the Russian by S. Vakhrameyev.
[2] G. Bianchi, R. J. Gardner, and M. Kiderlen, *Phase retrieval for characteristic functions of convex bodies and reconstruction from covariograms*. J. Amer. Math. Soc. **24**(2011), 293–343.
[3] K. Böröczky, E. Lutwak, D. Yang, G. Zhang, and Y. Zhao, *The dual Minkowski problem for symmetric convex bodies*. Adv. Math. **356**(2019), 106805.
[4] K. J. Böröczky, M. Henk, and H. Pollehn, *Subspace concentration of dual curvature measures of symmetric convex bodies*. J. Differential Geom. **109**(2018), 411–429.
[5] K. J. Böröczky, E. Lutwak, D. Yang, and G. Zhang, *The log-Brunn–Minkowski inequality*. Adv. Math. **231**(2012), 1974–1997.
[6] J. Bourgain and J. Lindenstrauss, *Projection bodies*. In: Geometric aspects of functional analysis (1986/87), Lecture Notes in Mathematics, 1317, Springer, Berlin, 1988, pp. 250–270.
[7] S. Boyd and L. Vandenberghe, *Convex optimization*, Cambridge University Press, Cambridge, 2004.
[8] S. Campi, *Recovering a centred convex body from the areas of its shadows: A stability estimate*. Ann. Mat. Pura Appl. **151**(1988), 289–302.
[9] S. Campi, R. J. Gardner, P. Gronchi, and M. Kiderlen, *Lightness functions*. Adv. Math. **231**(2012), no. 6, 3118–3146.
[10] C. Chen, Y. Huang, and Y. Zhao, *Smooth solutions to the* L_p*-dual Minkowski problem*. Math. Ann. **373**(2019), 953–976.
[11] R. J. Gardner, *Geometric tomography*. 2nd ed., Encyclopedia of Mathematics and Its Applications, 58, Cambridge University Press, New York, 2006.
[12] R. J. Gardner, M. Kiderlen, and P. Milanfar, *Convergence of algorithms for reconstructing convex bodies and directional measures*. Ann. Statist. **34**(2006), 1331–1374.
[13] R. J. Gardner and P. Milanfar, *Shape reconstruction from brightness functions*. In: Advanced signal processing algorithms, architectures, and implementations XI, Society of Photo-Optical Instrumentation Engineers (SPIE) Conference Series, 4474. International Society for Optics and Photonics, Bellingham, Washington, 2001, pp. 234–245.
[14] R. J. Gardner and P. Milanfar, *Reconstruction of convex bodies from brightness functions*. Discrete Comput. Geom. **29**(2003), 279–303.
[15] H. Groemer, *Geometric applications of Fourier series and spherical harmonics*, Encyclopedia of Mathematics and Its Applications, 61, Cambridge University Press, Cambridge, 1996.
[16] P. M. Gruber, *Convex and discrete geometry*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 336, Springer, Berlin, 2007.
[17] P. Guan and X.-N. Ma, *The Christoffel–Minkowski problem. I. Convexity of solutions of a Hessian equation*. Invent. Math. **151**(2003), 553–577.
[18] E. Hebey, *Sobolev spaces on Riemannian manifolds*, Lecture Notes in Mathematics, 1635, Springer, Berlin, 1996.

[19] Y. Huang, E. Lutwak, D. Yang, and G. Zhang, *Geometric measures in the dual Brunn–Minkowski theory and their associated Minkowski problems*. Acta Math. **216**(2016), 325–388.
[20] M. Kaasalainen and L. Lamberg, *Numerical solution of the Minkowski problem*. J. Comput. Appl. Math. **137**(2001), 213–227.
[21] M. Kaasalainen and L. Lamberg, *Inverse problems of generalized projection operators*. Inverse Problems **22**(2006), 749–769.
[22] Q.-R. Li, W. Sheng, and X.-J. Wang, *Flow by Gauss curvature to the Aleksandrov and dual Minkowski problems*. J. Eur. Math. Soc. **22**(2020), 893–923.
[23] J. J. Little, *An iterative method for reconstructing convex polyhedra from extended Gaussian images*. In: Proceedings of the Third AAAI Conference on Artificial Intelligence (Washington, D.C.), AAAI, 1983, pp. 247–250.
[24] E. Lutwak, D. Yang, and G. Zhang, *L_p dual curvature measures*. Adv. Math. **329**(2018), 85–132.
[25] J. Nocedal and S. J. Wright, *Numerical optimization*, Springer, New York, 1999.
[26] A. Poonawala, P. Milanfar, and R. J. Gardner, *Shape estimation from support and diameter functions*. J. Math. Imaging Vision **24**(2006), 229–244.
[27] B. Rubin, *Introduction to radon transforms*, Encyclopedia of Mathematics and Its Applications, 160, Cambridge University Press, New York, 2015.
[28] R. Schneider, *Convex bodies: the Brunn–Minkowski theory*, expanded edition, Encyclopedia of Mathematics and Its Applications, 151, Cambridge University Press, Cambridge, 2014.
[29] R. T. Seeley, *Spherical harmonics*. Amer. Math. Monthly 73(1966), 115–121.
[30] Y. Zhao, *Existence of solutions to the even dual Minkowski problem*. J. Differential Geom. **110**(2018), 543–572.

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