ESTIMATES FOR A REMAINDER TERM ASSOCIATED WITH THE SUM OF DIGITS FUNCTION

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1. Introduction. If $q(\geqslant 2)$ is a fixed integer it is well known that every positive integer k may be expressed uniquely in the form

$$k=\sum_{r=0}^{\infty} a_r(q,k)q^r \quad \text{where} \quad a_r(q,k)\in\{0,1,\ldots,q-1\}. \tag{1.1}$$

We introduce the 'sum of digits' function

$$\alpha(q,k)=\sum_{r=0}^{\infty} a_r(q,k). \tag{1.2}$$

Both the above sums are of course finite. Although the behaviour of $\alpha(q,k)$ is somewhat erratic, its average behaviour is more regular and has been widely studied.

For an integer $n>1$, let $A(q,n)=\sum_{k=1}^{n-1}\alpha(q,k)$, and define $A(q,1)=0$. In the particular case when $n=q^s$ $(s\geqslant 0)$ it is not difficult to prove that

$$A(q,q^s)=\tfrac{1}{2}(q-1)sq^s,$$

which suggests the asymptotic result

$$A(q,n)\sim\frac{\frac{1}{2}(q-1)}{\log q}n\log n \quad \text{as} \quad n\to\infty.$$

This was proved in 1940 by Bush [**2**], and in 1949 Mirsky [**7**] showed in addition that the error term is $O(n)$ but not $o(n)$, thereby improving a contemporary estimate of Bellman and Shapiro [**1**]. In 1952, Drazin and Griffiths [**4**] considered the more general problem of the average of

$$\alpha_t(q,k)=\sum_{r=0}^{\infty}\{a_r(q,k)\}^t, \quad \text{where} \quad t\in\mathbb{N}.$$

They obtained the main term and also gave bounds for the remainder term which are all precise in one direction, and in both directions when $t=1$ and $q=2$ or 3. In particular, for the case $q=2$ they proved that

$$-\frac{\log(4/3)}{\log 2}<\left\{A(2,n)-\frac{n\log n}{2\log 2}\right\}\Big/(n/2)\leqslant 0.$$

Equality holds on the right when $n=2^s$. Also if $n=n(s)$ is of either of the forms

$$1+2^2+2^4+\ldots+2^{2s} \quad \text{or} \quad 2(1+2^2+2^4+\ldots+2^{2s})$$

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then the

$$\lim_{s\to\infty}\left\{A(2, n)-\frac{n\log n}{2\log 2}\right\}\Big/(n/2)=-\frac{\log(4/3)}{\log 2},$$

ensuring that the above left hand inequality is best possible. These estimates have also been obtained by McIlroy [**6**] and Shiokawa [**8**].

In more recent times, there has been a great deal of work on generalisations of this and related problems, some probabilistic in nature. A paper of Stolarsky [**9**] in 1977, concerned with digital sums (the case $q=2$), contains a brief survey of the history of the problem and, very helpfully, cites sixty two references including the ones already mentioned.

In 1975, Delange [**3**] obtained a very elegant analytical form for the remainder term, involving a function which is continuous, nowhere differentiable and periodic with period 1, thereby generalizing an earlier result concerned with the case $q=2$ of Trollope [**10**]. The case of Cantor representations of integers was also considered by Trollope [**11**] and more recently by Kirschenhofer and Tichy [**5**]. Their investigation reduces to a study of

$$S(q, n)=\left\{A(q, n)-\tfrac{1}{2}(q-1)\left[\frac{\log n}{\log q}\right]n\right\}\Big/\frac{1}{2} \tag{1.3}$$

in the special case when the Cantor representation of an integer k becomes a representation of the form (1.1) for some q. With the usual notation, $\left[\dfrac{\log n}{\log q}\right]$ denotes the greatest integer $\leqslant \dfrac{\log n}{\log q}$. This suggests that, in the original digits problem, one might consider directly an estimate for $\dfrac{S(q, n)}{n}$ and that is the object of this paper. In particular, we obtain best possible upper and lower bounds when $q=2$ and 3. It is planned to consider later the cases $q=4$ and 5.

THEOREM 1. *If* $n\in\mathbb{N}$,

$$-\frac{2}{13}<\frac{S(2, n)}{n}<1. \tag{1.4}$$

THEOREM 2. *If* $n\in\mathbb{N}$,

$$-\frac{2}{7}<\frac{S(3, n)}{n}<2. \tag{1.5}$$

The method used to prove Theorems 1 and 2 involves expressing n in the special form n_m ($m\in\mathbb{N}$), to be described shortly. Then bounds are obtained for $S(q, n_m)/n_m$ in terms of q and m, from which Theorems 1 and 2 can be deduced.

Firstly we need to obtain an algebraic expression for $A(q, n)$. If $s\geqslant 2$,

$$A(q, q^s)=\sum_{1\leqslant r<q^{s-1}}\alpha(q, r)+\sum_{t=1}^{q-1}\sum_{tq^{s-1}\leqslant r<(t+1)q^{s-1}}\alpha(q, r).$$

Putting $r = tq^{s-1} + u$ in the second (inner) sum and using the fact that $\alpha(q, r) = t + \alpha(q, u)$ where $0 \leqslant u < q^{s-1}$, it follows easily that

$$A(q, q^s) = qA(q, q^{s-1}) + \tfrac{1}{2}(q-1)q^s.$$

If $s \geqslant 1$, an inductive proof now yields

$$A(q, q^s) = \tfrac{1}{2}(q-1)sq^s, \tag{1.6}$$

and more generally, if $1 \leqslant a < q$,

$$A(q, aq^s) = aA(q, q^s) + \tfrac{1}{2}a(a-1)q^s. \tag{1.7}$$

With a slight change of notation, every positive integer $n \not\equiv 0 \pmod q$ is of the form $n = n_m$ where

$$n_m = a_0q^{t_0} + a_1q^{t_0+t_1} + a_2q^{t_0+t_1+t_2} + \ldots + a_mq^{t_0+t_1+t_2+\ldots+t_m}, \tag{1.8}$$

for some $m \in \mathbb{N} \cup \{0\}$, $t_0 = 0$, positive integers $t_1, t_2, \ldots, t_m$ and non-zero coefficients $a_0, a_1, a_2, \ldots, a_m \in \{1, 2, \ldots, q-1\}$. Given such an integer n, for convenience of notation introduce

$$n_0 = a_0 \quad \text{and} \quad n_i = a_0 + a_1q^{t_1} + \ldots + a_iq^{t_1+\ldots+t_i} \tag{1.9}$$

for $1 \leqslant i \leqslant m$. Then

$$\begin{aligned} A(q, n_m) &= A(q, a_mq^{t_1+\ldots+t_m}) + \sum_{a_mq^{t_1+\ldots+t_m} \leqslant r < n_m} \alpha(q, r), \\ &= a_mA(q, q^{t_1+\ldots+t_m}) + \tfrac{1}{2}a_m(a_m-1)q^{t_1+\ldots+t_m} + a_mn_{m-1} + A(q, n_{m-1}), \end{aligned}$$

using (1.7), so that

$$A(q, n_m) - A(q, n_{m-1}) = a_mA(q, q^{t_1+\ldots+t_m}) + a_mn_{m-1} + \tfrac{1}{2}a_m(a_m-1)q^{t_1+\ldots+t_m}.$$

Iterating this formula and using the fact that $A(q, n_0) = \tfrac{1}{2}a_0(a_0-1)$ we obtain, on addition,

$$A(q, n_m) - \tfrac{1}{2}a_0(a_0-1) = \sum_{r=1}^{m} \{a_rA(q, q^{t_1+\ldots+t_r}) + a_rn_{r-1}\} + \tfrac{1}{2}\sum_{r=1}^{m} a_r(a_r-1)q^{t_1+\ldots+t_r}.$$

However, using (1.6),

$$\begin{aligned} \sum_{r=1}^{m} a_rA(q, q^{t_1+\ldots+t_r}) &= \tfrac{1}{2}(q-1)\sum_{r=1}^{m} a_r(t_1+\ldots+t_r)q^{t_1+\ldots+t_r} \\ &= \tfrac{1}{2}(q-1)\sum_{r=1}^{m}(t_1+\ldots+t_r)(n_r-n_{r-1}) = \tfrac{1}{2}(q-1)\sum_{r=1}^{m} t_r(n_m-n_{r-1}). \end{aligned}$$

Thus

$$\begin{aligned} A(q, n_m) = \tfrac{1}{2}(q-1)(t_1+\ldots+t_m)n_m &+ \sum_{r=1}^{m}(a_r - \tfrac{1}{2}(q-1)t_r\}n_{r-1} \\ &+ \tfrac{1}{2}a_0(a_0-1) + \tfrac{1}{2}\sum_{r=1}^{m} a_r(a_r-1)q^{t_1+\ldots+t_r}. \end{aligned}$$

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If $m \in \mathbb{N}$,

$$q^{t_1+\ldots+t_m} \leqslant n_m < q^{t_1+\ldots+t_m+1}$$

so that

$$t_1+\ldots+t_m = \left[\frac{\log n_m}{\log q}\right],$$

while if $m = 0$,

$$0 = \left[\frac{\log n_m}{\log q}\right].$$

Thus, from (1.3),

$$S(q, n_m) = \sum_{r=0}^{m} a_r(a_r - 1)q^{t_0+t_1+\ldots+t_r} + \sum_{r=1}^{m} \{2a_r - (q-1)t_r\}n_{r-1}. \tag{1.10}$$

It is easily verified that, if $\beta \in \mathbb{N}$,

$$\frac{S(q, q^{\beta}n_m)}{q^{\beta}n_m} = \frac{S(q, n_m)}{n_m}$$

so that there is no loss of generality in assuming that $n = n_m$ is of the form (1.8).

As already mentioned, it is our aim to prove Theorems 1 and 2 in a stronger form, and we now introduce

$$h_2(m) = \frac{2(2^{2m}-1)}{13\,.\,2^{2m}-1} \quad \text{and} \quad h_3(m) = \frac{6(3^m-1)}{7\,.\,3^{m+1}-5} \quad \text{for} \quad m \in \mathbb{N} \cup \{0\}.$$

Each of these functions is monotonic increasing. In fact $h_2(1) = 0{\cdot}1176\ldots$, $h_2(2) = 0{\cdot}1449\ldots$, $h_2(3) = 0{\cdot}1516\ldots$, $h_2(4) = 0{\cdot}1532\ldots$ and, as $m \to \infty$, $h_2(m) \to \frac{2}{13} = 0{\cdot}1538\ldots$. Also $h_3(1) = 0{\cdot}2068\ldots$, $h_3(2) = 0{\cdot}2608\ldots$, $h_3(3) = 0{\cdot}2775\ldots$ and, as $m \to \infty$, $h_3(m) \to \frac{2}{7} = 0{\cdot}2857\ldots$.

THEOREM 1*. *If* $m \geqslant 0$,

$$-h_2(m) \leqslant \frac{S(2, n_m)}{n_m} \leqslant 1 - \frac{m+1}{2^{m+1}-1}. \tag{1.11}$$

Equality holds on the right when $n_m = 1 + 2 + 2^2 + \ldots + 2^m$, *and on the left when* $n_m = 1 + 2^2 + 2^4 + \ldots + 2^{2(m-1)} + 2^{2(m+1)}$ $(m \geqslant 1)$.

THEOREM 2*. *If* $m \geqslant 0$,

$$-h_3(m) \leqslant \frac{S(3, n_m)}{n_m} \leqslant 2\left\{1 - \frac{m+1}{3^{m+1}-1}\right\}. \tag{1.12}$$

Equality holds on the right, when $n_m = 2(1 + 3 + 3^2 + \ldots + 3^m)$, *and on the left, for* $m \geqslant 3$, *when* $n_m = 2 + 3^2 + 3^3 + 3^4 + \ldots + 3^{m-1} + 3^m + 3^{m+2}$.

For each of these theorems it is the right hand inequality which is the easiest to establish. In fact we can prove a general theorem in this respect.

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THEOREM 3*. *If $q \geqslant 2$ and $m \geqslant 0$,*

$$\frac{S(q, n_m)}{n_m} \leqslant (q-1)\left\{1 - \frac{m+1}{q^{m+1}-1}\right\}, \tag{1.13}$$

with equality when $n_m = (q-1)(1+q+q^2+\ldots+q^m) = q^{m+1}-1$. Since every positive integer is of the form n_m or $q^\beta n_m$ for some $\beta \in \mathbb{N}$, it is clear that the starred theorems give rise to Theorems 1 and 2.

I should like to express my thanks to my colleague Dr G. M. Phillips for computational work which enabled the value of $h_3(m)$, upon which the proof of Theorem 2* depends, to be determined for small values of m. I should also like to thank the referee for some very helpful comments.

2. The proofs of each of Theorems 1*–3* are inductive starting with the small values of m. The following identities, derived from (1.10), will be of later use.

$$S(q, n_m) = S(q, n_{m-1}) + a_m(a_m-1)q^{t_1+\ldots+t_m} + \{2a_m - (q-1)t_m\}n_{m-1} \qquad (m \geqslant 1); \tag{2.1}$$

$$S(q, n_m) = a_0(a_0-1) + a_0 \sum_{r=1}^{m} \{2a_r - (q-1)t_r\}$$
$$+ q^{t_1}S(q, a_1 + a_2q^{t_2} + a_3q^{t_2+t_3} + \ldots + a_mq^{t_2+t_3+\ldots+t_m}) \qquad (m \geqslant 1); \tag{2.2}$$

and, for any integer l, with $2 \leqslant l \leqslant m-1$,

$$S(2, n_m) = \sum_{j=1}^{l} \left\{2^{t_0+t_1+\ldots+t_{j-1}} \sum_{r=j}^{m} (2-t_r)\right\}$$
$$+ 2^{t_1+\ldots+t_{l-1}+t_l}S(2, 1 + 2^{t_{l+1}} + 2^{t_{l+1}+t_{l+2}} + \ldots + 2^{t_{l+1}+t_{l+2}+\ldots+t_m}) \tag{2.3}$$

where $t_0 = 0$, as before.

LEMMA. *For $m \geqslant 1$ and n_m as in* (1.8) *and* (1.9),

$$\frac{n_{m-1}}{n_m} < \frac{1}{1 + a_mq^{t_m-1}}. \tag{2.4}$$

Proof. For $0 \leqslant i \leqslant m-1$ we have $0 < a_i \leqslant q-1$, giving

$$n_{m-1} \leqslant (q-1)(1 + q^{t_1} + q^{t_1+t_2} + \ldots + q^{t_1+t_2+\ldots+t_{m-1}}),$$
$$\leqslant (q-1)\left[\frac{q^{t_1+\ldots+t_{m-1}+1}-1}{q-1}\right] < q^{t_1+\ldots+t_{m-1}+1}.$$

Hence

$$\frac{n_{m-1}}{n_m} = \frac{n_{m-1}}{n_{m-1} + a_mq^{t_1+\ldots+t_m}},$$
$$< \frac{q^{t_1+\ldots+t_{m-1}+1}}{q^{t_1+\ldots+t_{m-1}+1} + a_mq^{t_1+\ldots+t_m}},$$
$$= \frac{1}{1 + a_mq^{t_m-1}}.$$

3. Proof of Theorem 3*. If $m=0$, we have $n_m=n_0=a_0$ so that

$$\frac{S(q,n_0)}{n_0}=\frac{a_0(a_0-1)}{a_0}=a_0-1\leqslant q-2=(q-1)\left\{1-\frac{1}{q-1}\right\},$$

which is the result stated.

Now choose $m\geqslant 1$ and assume that

$$\frac{S(q,n_{m-1})}{n_{m-1}}\leqslant (q-1)\left\{1-\frac{m}{q^m-1}\right\}. \tag{3.1}$$

By (2.1),

$$\begin{aligned}S(q,n_m)&=S(q,n_{m-1})+a_m(a_m-1)q^{t_1+\dots+t_m}+\{2a_m-(q-1)t_m\}n_{m-1},\\ &\leqslant (q-1)\left\{1-\frac{m}{q^m-1}\right\}n_{m-1}+(a_m-1)(n_m-n_{m-1})+\{2a_m-(q-1)t_m\}n_{m-1},\\ &=(a_m-1)n_m+\left\{a_m+q-(q-1)t_m-\frac{m(q-1)}{q^m-1}\right\}n_{m-1}, \qquad (3.2)\\ &\leqslant (a_m-1)n_m+\left\{a_m+1-\frac{m(q-1)}{q^m-1}\right\}n_{m-1}, \qquad \text{since } t_m\geqslant 1.\end{aligned}$$

Using (2.4) we see that

$$\begin{aligned}\frac{S(q,n_m)}{n_m}&\leqslant a_m-1+\left\{a_m+1-\frac{m(q-1)}{q^m-1}\right\}\frac{1}{1+a_m},\\ &=a_m-\frac{m(q-1)}{(q^m-1)(1+a_m)}.\end{aligned}$$

If $1\leqslant a_m\leqslant q-2$ we have

$$\begin{aligned}\frac{S(q,n_m)}{n_m}&\leqslant q-2-\frac{m}{q^m-1},\\ &=(q-1)\left\{1-\frac{m+1}{q^{m+1}-1}\right\}-\left\{1+\frac{m}{q^m-1}-\frac{(q-1)(m+1)}{q^{m+1}-1}\right\}.\end{aligned}$$

As $\dfrac{(q-1)(m+1)}{q^{m+1}-1}<1$, (1.13) follows in this case.

Now suppose that $a_m=q-1$ and $t_m\geqslant 2$. From (3.2),

$$\frac{S(q,n_m)}{n_m}\leqslant q-2+\left\{1-\frac{m(q-1)}{q^m-1}\right\}\frac{n_{m-1}}{n_m}.$$

By (2.4),

$$\frac{n_{m-1}}{n_m}\leqslant\frac{1}{1+(q-1)q}<\frac{1}{(q-1)q}.$$

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Thus

$$\frac{S(q, n_m)}{n_m} \leqslant q-2+\frac{1}{(q-1)q}-\frac{m}{q(q^m-1)}.$$

It follows that (1.13) will hold if we can prove that

$$q-2+\frac{1}{(q-1)q}-\frac{m}{q(q^m-1)} \leqslant q-1-\frac{(q-1)(m+1)}{q^{m+1}-1}$$

or equivalently

$$\frac{(q-1)(m+1)}{q^{m+1}-1}-\frac{m}{q(q^m-1)} \leqslant \frac{q(q-1)-1}{q(q-1)}. \tag{3.3}$$

Since $q^{m+1}-1>q(q^m-1)$ we have

$$\frac{(q-1)(m+1)}{q^{m+1}-1}-\frac{m}{q(q^m-1)} < \frac{(q-1)(m+1)-m}{q(q^m-1)} < \frac{(q-1)(m+1)-m}{m(q-1)q},$$
$$\leqslant \frac{q(q-1)-1}{q(q-1)} \quad \text{since} \quad \frac{m+1}{m} \leqslant q.$$

Hence (3.3) holds.

Thus we are left with the case $a_m = q-1$ and $t_m = 1$. If $m = 1$, it follows from (3.2) that

$$\frac{S(q, n_1)}{n_1} \leqslant q-2+(q-1)\frac{n_0}{n_1},$$

where

$$\frac{n_0}{n_1} = \frac{a_0}{a_0+(q-1)q} \leqslant \frac{q-1}{q-1+(q-1)q} = \frac{1}{1+q}.$$

Hence

$$\frac{S(q, n_1)}{n_1} \leqslant q-2+\frac{q-1}{q+1} = (q-1)\left\{1-\frac{2}{q^2-1}\right\},$$

as required.

Suppose now that for some integer l with $2 \leqslant l \leqslant m$,

$$t_m = t_{m-1} = \ldots = t_l = 1, \qquad t_{l-1} \geqslant 2 \quad \text{and} \quad a_m = q-1. \tag{3.4}$$

We shall now prove that (1.13) holds. To do this requires an improved upper bound for n_{m-1}/n_m. For, putting $a_m = q-1$ and $t_m = 1$ in (3.2) we see that

$$\frac{S(q, n_m)}{n_m} \leqslant q-2+\left\{q-\frac{m(q-1)}{q^m-1}\right\}\frac{n_{m-1}}{n_m}, \tag{3.5}$$

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and (1.13) will follow if we can prove that

$$q-2+\left\{q-\frac{m(q-1)}{q^m-1}\right\}\frac{n_{m-1}}{n_m}\leqslant (q-1)\left\{1-\frac{m+1}{q^{m+1}-1}\right\}.$$

On rearranging, this inequality reduces to

$$\frac{n_{m-1}}{n_m}\leqslant\frac{q^m-1}{q^{m+1}-1}. \tag{3.6}$$

To establish (3.6) we make use of (3.4). We have

$$\begin{aligned} n_{m-1} &\leqslant (q-1)\{1+q^{t_1}+q^{t_1+t_2}+\ldots+q^{t_1+t_2+\ldots+t_{l-1}}\} \\ &\quad +(q-1)q^{t_1+\ldots+t_{l-1}+1}(1+q+q^2+\ldots+q^{m-l-1}), \\ &\leqslant (q-1)\left\{\frac{q^{t_1+\ldots+t_{l-1}+1}-1}{q-1}-q^{t_1+\ldots+t_{l-1}-1}\right\} \\ &\quad +(q-1)q^{t_1+\ldots+t_{l-1}+1}\frac{(q^{m-l}-1)}{q-1}, \\ &= q^{t_1+\ldots+t_{l-1}-1}(q^{m-l+2}-q+1)-1, \\ &< q^{t_1+\ldots+t_{l-1}-1}(q^{m-l+2}-q+1). \end{aligned}$$

Hence

$$\begin{aligned} \frac{n_{m-1}}{n_m} &= \frac{n_{m-1}}{n_{m-1}+a_m q^{t_1+\ldots+t_m}}, \\ &< \frac{q^{t_1+\ldots+t_{l-1}-1}(q^{m-l+2}-q+1)}{q^{t_1+\ldots+t_{l-1}-1}(q^{m-l+2}-q+1)+(q-1)q^{t_1+\ldots+t_{l-1}+m-l+1}} \\ &= \frac{q^{m-l+2}-q+1}{q^{m-l+3}-q+1}. \end{aligned}$$

Thus (3.5) will follow provided that

$$\frac{q^{m-l+2}-q+1}{q^{m-l+3}-q+1}\leqslant\frac{q^m-1}{q^{m+1}-1},$$

or equivalently, on rearranging,

$$q^{m-l+2}\{q^{l-2}(q-1)-1\}\geqslant 0,$$

and this is clearly true for $q\geqslant 2$ and $2\leqslant l\leqslant m$.

Thus (1.13) is now established except possibly when

$$m\geqslant 2,\qquad a_m=q-1\quad\text{and}\quad t_1=t_2=\ldots=t_m=1.$$

However in this case (3.5) holds and (3.6) is easily verified, using the fact that $n_{m-1}\leqslant q^m-1$.

4. Proof of Theorem 1*. For the case $q=2$, it suffices to prove that, if $n_m = 2^{t_0} + 2^{t_0+t_1} + \ldots + 2^{t_0+t_1+\ldots+t_m}$, where $t_0=0,\ t_1, \ldots, t_m \in \mathbb{N}$ and $m \geq 0$ then

$$\frac{S(2, n_m)}{n_m} \geq -h_2(m) \quad \text{where} \quad h_2(m) = \frac{2(2^{2m}-1)}{13 \cdot 2^{2m} - 1}. \tag{4.1}$$

When $m=0$, it is clear from (1.10) that $S(2, n_m) = S(2, n_0) = 0 = -h_2(0)$. Thus we can assume that $m \geq 1$. As before, introduce

$$n_0 = 1 \quad \text{and} \quad n_i = 1 + 2^{t_1} + 2^{t_1+t_2} + \ldots + 2^{t_1+t_2+\ldots+t_i} \quad (1 \leq i \leq m).$$

Then (1.10) takes the simple form

$$S(2, n_m) = \sum_{r=1}^{m} (2 - t_r) n_{r-1}.$$

If $m=1$,

$$\frac{S(2, n_1)}{n_1} = \frac{2 - t_1}{1 + 2^{t_1}},$$

and it is easily verified that for $t \in \mathbb{N}$ this function takes a minimum value namely $-\frac{2}{17} = -h_2(1)$, when $t_1 = 4$.

Now choose $m \geq 2$ and assume that for all integers m' satisfying $1 \leq m' \leq m-1$,

$$\frac{S(2, n_{m'})}{n_{m'}} \geq -h_2(m') > -\tfrac{2}{13}. \tag{4.2}$$

Using (2.1) with $q=2$ and $a_m = 1$, we have

$$S(2, n_m) = S(2, n_{m-1}) + (2 - t_m) n_{m-1},$$

giving

$$\frac{S(2, n_m)}{n_m} = \left\{ \frac{S(2, n_{m-1})}{n_{m-1}} + 2 - t_m \right\} \frac{n_{m-1}}{n_m}.$$

By the inductive hypothesis and Theorem 3*,

$$-\frac{2}{13} < \frac{S(2, n_{m-1})}{n_{m-1}} < 1,$$

and, by (2.4),

$$\frac{n_{m-1}}{n_m} < \frac{1}{1 + 2^{t_m - 1}} \leq \tfrac{1}{2}.$$

Thus if $t_m = 1$ we have

$$\frac{S(2, n_m)}{n_m} > 0 > -h_2(m),$$

and if $t_m=2$ we have

$$\frac{S(2,n_m)}{n_m}>-\tfrac{1}{2}h_2(m-1)>-h_2(m).$$

If $t_m \geqslant 3$,

$$\frac{S(2,n_m)}{n_m}>-(\tfrac{2}{13}+t_m-2)\cdot\frac{1}{1+2^{t_m-1}},$$

$$=-\left\{\frac{13t_m-24}{13(1+2^{t_m-1})}\right\}.$$

The function $\dfrac{13t_m-24}{13(1+2^{t_m-1})}$ is monotonic decreasing for $t_m \geqslant 6$ with the value $0{\cdot}1258\ldots$ ($<h_2(m)$ for $m \geqslant 2$) when $t_m=6$. Hence (4.1) follows if $t_m=1$, 2 or if $t_m \geqslant 6$. Assume henceforth that

$$t_m=3,\ 4 \text{ or } 5. \tag{4.3}$$

By (2.2), with $q=2$, $a_0=a_1=\ldots=a_m=1$, we have

$$S(2,n_m)=\sum_{r=1}^{m}(2-t_r)+2^{t_1}S(2,1+2^{t_2}+2^{t_2+t_3}+\ldots+2^{t_2+t_3+\ldots+t_m}).$$

Applying the induction hypothesis again gives

$$S(2,n_m)\geqslant\sum_{r=1}^{m}(2-t_r)-2^{t_1}h_2(m-1)\{1+2^{t_2}+2^{t_2+t_3}+\ldots+2^{t_2+\ldots+t_m}\},$$

$$=\sum_{r=1}^{m}(2-t_r)-h_2(m-1)\{n_m-1\}.$$

Thus (4.1) will follow provided that

$$\sum_{r=1}^{m}(2-t_r)+\{h_2(m)-h_2(m-1)\}n_m+h_2(m-1)\geqslant 0. \tag{4.4}$$

If $s\in\mathbb{N}$,

$$h_2(m)-h_2(m-s)=\frac{3(2^{2s}-1)2^{2(m-s)+3}}{(13\,.\,2^{2m}-1)(13\,.\,2^{2(m-s)}-1)}. \tag{4.5}$$

Hence (4.4) is equivalent to

$$\sum_{r=1}^{m}(2-t_r)+\frac{9\,.\,2^{2m+1}}{(13\,.\,2^{2m}-1)(13\,.\,2^{2(m-1)}-1)}n_m+h_2(m-1)\geqslant 0. \tag{4.6}$$

Clearly (4.6) holds $\forall t_1,\ldots,t_m>0$ with $\sum_{r=1}^{m}(2-t_r)\geqslant 0$. Thus assume now that

$$\sum_{r=1}^{m}t_r=2m+1+k \quad \text{where} \quad k\geqslant 0. \tag{4.7}$$

Condition (4.6) then takes the form

$$\frac{9 \,.\, 2^{2m+1}}{(13 \,.\, 2^{2m}-1)(13 \,.\, 2^{2(m-1)}-1)} n_m + h_2(m-1) \geqslant k+1. \tag{4.8}$$

If $m=2$, we have $n_m = 1 + 2^{k+5-t_2} + 2^{k+5} > 1 + 2^{k+5}$, so that (4.8) will follow provided that

$$\frac{9 \,.\, 2^5(1+2^{k+5})}{10{,}557} + h_2(1) > k+1.$$

The only integral value of $k \geqslant 0$ for which this inequality does not hold is $k=1$. In this case, from (4.7), $t_1 + t_2 = 6$. By (4.3), therefore,

$$(t_1, t_2) = (3, 3), (2, 4) \text{ or } (1, 5).$$

Then $\dfrac{S(2, n_2)}{n_2} = -0{\cdot}1369\ldots, \ -0{\cdot}1449\ldots (= -h_2(2))$ or $-0{\cdot}1194\ldots$ in each of these cases respectively, and the case $m=2$ is proved.

Assume henceforth that $m \geqslant 3$. By (4.7),

$$n_m > 2^{2m+k+1-t_m}(1+2^{t_m}).$$

Since $(13 \,.\, 2^{2m}-1)(13 \,.\, 2^{2(m-1)}-1) < 169 \,.\, 2^{4m-2}$, (4.8) will follow if we can prove that

$$\frac{9 \,.\, 2^{2m+1}}{169 \,.\, 2^{4m-2}} \,.\, 2^{2m+k+1-t_m}(1+2^{t_m}) + h_2(m-1) \geqslant k+1,$$

that is

$$\tfrac{9}{169} \,.\, 2^{k+4-t_m}(1+2^{t_m}) + h_2(m-1) \geqslant k+1.$$

It is easily verified that this inequality holds when $t_m = 3$ for all integers $k \geqslant 0$ and, when $t_m = 4$ or 5, it holds for all integers $k \geqslant 0$ except $k=1$. Hence we are left with the cases

$$m \geqslant 3, \qquad t_m = 4 \text{ or } 5 \quad \text{and} \quad \sum_{r=1}^{m} t_r = 2m+2. \tag{4.9}$$

Now use identity (2.3), with $l=2$, together with the induction hypothesis on $S(2, 1+2^{t_3}+2^{t_3+t_4}+\ldots+2^{t_3+\cdots+t_m})$ to obtain

$$S(2, n_m) \geqslant \sum_{r=1}^{m} (2-t_r) + 2^{t_1} \sum_{r=2}^{m} (2-t_r) - h_2(m-2)\{n_m - 1 - 2^{t_1}\}.$$

Then (4.1) will follow if we can show that

$$\sum_{r=1}^{m} (2-t_r) + 2^{t_1} \sum_{r=2}^{m} (2-t_r) + \{h_2(m) - h_2(m-2)\} n_m + (1+2^{t_1}) h_2(m-2) \geqslant 0.$$

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Using (4.5) with $s=2$ and (4.9) this inequality is equivalent to

$$\frac{45\,.\,2^{2m-1}}{(13\,.\,2^{2m}-1)(13\,.\,2^{2(m-2)}-1)}n_m+(1+2^{t_1})h_2(m-2)\geqslant 2+2^{t_1}(4-t_1). \tag{4.10}$$

If $t_1\geqslant 5$, $2+2^{t_1}(4-t_1)<0$ and (4.10) follows easily. Hence, in addition to (4.9), assume that

$$t_1=1,\ 2,\ 3 \text{ or } 4. \tag{4.11}$$

If $m=3$, (4.9) and (4.11) are only satisfied when

$$(t_1, t_2, t_3)=(1,3,4),\ (1,2,5),\ (2,2,4),\ (2,1,5) \quad\text{and}\quad (3,1,4).$$

The corresponding values of $\dfrac{S(2, n_3)}{n_3}$ are

$$-0{\cdot}1454,\ldots,\quad -0{\cdot}1198\ldots,\quad -0{\cdot}1516\ldots(=-h_2(3)),\quad -0{\cdot}1263\ldots\quad\text{and}\quad -0{\cdot}1494\ldots$$

respectively, thus settling the case $m=3$ of the theorem.

Assume henceforth that $m\geqslant 4$. Clearly

$$n_m>2^{2m+2-t_m}(1+2^{t_m}).$$

Accordingly (4.10) will follow if we can show that

$$\frac{45\,.\,2^{2m-1}}{169\,.\,2^{4m-4}}\cdot 2^{2m+2-t_m}(1+2^{t_m})+(1+2^{t_1})h_2(m-2)\geqslant 2+2^{t_1}(4-t_1),$$

that is

$$\tfrac{45}{169}\,.\,2^{5-t_m}(1+2^{t_m})+(1+2^{t_1})h_2(m-2)\geqslant 2+2^{t_1}(4-t_1).$$

Equivalently, this is the condition

$$2+2^{t_1}(4-t_1)-(1+2^{t_1})h_2(m-2)\leqslant\begin{cases}9{\cdot}0532\ldots & \text{if}\quad t_m=4\\ 8{\cdot}7869\ldots & \text{if}\quad t_m=5\end{cases},$$

which holds for $t_1=1$, 3 or 4. Thus it now remains to consider the cases

$$m\geqslant 4,\qquad t_1=2,\quad t_m=4 \text{ or } 5\quad\text{and}\quad \sum_{r=1}^{m}t_r=2m+2. \tag{4.12}$$

In the following we choose the maximal integer l satisfying

$$3\leqslant l\leqslant m-1\quad\text{and}\quad t_1=t_2=\ldots=t_{l-2}=2. \tag{4.13}$$

Using identity (2.3) and the inductive hypothesis for

$$S(2,\,1+2^{t_{l+1}}+\ldots+2^{t_{l+1}+\cdots+t_m})$$

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we have

$$S(2, n_m) \geqslant \sum_{r=1}^{m} (2-t_r) + 2^{t_1} \sum_{r=2}^{m} (2-t_r) + \ldots + 2^{t_1+t_2+\ldots+t_{l-1}} \sum_{r=l}^{m} (2-t_r)$$
$$- h_2(m-l)\{n_m - 1 - 2^{t_1} - 2^{t_1+t_2} - \ldots - 2^{t_1+t_2\ldots+t_{l-1}}\}.$$

Hence

$$S(2, n_m) \geqslant -h_2(m)n_m$$

provided that

$$\sum_{r=1}^{m} (2-t_r) + 2^{t_1} \sum_{r=2}^{m} (2-t_r) + \ldots + 2^{t_1+t_2+\ldots+t_{l-1}} \sum_{r=l}^{m} (2-t_r)$$
$$+ \{h_2(m) - h_2(m-l)\}n_m + \{1 + 2^{t_1} + 2^{t_1+t_2} + \ldots + 2^{t_1+t_2+\ldots+t_{l-1}}\}h_2(m-l) \geqslant 0$$

With (4.5), (4.12) and (4.13), this condition takes the form

$$-\tfrac{2}{3}(2^{2l-2}-1) + (t_{l-1}-4)2^{2l-4+t_{l-1}} + \frac{3(2^{2l}-1)2^{2(m-l)+3}}{(13\,.\,2^{2m}-1)(13\,.\,2^{2(m-l)}-1)}n_m$$
$$+ \{\tfrac{1}{3}(2^{2l-2}-1) + 2^{2l-4+t_{l-1}}\}h_2(m-l) \geqslant 0. \quad (4.14)$$

If

$$t_{l-1} \geqslant 5, \qquad (t_{l-1}-4)2^{2l-4+t_{l-1}} \geqslant 2^{2l+1} > \tfrac{2}{3}(2^{2l-2}-1)$$

and (4.14) follows easily. Thus, in addition to the conditions of (4.13), we can assume that

$$t_{l-1} = 1,2,3 \text{ or } 4. \quad (4.15)$$

Obviously,

$$n_m > 2^{2m+2-t_m}(1+2^{t_m}) \geqslant 33\,.\,2^{2m-3} \text{ for } t_m = 4 \text{ or } 5.$$

Thus

$$\frac{3(2^{2l}-1)2^{2(m-l)+3}}{(13\,.\,2^{2m}-1)(13\,.\,2^{2(m-l)}-1)}n_m > \tfrac{99}{169}(2^{2l}-1),$$

and (4.14) is a consequence of

$$T_1(l, t_{l-1}) + T_2(l) + T_3(l, t_{l-1})h_2(m-l) > 0 \quad (4.16)$$

subject to $3 \leqslant l \leqslant m-1$ and $t_{l-1} \in \{1, 2, 3, 4\}$, where

$$T_1(l, t_{l-1}) = -\tfrac{2}{3}(2^{2l-2}-1) + (t_{l-1}-4)2^{2l-4+t_{l-1}}, \qquad T_2(l) = \tfrac{99}{169}(2^{2l}-1)$$

and $T_3(l, t_{l-1}) = \frac{1}{3}(2^{2l-2}-1) + 2^{2l-4+t_{l-1}}$.

It is easily verified that

$$T_1(l, 1) + T_2(l) = (\tfrac{99}{169} - \tfrac{13}{24})2^{2l} + \tfrac{41}{507} > 0,$$

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and (4.16) follows. Also

$$T_1(l,3)+T_2(l)+T_3(l,3)h_2(m-l)=-\tfrac{41}{507}(2^{2l}-1)+\tfrac{1}{3}(7\,.\,2^{2l-2}-1)h_2(m-l)$$
$$=\tfrac{7}{3}\,.\,2^{2l-2}\{h_2(m-l)-\tfrac{164}{1183}\}+\tfrac{1}{3}\{\tfrac{41}{169}-h_2(m-l)\}.$$

If $m-l\geqslant 2$, $h_2(m-l)\geqslant h_2(2)>\frac{164}{1183}$ and $\frac{41}{169}>\frac{2}{13}>h_2(m-l)$, so that (4.16) follows in this case. If $m-l=1$ and $t_{l-1}=t_{m-2}=3$, the conditions of (4.12) and (4.13) can only be satisfied if $t_{m-1}+t_m=5$, whence $(t_{m-1},t_m)=(1,4)$. Thus

$$t_1=\ldots=t_{m-3}=2,\qquad t_{m-2}=3,\quad t_{m-1}=1\quad\text{and}\quad t_m=4,$$

and it may be verified that, in this case,

$$-\frac{S(2,n_m)}{n_m}=\frac{2(2^{2m}-1)}{211\,.\,2^{2m-4}-1}<h_2(m).$$

We have, too

$$T_1(l,4)+T_2(l)=\tfrac{1}{507}(425\,.\,2^{2l-1}+41)>0$$

and (4.16) follows again.

Now it remains to consider the case $t_{l-1}=2$. In this case we have $l=m-1$, since $l\leqslant m-2$ (l chosen maximal) implies $t_{l-1}\neq 2$. For $l=m-1$ only the following two cases are possible, because of (4.12) and (4.13):

$$(\alpha)\ \ t_1=t_2=\ldots=t_{m-2}=2,\qquad t_m=5\quad\text{and}\quad t_{m-1}=1$$

and

$$(\beta)\ \ t_1=t_2=\ldots=t_{m-2}=2,\qquad t_m=4\quad\text{and}\quad t_{m-1}=2.$$

For case (α) it may be verified that

$$-\frac{S(2,n_m)}{n_m}=\frac{13\,.\,2^{2m-3}-2}{101\,.\,2^{2m-3}-1}<h_2(m),$$

and, for case (β), we have

$$-\frac{S(2,n_m)}{n_m}=h_2(m),$$

giving the critical form for n_m.

5. Proof of Theorem 2*. For the case $q=3$ we have to prove that, if $n_m=a_0 3^{t_0}+a_1 3^{t_0+t_1}+\ldots+a_m 3^{t_0+t_1+\ldots+t_m}$ where $t_0=0$, $t_1,\ldots,t_m\in\mathbb{N}$ and $a_0,a_1,\ldots,a_m\in\{1,2\}$ then, for $m\geqslant 0$,

$$\frac{S(3,n)}{n_m}\geqslant -h_3(m)\quad\text{where}\quad h_3(m)=\frac{6(3^m-1)}{7\,.\,3^{m+1}-5}.\tag{5.1}$$

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With the usual notation,

$$n_0 = a_0 \quad \text{and} \quad n_1 = a_0 + a_1 3^{t_1} + a_2 3^{t_1+t_2} + \dots + a_i 3^{t_1+t_2+\dots+t_i} \quad (1 \leq i \leq m),$$

and (1.10) takes the form

$$S(3, n_m) = \sum_{r=0}^{m} a_r(a_r - 1)3^{t_0+t_1+\dots+t_r} + 2\sum_{r=1}^{m} (a_r - t_r)n_{r-1}.$$

If $m = 0$, $S(3, n_0) = a_0(a_0 - 1) \geq 0 = -h_3(0)$ and (5.1) holds with equality when $n_0 = a_0 = 1$.

We prove the case $m = 1$ separately, before using an inductive proof for the general case. However it is useful first to obtain two preliminary results.

Lemma 5.1. *If* $m \geq 0$,

$$\frac{S(3, n_m)}{n_m} > -1.$$

Proof. If $m = 0$, $\dfrac{S(3, n_0)}{n_0} = a_0 - 1 \geq 0$. Thus choose $m \geq 1$ and assume that

$$\frac{S(3, n_{m'})}{n_{m'}} > -1 \text{ for all integers } m' \quad \text{with} \quad 0 \leq m' \leq m - 1.$$

By (2.1)

$$\begin{aligned} S(3, n_m) &= S(3, n_{m-1}) + a_m(a_m - 1)3^{t_1+\dots+t_m} + 2(a_m - t_m)n_{m-1}, \\ &> -n_{m-1} + 2(1 - t_m)n_{m-1}, \end{aligned}$$

using the inductive hypothesis and $a_m \geq 1$. It follows that

$$\frac{S(3, n_m)}{n_m} > -1$$

provided that

$$\frac{n_{m-1}}{n_m} < \frac{1}{2t_m - 1}.$$

By (2.4),

$$\frac{n_{m-1}}{n_m} \leq \frac{1}{1 + 3^{t_m - 1}},$$

and it is easily verified that

$$\frac{1}{1 + 3^{t_m - 1}} < \frac{1}{2t_m - 1} \text{ for all integers } t_m \geq 1.$$

Lemma 5.2. *If* $m \geq 0$ *and* $a_m = 2$, *then* $S(3, n_m) > 0$.

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Proof. If $m=0$, $S(3,n_0)=a_0(a_0-1)=2$. Thus assume that $m\geqslant 1$. By (2.1), if $a_m=2$ we have

$$S(3,n_m)=S(3,n_{m-1})+2\,.\,3^{t_1+\dots+t_m}+2(2-t_m)n_{m-1},$$
$$>(3-2t_m)n_{m-1}+2\,.\,3^{t_1+\dots+t_m},$$

using Lemma 5.1.

Thus if $t_m=1$, $S(3,n_m)>0$. If $t_m\geqslant 2$,

$$S(3,n_m)=2\,.\,3^{t_1+\dots+t_m}-(2t_m-3)n_{m-1}.$$

But $n_{m-1}<3^{t_1+\dots+t_{m-1}+1}$, and it is easily verified that

$$(2t_m-3)3^{t_1+\dots+t_{m-1}+1}<2\,.\,3^{t_1+\dots+t_m}\text{ for }t_m\geqslant 2,$$

giving $S(3,n_m)>0$.

Proof of $\dfrac{S(3,n_1)}{n_1}\geqslant -\frac{6}{29}$. We have $n_1=a_0+a_13^{t_1}$ and $S(3,n_1)=a_0(a_0-1)+a_1(a_1-1)3^{t_1}+2(a_1-t_1)a_0$. By Lemma 5.2, we can assume that $a_1=1$.

If $a_0=1$, we have

$$\frac{S(3,n_1)}{n_1}=\frac{2(1-t_1)}{1+3^{t_1}}.$$

For $t_1=1,2$ and 3, $\dfrac{S(3,n_1)}{n_1}$ takes the values 0, $-\frac{1}{5}$ and $-\frac{1}{7}$ respectively, and thereafter continues to increase towards 0 as $t_1\to\infty$.

If $a_0=2$, we have

$$\frac{S(3,n_1)}{n_1}=\frac{2(3-2t_1)}{2+3^{t_1}}.$$

For $t_1=1$, 2 and 3, $\dfrac{S(3,n_1)}{n_1}$ takes the values $\frac{2}{5}$, $-\frac{2}{11}$ and $-\frac{6}{29}$ respectively, and then continues to increase towards 0 as $t_1\to\infty$. Hence

$$\frac{S(3,n_1)}{n_1}\geqslant -\tfrac{6}{29}\text{ with equality only when }n_1=2+3^3.$$

Proof of $\dfrac{S(3,n_m)}{n_m}\geqslant -h_3(m)$, $(m\geqslant 2)$. Assume that $\dfrac{S(3,n_{m'})}{n_{m'}}\geqslant -h_3(m')$ for all integers m' satisfying $1\leqslant m'\leqslant m-1$. By Lemma 5.2, we can take $a_m=1$ and then, by (2.1), we have

$$S(3,n_m)=S(3,n_{m-1})+2(1-t_m)n_{m-1}.$$

If $t_m=1$,

$$\frac{S(3,n_m)}{n_m}=\frac{S(3,n_{m-1})}{n_{m-1}}\cdot\frac{n_{m-1}}{n_m}.$$

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and the induction hypothesis, together with (2.4), yields

$$\frac{S(3, n_m)}{n_m} \geqslant -\tfrac{1}{2}h_3(m-1) > -h_3(m).$$

If $t_m \geqslant 2$, we have on applying the induction hypothesis

$$\frac{S(3, n_m)}{n_m} \geqslant -\{h_3(m-1) + 2(t_m - 1)\}\frac{n_{m-1}}{n_m},$$
$$\geqslant -\frac{\{h_3(m-1) + 2(t_m-1)\}}{1 + 3^{t_m - 1}}.$$

As $h_3(m-1) < \frac{2}{7}$, we have $\frac{S(3, n_m)}{n_m} > -f(t_m)$ where $f(t) = \frac{2(t-1) + \frac{2}{7}}{1 + 3^{t-1}}$. Now $f(2) = \frac{4}{7}$, $f(3) = \frac{3}{7}$, $f(4) = \frac{11}{49} < h_3(2)$ and $f(t)$ continues to decrease as t increases, so that (5.1) follows if $t_m \geqslant 4$. Thus we only need consider the cases when $t_m = 2$ or 3.

By (2.2), we have

$$S(3, n_m) = a_0(a_0 - 1) + 2a_0 \sum_{r=1}^{m} (a_r - t_r) + 3^{t_1} S(3, a_1 + a_2 3^{t_2} + a_3 3^{t_2 + t_3} + \ldots + a_m 3^{t_2 + t_3 + \ldots + t_m}),$$

and applying the induction hypothesis once again we see that

$$S(3, n_m) \geqslant a_0(a_0 - 1) + 2a_0 \sum_{r=1}^{m} (a_r - t_r) - (n_m - a_0)h_3(m-1).$$

Thus $S(3, n_m) \geqslant -h_3(m)$ provided that

$$a_0(a_0 - 1) + 2a_0 \sum_{r=1}^{m} (a_r - t_r) + \{h_3(m) - h_3(m-1)\}n_m + a_0 h_3(m-1) \geqslant 0. \tag{5.2}$$

Since $0 \leqslant h_3(m-1) < h_3(m)$ for $m \geqslant 1$, this inequality is easily satisfied when $\sum_{r=1}^{m} (a_r - t_r) \geqslant 0$. Thus suppose henceforth that

$$a_m = 1, \qquad t_m = 2 \text{ or } 3 \quad \text{and} \quad \sum_{r=1}^{m} (a_r - t_r) = -1 - k \quad \text{where} \quad k \geqslant 0. \tag{5.3}$$

Then (5.2) takes the form

$$\{h_3(m) - h_3(m-1)\}n_m + a_0 h_3(m-1) \geqslant a_0(3 - a_0 + 2k). \tag{5.4}$$

The case $m = 2$. We have $h_3(m) - h_3(m-1) = \frac{6}{23} - \frac{6}{29} = \frac{36}{667}$, and $n_m = n_2 \geqslant 1 + 3^{t_1} + 3^{t_1 + t_2}$ where, from (5.3),

$$t_1 = \begin{cases} a_1 + k & \text{if} \quad t_2 = 2, \\ a_1 + k - 1 & \text{if} \quad t_2 = 3. \end{cases} \tag{5.5}$$

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If $t_2=2$, (5.4) will follow provided that

$$\tfrac{36}{667}(1+3^{a_1+k}+3^{a_1+k+2})+\tfrac{6}{29}a_0 \geq a_0(3-a_0+2k). \tag{5.6}$$

For $a_0=1$, (5.6) holds except when $k=0$ and $a_1=1$. In this case, $n_2=1+3+3^3$ and $-\dfrac{S(n_2)}{n_2}=\dfrac{8}{31}<\dfrac{6}{23}$. For $a_0=2$, (5.6) holds except when $k=1$ and $a_1=1$. Then $n_2=2+3^2+3^4$ and $-\dfrac{S(3,n_2)}{n_2}=\dfrac{6}{23}$, giving rise to the critical case.

If $t_2=3$, (5.4) will follow provided that

$$\tfrac{36}{667}(1+3^{a_1+k-1}+3^{a_1+k+2})+\tfrac{6}{29}a_0 \geq a_0(3-a_0+2k). \tag{5.7}$$

For $a_0=1$, (5.7) holds except when $k=0$ and $a_1=1$. But, from (5.5), this implies that $t_1=0$ so that this possibility is excluded. For $a_0=2$, (5.7) holds except when $k=0$ or 1 and $a_1=1$. From (5.5), $k=0$ and $a_1=1$ imply once again that $t_1=0$. If $k=1$ and $a_1=1$ we have $n_2=2+3+3^4$ and $-\dfrac{S(n_2)}{n_2}=\dfrac{9}{43}<\dfrac{6}{23}$.

The case $m \geq 3$. We have

$$h_3(m)-h_3(m-1)=\frac{64\,.\,3^m}{(7\,.\,3^{m+1}-5)(7\,.\,3^m-5)} \geq \frac{64}{49\,.\,3^{m+1}},$$

and

$$\begin{aligned} n_m &\geq 1+3^{t_1}+3^{t_1+t_2}+\ldots+3^{t_1+t_2+\ldots+t_{m-2}}+(1+3^{t_m})\,.\,3^{t_1+\ldots+t_{m-1}}, \\ &\geq \tfrac{1}{2}\{3^{m-1}-1+2(1+3^{t_m})3^{t_1+\ldots+t_{m-1}}\}. \end{aligned}$$

Thus

$$\{h_3(m)-h_3(m-1)\}n_m \geq \tfrac{32}{441}\left\{2(1+3^{t_m})3^{t_1+\ldots+t_{m-1}-m+1}+1-\frac{1}{3^{m-1}}\right\},$$

and (5.4) will hold if we can prove that

$$\tfrac{32}{441}\{2(1+3^{t_m})3^{t_1+\ldots+t_{m-1}-m+1}+1-\tfrac{1}{9}\} \geq a_0\left(3-a_0+2k-\tfrac{6}{23}\right). \tag{5.8}$$

From (5.3) we have the condition

$$t_1+\ldots+t_{m-1}=\begin{cases} a_1+\ldots+a_{m-1}+k & \text{if } t_m=2, \\ a_1+\ldots+a_{m-1}+k-1 & \text{if } t_m=3. \end{cases} \tag{5.9}$$

Suppose first that

$$a_1+\ldots+a_{m-1}=m-1 \quad \text{or equivalently} \quad a_1=\ldots=a_{m-1}=1. \tag{5.10}$$

Then (5.8) is equivalent to

$$T_1(a_0,k) \leq \begin{cases} T_2(k) & \text{if } t_m=2, \\ T_3(k) & \text{if } t_m=3, \end{cases} \tag{5.11}$$

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where

$$T_1(a_0, k) = a_0(3 - a_0 + 2k - \tfrac{6}{23}), \qquad T_2(k) = \tfrac{32}{441}(20 \,.\, 3^k + \tfrac{8}{9})$$

and

$$T_3(k) = \tfrac{32}{441}(56 \,.\, 3^{k-1} + \tfrac{8}{9}).$$

The small values of k give rise to the following values of T_1, T_2 and T_3:

$T_1(1, 0) = 1{\cdot}73\ldots$	$T_2(0) = 1{\cdot}51\ldots$
$T_1(2, 0) = 1{\cdot}47\ldots$	$T_3(0) = 1{\cdot}41\ldots$
$T_1(1, 1) = 3{\cdot}73\ldots$	$T_2(1) = 4{\cdot}41\ldots$
$T_1(2, 1) = 5{\cdot}47\ldots$	$T_3(1) = 4{\cdot}12\ldots$
$T_1(1, 2) = 5{\cdot}73\ldots$	$T_2(2) = 13{\cdot}12\ldots$
$T_1(2, 2) = 9{\cdot}47\ldots$	$T_3(2) = 12{\cdot}25\ldots$
$T_1(1, 3) = 7{\cdot}73\ldots$	$T_2(3) = 39{\cdot}24\ldots$
$T_1(2, 3) = 13{\cdot}47\ldots$	$T_3(3) = 36{\cdot}63\ldots$

As k increases, the values of $T_2(k)$ and $T_3(k)$ increase exponentially while those of $T_1(a_0, k)$ increase only linearly, and it is not difficult to prove that $T_1(a_0, k) < T_i(k)$ if $i = 2$ or 3 for all $k \geqslant 4$, and (5.11) holds. From inspection of the above table, we see that (5.11) is true except in the following cases:

(i) $$k = 0 : (a_0, t_m) = (1, 2), (1, 3) \text{ or } (2, 3)$$

and

(ii) $$k = 1 : (a_0, t_m) = (2, 2) \text{ or } (2, 3).$$

However, from (5.9) and (5.10), it is not possible to have $t_m = 3$ when $k = 0$ since this would imply that $t_1 + \ldots + t_{m-1} = m - 2$. Thus case (i) reduces to

(i)′ $$k = 0 : (a_0, t_m) = (1, 2).$$

From (5.9) and (5.10), this implies that $(a_0, a_1, \ldots, a_m) = (1, 1, \ldots, 1)$ and $(t_1, \ldots, t_{m-1}, t_m) = (1, \ldots, 1, 2)$. Hence

$$n_m = 1 + 3 + 3^2 + \ldots + 3^{m-1} + 3^{m+1} = \tfrac{1}{2}(3^m - 1) + 3^{m+1} = \tfrac{1}{2}(7 \,.\, 3^m - 1)$$

and $S(3, n_m) = 2(1 - 2) \,.\, \tfrac{1}{2}(3^m - 1) = -(3^m - 1)$, giving

$$-\frac{S(3, n_m)}{n_m} = \frac{2(3^m - 1)}{7 \,.\, 3^m - 1} < \frac{6(3^m - 1)}{7 \,.\, 3^{m+1} - 5}.$$

Now consider case (ii). If $k = 1$ and $t_m = 3$, we see from (5.9) and (5.10) that $t_1 + \ldots + t_{m-1} = m - 1$, giving $t_1 = \ldots = t_{m-1} = 1$. Thus, if $(a_0, t_m) = (2, 3)$, we have

$$(a_0, a_1, \ldots, a_m) = (2, 1, \ldots, 1) \quad \text{and} \quad (t_1, \ldots, t_{m-1}, t_m) = (1, \ldots, 1, 3).$$

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This gives

$$n_m = 2 + 3 + 3^2 + \ldots + 3^{m-1} + 3^{m+2} = \tfrac{1}{2}(3^m + 1) + 3^{m+2} = \tfrac{1}{2}(19 \,.\, 3^m + 1)$$

and

$$S(3, n_m) = 2 + 2(1-3) \,.\, \tfrac{1}{2}(3^m + 1) = -2 \,.\, 3^m.$$

Hence

$$-\frac{S(3, n_m)}{n_m} = \frac{4 \,.\, 3^m}{19 \,.\, 3^m + 1} < \frac{6(3^m - 1)}{7 \,.\, 3^{m+1} - 5} \Leftrightarrow (15 \,.\, 3^m + 1)(3^m - 3) > 0$$

which is true for $m \geqslant 1$.

If $k = 1$ and $(a_0, t_m) = (2, 2)$ we have, from (5.9) and (5.10), $(a_0, a_1, \ldots, a_m) = (2, 1, \ldots, 1)$ and $t_1 + \ldots + t_{m-1} = m$. If $t_1 = 2$ then $t_2 = \ldots = t_{m-1} = 1$, and we have

$$n_m = 2 + 3^2 + 3^3 + \ldots + 3^m + 3^{m+2} = \tfrac{1}{2}(3^{m+1} - 5) + 3^{m+2} = \tfrac{1}{2}(7 \,.\, 3^{m+1} - 5)$$

and

$$S(3, n_m) = 2 + 2(1-2) \,.\, 2 + 2(1-2) \,.\, \tfrac{1}{2}(3^{m+1} - 5) = -3(3^m - 1).$$

Thus

$$\frac{S(3, n_m)}{n_m} = -\frac{6(3^m - 1)}{7 \,.\, 3^{m+1} - 5},$$

and this is the critical case. Alternatively if $t_1 = 1$, there is some r with $2 \leqslant r \leqslant m - 1$ such that

$$t_1 = \ldots = t_{r-1} = 1, \; t_r = 2, \qquad t_{r+1} = \ldots = t_{m-1} = 1.$$

[If $r = m - 1$, this condition should read $t_1 = \ldots = t_{r-1} = 1$, $t_r = 2$.] In this case

$$n_m = 2 + 3 + \ldots + 3^{r-1} + 3^{r+1} + \ldots + 3^m + 3^{m+2},$$

so that

$$n_{r-1} = \tfrac{1}{2}(3^r + 1), \qquad n_{m-1} = \tfrac{1}{2}(3^{m+1} - 2 \,.\, 3^r + 1)$$

and

$$n_m = \tfrac{1}{2}(7 \,.\, 3^{m+1} - 2 \,.\, 3^r + 1).$$

Also it may be verified that $S(3, n_m) = -(3^{m+1} - 3^r)$, giving

$$-\frac{S(3, n_m)}{n_m} = \frac{6(3^m - 3^{r-1})}{7 \,.\, 3^{m+1} - 2 \,.\, 3^r + 1} \leqslant \frac{6(3^m - 1)}{7 \,.\, 3^{m+1} - 5}$$
$$\Leftrightarrow (5 \,.\, 3^{m+1} + 1)(3^{r-1} - 1) \geqslant 0$$

which is true.

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It remains to observe that when

$$a_1 + \ldots + a_{m-1} = m - 1 + u \quad \text{where} \quad u \geqslant 1,$$

the values of $T_2(k)$ and $T_3(k)$ in (5.11) are replaced by $T_2(k+u)$ and $T_3(k+u)$ while those of $T_1(a_0, k)$ remain unaltered. Inspection of the tabulated values shows that the inequalities (5.11) are always satisfied. Hence the theorem is proved.

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