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EQUIVARIANT POLYNOMIAL AUTOMORPHISMS OF Θ-REPRESENTATIONS

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ABSTRACT. We show that every equivariant polynomial automorphism of a Θ-representation and of the reduction of an irreducible Θ-representation is a multiple of the identity.

1. **Introduction.** Given a representation V of an algebraic group G over $\mathbb{C}$ we ask the question: What is $\mathrm{Aut}_G(V)$, the group of polynomial automorphisms that commute with the linear G-action. For many reducible representations nonlinear equivariant automorphisms exist: Consider for example the SL_2-module $R_2 \oplus R_4$ where R_j denotes the binary forms of degree j. The map $(p, q) \longmapsto (p, q + p^2)$ is an SL_2-equivariant automorphism. For more information on SL_2-automorphisms of R_j see [13].

In order to determine $\mathrm{Aut}_G(V)$ for a simple G-module it suffices to assume G is semisimple. First replace G by the reductive group $G/\mathcal{R}(G)$ since the radical $\mathcal{R}(G)$ acts trivially on a simple module, and note that if there exists a one-dimensional subgroup of the center acting nontrivially, every automorphism commuting with this action therefore induces an automorphism on a projective space which is linear [6, II. Example 7.1.1].

In this work we investigate $\mathrm{Aut}_G(V)$ for the so-called *Θ-representations* $G \longrightarrow \mathrm{GL}(V)$ which are defined as follows: Given a $\mathbb{Z}_m$-graduation on a simple Lie algebra $\mathfrak{g} = \oplus_{j\in\mathbb{Z}_m}\mathfrak{g}_j$ (with $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$) the induced $\mathfrak{g}_0$-operation on $\mathfrak{g}_1$ defines a G-module structure on $\mathfrak{g}_1$ (called Θ-representation) where G is a connected reductive group with Lie algebra $\mathfrak{g}_0$ (see 3 for details). These representations which were classified by Kac ([8], [7]) have some properties of the adjoint representations. We call the representation of the commutator subgroup (G, G) on $\mathfrak{g}_1$ the *reduction of the Θ-representation*. The main result of this work is:

THEOREM (3.3).
(a) The automorphism group of a Θ-representation $G \longrightarrow \mathrm{GL}(V)$ *of a semisimple group* G *is* $\mathbb{C}^* \,\mathrm{id}_V$.
(b) The automorphism group of the reduction of an irreducible Θ-representation is also $\mathbb{C}^* \,\mathrm{id}_V$.

The question arises whether there is a simple module with nonlinear automorphisms. In [14] it is shown that the natural $\mathrm{SL}_3 \times \mathrm{SL}_5 \times \mathrm{SL}_{13}$-representation has an automorphism group of dimension 2. This is the lowest dimensional module with an open orbit and nonlinear equivariant automorphisms.

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Theorem 3.3 is proved case by case to some extent. We distinguish between several types of Θ-representations such as adjoint representations, or more generally the ones with finite $\mathrm{Nor}_G(H)/H$ (where H denotes a generic isotropy group). We separately look at the prehomogeneous Θ-representations, and finally the ones without any of the properties above. The biggest class of Θ-representations ($\bar{N} := \mathrm{Nor}_G(H)/H$ finite) can be handled by a general statement (Lemma 3.1). All the remaining ones are checked case by case to have no nonlinear equivariant automorphisms (Sections 5 and 6). However, the embedding of a generic stabilizer H of the Θ-representation V and its fixed point space V^H is of great importance. It is given for many examples of Θ-representations. In fact, if $\mathrm{Aut}_{\bar{N}}(V^H)$ only consists of linear automorphisms, then so does $\mathrm{Aut}_G(V)$ (see proof of 2.3). For few of the Θ-representations (6.1, 6.2) the method of restitution of multilinear invariants is used [10, Section 6].

The automorphism group of a G-module is related to a rationality question of the linearization problem: For a (finite) Galois field extension $k \subset K$ in characteristic 0 the non-abelian cohomology $\mathrm{H}^1\big(\mathrm{Gal}(K/k), \mathrm{Aut}_{G_K}(V_K)\big)$ is the set of isomorphism classes of G_k-actions on the space V_k (defined over k) becoming G_K-isomorphic to the G_K-module V_K by field extension [14, Appendix], [22, III. 1]. If $\mathrm{Aut}_{G_K}(V_K) = K^* \,\mathrm{id}_{V_K}$, then $\mathrm{H}^1\big(\mathrm{Gal}(K/k), \mathrm{Aut}_{G_K}(V_K)\big) = 0$ which shows that every G_k-action on the affine space $\mathbb{A}^n_k$ which is G_K-isomorphic to V_K is also linearizable over the subfield k.

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2. **Remarks on G-modules with closed generic orbit.** Let G be a reductive group and V a finite dimensional G-module. By a theorem of Matsushima the stabilizer G_v, $v \in V$ where $Gv \subset V$ is a closed orbit, is a reductive group [17], [16, I.2.].

For a closed subgroup $H \subset G$ the subgroup $\mathrm{Nor}_G(H) := \{g \in G \mid gHg^{-1} = H\}$ is called the normalizer of H and define $\bar{N} := \mathrm{Nor}_G(H)/H$. It induces a linear $\bar{N}$-action on the fixed point space $V^H = \{v \in V \mid hv = v \;\forall h \in H\}$.

The set of conjugacy classes (G_v) where $Gv \subset V$ is a closed orbit, is partially ordered, that is $(G_1) \leq (G_2)$ if G_1 is conjugate to a subgroup of G_2. There is a unique minimal isotropy class (H) of the above set, called the principal isotropy class [16]. Let $H \subset G$ now be a principal isotropy group, *i.e.*, (H) is minimal. If G is semisimple and $\bar{N}$ finite, then it follows from a theorem of Kraft-Petrie-Randall [11, Corollary 5.5] that $V^H/\bar{N} \cong \mathbb{C}^r$ for some $r \in \mathbb{N}$. By Chevalley's Theorem $\bar{N}$ therefore acts on V^H as a finite reflection group (*cf.* for example [23, Theorem p. 76]).

DEFINITION. A set of hyperplanes $\{H_i \subset \mathbb{C}^n\}_{i \in I}$ is said to be in general position if $\bigcap_{i \in I} H_i = \{0\}$.

LEMMA 2.1. *Let $\varphi\colon \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a polynomial automorphism. If φ stabilizes every element of a set of hyperplanes $H_i := \mathcal{Z}(l_i)$, $i \in I$ in general position, then φ is diagonalizable; in particular φ is linear.*

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PROOF. Consider the induced (linear) automorphism on the regular functions of $\mathbb{C}^n$ denoted by $\varphi^*\colon \mathbb{C}[\mathbb{C}^n] \longrightarrow \mathbb{C}[\mathbb{C}^n]$. We have that $\varphi^*(l_i)(v) = l_i\big(\varphi(v)\big) = 0$ for any $v \in H_i$, consequently $\varphi^*(l_i) \in \mathbb{C}l_i$. Since the hyperplanes are in general position there is a basis $l_1, \ldots, l_n$ of $(\mathbb{C}^n)^*$ (after renumbering). This means φ is diagonal with respect to the dual basis of $l_1, \ldots, l_n$. ∎

REMARK 2.2. If V is a simple G-module, then by 2.1 every $\sigma \in \mathrm{Aut}_G(V)$ which stabilizes a hyperplane is a homothety. A general $\sigma \in \mathrm{Aut}_G(V)$ preserves every line $\mathbb{C}(gv)$ where v is a highest weight vector and $g \in G$, since $\mathbb{C}v$ is the fixed point space V^U of a maximal unipotent subgroup $U \subset G$. In fact, $u\sigma(\mathbb{C}v) = \sigma(u\mathbb{C}v) = \sigma(\mathbb{C}v)$ for all $u \in U$, so $\sigma|_{\mathbb{C}v} = \lambda\,\mathrm{id}_{\mathbb{C}v}$ for some $\lambda \in \mathbb{C}^*$, and by equivariance $\sigma|_{\mathbb{C}gv} = \lambda\,\mathrm{id}_{\mathbb{C}gv}$. For every $x \in V^*$ this implies that $\sigma^*(x)(gv) = x\big(\sigma(gv)\big) = x(\lambda gv)$. However, $\sigma^*(x)$ may not be a multiple of x, for we cannot show $\sigma^*(x)(w) = x(\lambda w)$ for all $w \in V$. It would need the fact $\sigma^*(x)(g_1v + g_2v) = x\big(\sigma(g_1v + g_2v)\big) = x\big(\sigma(g_1v) + \sigma(g_2v)\big)$, but σ is not linear.

THEOREM 2.3. *Let G be a semisimple group, V a simple G-module and $H \subset G$ a principal isotropy group. If the generic orbit is closed and $\bar{N} = \mathrm{Nor}_G(H)/H$ is finite then* $\mathrm{Aut}_G(V) = \mathbb{C}^*\,\mathrm{id}_V$.

PROOF. Let $H_1, \ldots, H_t \subset V^H$ be the hyperplanes associated to the generating reflections $s_1, \ldots, s_t$ of $\bar{N}$. Suppose $V_1 := \bigcap_{i=1}^t H_i \neq \{0\}$. $V_1 \subset V^H$ is $\bar{N}$-stable, and let V_2 be an $\bar{N}$-stable complement in V^H. Take an $x \in (V^H)^*$, $x \neq 0$ which vanishes on V_2. It is easy to see that ${}^s x(v_j) = x(v_j)$ for all $v_j \in V_j$ and $s \in \bar{N}$, $j = 1, 2$. Hence $x \in \mathbb{C}[V^H]^{\bar{N}}$ which is isomorphic to $\mathbb{C}[V]^G$ by a theorem of Luna-Richardson. This means there is a nontrivial G-fixed point in V^* which is impossible since V^* is simple. It follows that the hyperplanes $H_1, \ldots, H_t$ are in general position. So by the Lemma 2.1 above $\sigma|_{V^H}$ is linear.

We obtain the relation $\sigma \circ \lambda\,\mathrm{id}_V - \lambda\,\mathrm{id}_V \circ \sigma = 0$ on GV^H, even on V since H is a generic stabilizer, *i.e.*, $\overline{GV^H} = V$. So σ induces an automorphism on the projective space $\mathbb{P}V$ which has to be linear [6, II. Example 7.1.1]. Schur's Lemma finishes the proof. ∎

The essential point in the proof is the general position of the hyperplanes H_j. A G-module V without nontrivial G-fixed points also guarantees this property. So we state the following corollary:

COROLLARY 2.4. *Let G be a semisimple group and V a G-module. Let the generic orbit be closed and $\bar{N}$ finite (thus a finite reflection group). If the hyperplanes, associated to the generators of $\bar{N}$ are in general position, then* $\mathrm{Aut}_G(V)$ *only consists of linear automorphisms. In particular, if $V^G = \{0\}$, then all automorphisms in* $\mathrm{Aut}_G(V)$ *are linear.*

These statements show that the adjoint representation of a semisimple group G only admits linear automorphisms. In fact, the generic isotropy group is a maximal torus and the generic orbit is closed. The Weyl group $\bar{N} := \mathrm{Nor}_G(T)/T$ acts on $(\mathrm{Lie}\,G)^T = \mathrm{Lie}\,T$ by reflections. The hyperplanes of the associated generators of $\bar{N}$ have trivial intersection. The adjoint representation of G is simple if and only if $\mathrm{Lie}\,T$ is a simple $\bar{N}$-module and this is equivalent to G being a simple group. So by Corollary 2.4 one obtains (*cf.* [1, 2.2 Proposition]):

THEOREM 2.5. *Let G be a semisimple group. Every G-equivariant automorphism of the adjoint representation is linear. In particular, such an automorphism is a multiple of the identity in case G is simple.*

3**. Introduction to Θ-representations.** For many aspects adjoint representations are the 'nicest' representations. A class of nice representations which contains the adjoint representations, is the set of Θ-representations. They fulfill two important properties which also hold for the adjoint representations: coregularity (the algebra of invariant functions has algebraically independent homogeneous generators) and visibility (any fiber of the corresponding quotient map has the same dimension) [15].

Let $(\mathfrak{g}, \Theta)$ (or $(\mathfrak{g}, m)$) denote the $\mathbb{Z}_m$-graded Lie algebra

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}_m} \mathfrak{g}_j$$

where $m \in \{1, 2, 3, \ldots\} \cup \{\infty\}$ and $\mathbb{Z}_\infty := \mathbb{Z}$. Let Θ denote the corresponding linear automorphism

$$\Theta(x) = \varepsilon^j x, \quad x \in \mathfrak{g}_j, \text{ where } \varepsilon = e^{2\pi i/m}, \text{ if } m \neq \infty$$

and

$$\Theta_t(x) = t^j x, \quad x \in \mathfrak{g}_j, \text{ where } t \in \mathbb{C}^*, \text{ if } m = \infty.$$

There is a one-to-one correspondence between the isomorphism classes of $\mathbb{Z}_m$-gradings on $\mathfrak{g}$ and the classes of conjugate automorphisms of period m of $\mathfrak{g}$ if $m \neq \infty$, respectively the one-dimensional tori in the automorphism group of $\mathfrak{g}$ if $m = \infty$.

Let $(\mathfrak{g}, \Theta)$ now be a simple $\mathbb{Z}_m$-graded Lie algebra. The adjoint representation of $\mathfrak{g}$ induces by restriction a $\mathfrak{g}_0$-module $\mathfrak{g}_1$; the adjoint group G_0 of the Lie algebra $\mathfrak{g}_0$ is a connected algebraic group, called Θ-*group* (*cf.* [24] and [8]).

Set $G := G_0$, $V := \mathfrak{g}_1$ and let θ be the restriction of the adjoint representation Ad to G, *i.e.*,

$$\theta := \mathrm{Ad}\,|_G \colon G \longrightarrow \mathrm{GL}(V).$$

θ is called the Θ-*representation* of $(\mathfrak{g}, \Theta)$.

The semisimple elements in $\mathfrak{g}$ are precisely the elements of closed orbits of the adjoint representation. This is still true for the Θ-representation θ of a reductive graded Lie algebra $(\mathfrak{g}, \Theta)$: an element $x \in \mathfrak{g}_1 \subset \mathfrak{g}$ is semisimple if and only if Gx is a closed orbit [24, Section 2.4. Proposition 3]. An abelian maximal subspace $\mathfrak{c} \subset V$ consisting of semisimple elements is called a Cartan subspace. Every closed orbit in V intersects any fixed Cartan subspace [24, Corollary p. 473].

The notion of the Weyl group of an adjoint representation can be carried over to the Θ-representations: Let $\mathrm{Nor}_G(\mathfrak{c}) := \{g \in G \mid \theta(g)\mathfrak{c} = \mathfrak{c}\}$ and $Z_G(\mathfrak{c}) := \{g \in G \mid \theta(g)x = x \ \forall x \in \mathfrak{c}\}$, then $W := \mathrm{Nor}_G(\mathfrak{c})/Z_G(\mathfrak{c})$ is a finite reflection group ([24, Section 3.4.

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Prop. 3, Section 6.1. Thm. 8]) called the Weyl group of the graded Lie algebra $(\mathfrak{g}, \Theta)$. The (geometric) quotient $\mathfrak{c}/W$ of the induced W-module $\mathfrak{c}$ is isomorphic to $V/\!/G$ [24, Section 4.4. Theorem 7], thus we obtain an isomorphism on the invariant polynomial functions $\mathbb{C}[V]^G \cong \mathbb{C}[\mathfrak{c}]^W$ which is induced by the restriction map. This implies $\dim \mathfrak{c} = \dim V/\!/G$.

We determine $\mathrm{Aut}_G(V)$ for all irreducible Θ-representations (G, V) of simple graded Lie algebras $(\mathfrak{g}, m)$. The latter were classified by Kac (*cf.* [8], [24], [7]). So from now on let $\mathfrak{g}$ be simple. If $m = \infty$ then $\mathbb{C}[V]^G = \mathbb{C}$ (and $\mathfrak{c} = 0$) since $\theta(G)$ contains $\mathbb{C}^* \operatorname{id}_V$ induced by the automorphisms Θ_t, $t \in \mathbb{C}^*$. In fact, all derivations of $\mathfrak{g}$ are inner, so $t \longmapsto \Theta_t$ corresponds to a one-dimensional torus in the adjoint group G_0. So in case $m = \infty$ every G-automorphism induces an automorphism on the projective space $\mathbb{P}V$ since it commutes with $\mathbb{C}^* \operatorname{id}_V \subset \theta(G)$, *i.e.*, $\mathrm{Aut}_G(V)$ only contains linear elements [6, II. Example 7.1.1]. We therefore consider V as a (G, G)-module called the *reduction of the* Θ*-representation*. Note that Popov and Vinberg call it the reduced Θ-representation (*cf.* [19, 8.5]). In Table 4.4 where all (irreducible) Θ-representations will be listed, the reduction of the Θ-representation is taken for the Θ-type $(\mathfrak{g}, \infty)$.

Interestingly, if the Θ-group G is semisimple, V is automatically a simple G-module ([24, Section 8.3. Proposition 18]). Among several methods to find $\mathrm{Aut}_G(V)$ Theorem 2.3 is the most important one. So we start looking more closely at Θ-representations with generically closed orbits.

LEMMA 3.1. *Let $(\mathfrak{g}, \Theta)$ be a simple $\mathbb{Z}_m$-graded Lie algebra where the associated Θ-representation (G, V) has generically closed orbits. Let G_Θ be a connected algebraic group with* $\mathrm{Lie}(G_\Theta) = \mathfrak{g}$ *and $\mathfrak{c} \subset V$ denote a Cartan subspace, then:*

(a) $H := Z_G(\mathfrak{c}) = Z_{G_\Theta}(\mathfrak{c}) \cap G$ is a generic isotropy group.

(b) $\mathfrak{c} \subseteq V^H$; moreover, $\mathfrak{c} = V^H$ (or equivalently $\dim V/\!/G = \dim V^H$) if and only if $\overline{N} := \mathrm{Nor}_G(H)/H$ is a finite group.

(c) If G is semisimple and $\mathfrak{c} = V^H$, then $\mathrm{Aut}_G(V) = \mathbb{C}^* \operatorname{id}_V$.

PROOF. (a) Since the generic orbit is closed, it consists of semisimple elements and intersects $\mathfrak{c}$. Let $x \in \mathfrak{c}$ be a generic element, then $Z_{G_\Theta}(x) \cap G$ is a generic isotropy group. Using [24, Section 3.2] we see that $H = Z_{G_\Theta}(\mathfrak{c}) \cap G = Z_{G_\Theta}(x) \cap G$ (recall that $Z_{G_\Theta}(\mathfrak{c})$ is connected).

(b) Clearly $\mathfrak{c} \subseteq V^H$. If $\mathfrak{c} = V^H$, then it is easy to see that $\mathrm{Nor}_G(H) = \mathrm{Nor}_G(V^H) := \{g \in G \mid (\mathrm{Ad}\, g)v \in V^H \ \forall v \in V^H\}$. So $\overline{N} = \mathrm{Nor}_G(H)/H = W$ is finite. For the converse set $N := \mathrm{Nor}_G(\mathfrak{c})$. Since $G\mathfrak{c} \subset V$ is dense $\dim V = \dim(G \times^N \mathfrak{c}) = \dim G + \dim \mathfrak{c} - \dim \mathrm{Nor}_G(\mathfrak{c})$, and analogously $\dim V = \dim G + \dim V^H - \dim \mathrm{Nor}_G(H)$. Therefore $\dim \mathfrak{c} = \dim V^H$ since both, $W = N/H$ and $\mathrm{Nor}_G(H)/H$ are finite. Recall that $G \times^N \mathfrak{c}$ is the (geometric) quotient of $G \times \mathfrak{c}$ by the group N; it is acting by $n(g, x) = (gn^{-1}, nx)$ where $n \in N$ and $(g, x) \in G \times \mathfrak{c}$.

(c) now follows from (b) and Theorem 2.3. ∎

REMARK 3.2. Popov and Vinberg state in [19, 8.5] that $V^{Z_G(\mathfrak{c})} = \mathfrak{c}$ for $m < \infty$. This is a mistake. In fact, consider for example the Θ-representation $(E_6^{(1)}, 2)$ (N° 29 in Table 4.4).

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In 6.3 we show that $\dim \mathfrak{c} = \dim V/\!\!/G = 2$ and $\dim V^H = 16$ where H denotes a generic stabilizer.

The main result of this work is:

THEOREM 3.3.

(a) The automorphism group of a Θ-representation $G \longrightarrow \mathrm{GL}(V)$ of a semisimple group G is $\mathbb{C}^* \,\mathrm{id}_V$.

(b) The automorphism group of the reduction of an irreducible Θ-representation is also $\mathbb{C}^* \,\mathrm{id}_V$.

Recall that every Θ-representation is irreducible in case G is semisimple [24, Section 8.3. Proposition 18]. If G is reductive (and not semisimple), then the automorphism group of a Θ-representation is $\mathbb{C}^* \,\mathrm{id}_V$, because the center of G acts as scalar transformations on V. In this case $\mathbb{C}[V]^G = \mathbb{C}$ and the Θ-representation is of type $(\mathfrak{g}, \infty)$ (*cf.* [8, Proposition 3.1.I.] and [24, Section 8.3.]).

REMARK 3.4. Unfortunately, Theorem 3.3 is not valid for reductions of reducible Θ-representations. The $G := \mathrm{SL}_m \times \mathrm{SP}_{2n} \times T_1$-module $V := (\mathbb{C}^m)^* \oplus (\mathbb{C}^m \otimes \mathbb{C}^{2n})$ defined by

$$(g, s, t).(x, v \otimes w) := \left(t^{2mn} \cdot (g^t)^{-1} x, t^{-m}(gv \otimes sw)\right)$$

is the reduction of the reducible Θ-representation (C_{m+n+1}, ∞). Its automorphism group $\mathrm{Aut}_G(V)$ is 3-dimensional if $2(\frac{2n-1}{m}+1) \in \mathbb{Z}$ whereas the group of linear G-automorphisms is 2-dimensional. The proof is different from the methods for proving 3.3. Moreover, it is quite lengthy, it uses the Littlewood-Richardson Theorem. I refer to my Ph.D. thesis [12, 7.8].

For convenience we give the complete list of the irreducible Θ-representations, resp. of the reductions of them. All data not computed in this work, is taken from [8, Table II, III], corrections in [3]. For a complete table with the degrees of the homogeneous generating invariants see [15]. In case $m = \infty$ the group G in Table 4.4 always denotes the corresponding reduction of the Θ-group described as above. Without confusion they will also be called Θ-groups. Thus G is always a semisimple group.

The following notations are used in Table 4.4: For G acting on a vector space V we denote by $S^i G$ ($\wedge^i G$, respectively) the G-module of the i-th symmetric (exterior, respectively) power of V. The highest irreducible component of $S^i G$ is denoted by $S^i_0 G$ and analogously for $\wedge^i_0 G$. The column labeled by $\mathfrak{h}$ contains the Lie algebra type of a generic stabilizer unless $\mathfrak{h} = 0$, where the finite isotropy group is given after dividing with the kernel of the representation. $\mathfrak{A}_k$ denotes the group of even permutations of k elements. The explicit decomposition of the finite generic stabilizers as semidirect products is omitted. A, B, C, D, E, F_4, G_2 denote the simple Lie algebras indexed by their rank. $\mathfrak{t}_k$ is the Lie algebra of a k-dimensional torus and $\mathfrak{u}_j$ is a j-dimensional nilpotent Lie algebra.

The rubric 'method' describes how $\mathrm{Aut}_G(V) = \mathbb{C}^* \,\mathrm{id}_V$ is verified: The expression 'prehom.' means that the corresponding module is prehomogeneous, *i.e.*, it has a dense orbit. They are handled in Proposition 5.1. The 'adjoint' representations have been settled in 2.5. 'Finite $\bar{N}$' says that $\mathrm{Nor}_G(H)/H$ is finite, so we can make use of 3.1, respectively of Theorem 2.3 in case of a reduced Θ-representation with one-dimensional quotient. In some cases the tables of Élashvili [4], [5] are used to check $\dim V/\!/G = \dim V^H$ (which is equivalent to the finiteness of $\bar{N} = \mathrm{Nor}_G(H)/H$), but mostly we refer to later computations. The abbreviation 'restitution' stands for the restitution of multilinear covariants [10, Section 6] which is explicitly verified for $\mathrm{SO}_n \otimes \mathrm{SP}_{2m}$ in 6.1.

REMARK 3.5. If the generic isotropy group H is reductive, then G/H is affine, and therefore the generic orbit is closed [9, II.4.3. Satz 6]. All Θ-representations with $\dim V/\!/G > 0$ have generically closed orbits except N° 4b in Table 4.4.

4. **Equivariant automorphisms of Θ-representations with finite $\bar{N}$.** In this section we give details of Θ-representations $G \longrightarrow \mathrm{GL}(V)$ with $\dim V^H = \dim V/\!/G$ or equivalently with finite $\bar{N}$ in order to apply Lemma 3.1. This shows that every G-automorphism is a homothety. The finiteness of $\bar{N}$ for some Θ-representations was shown by Élashvili [4], [5] as pointed out in Table 4.4. So, for the examples not referred to the literature we briefly indicate the representation space V, the embedding of a generic stabilizer $H \subset G$ as well as the fixed point space V^H. The corresponding Θ-group is always denoted by G and its Lie algebra by $\mathfrak{g}$. For the verification of a stabilizer $H = G_v$, $v \in V$ to be generic, we sometimes use the equivalent condition that $\{u \in V^H \mid G_u = H\}$ is dense in V^H and $V = (\mathrm{Lie}\, G).v + V^H$ (see [19, Theorem 7.3]). The equality $\dim G + \dim V^H - \dim \mathrm{Nor}_G(H) = \dim V$ (*i.e.*, $\overline{GV^H} = V$) also implies that $\{g \in G \mid gv = v\ \forall v \in V^H\}$ is a generic isotropy group.

4.1. $\mathrm{SL}_n \otimes \mathrm{SL}_n$. The representation space is the set of $n \times n$-matrices M_n, and $H = G_{E_n} = \{(A, A) \in G \mid A \in \mathrm{SL}_n\}$. So $\mathrm{M}_n^H = \mathbb{C}E_n$.

4.2. $\mathrm{SL}_n \otimes \mathrm{SO}_m$, $3 \leq n = m$ *and* $1 \leq n < m$. Let V denote the space of $n \times m$-matrices $\mathrm{M}_{n\times m}$. Let $M_0 := (E_n \,|\, 0) \in V$, then $H = G_{M_0} = \left\{\left(A, \begin{pmatrix} A & \\ & B\end{pmatrix}\right) \mid A \in \mathrm{SO}_n, B \in \mathrm{SO}_{m-n}\right\}$ and $V^H = \mathbb{C}M_0$.

4.3. $S_0^2\,\mathrm{SO}_n$, $n > 4$. This representation is the SO_n-conjugation on $V = \mathrm{Sym}_n / \mathbb{C}E_n$ where Sym_n denotes the symmetric $n \times n$-matrices. Let $A := \mathrm{diag}(1, 2, \ldots, n)$ then $H = G_A = \{S = \mathrm{diag}(\pm 1, \ldots, \pm 1) \mid \det S = 1\} \cong (\mathbb{Z}_2)^{n-1}$. One obtains $\dim V^H = n - 1 = \dim V/\!/G$.

4.4. $\mathrm{SO}_n \otimes \mathrm{SO}_m$, $n \geq m > 2$. The composition $V = \mathrm{M}_{n\times m} \xrightarrow{\pi_{\mathrm{SO}_n}} S^2\mathbb{C}^m \xrightarrow{\pi_{\mathrm{SO}_m}} \mathbb{C}^m$ is the $G = \mathrm{SO}_n \times \mathrm{SO}_m$-quotient where π_L denotes the quotient by the group L. The matrix $A_0 := \begin{pmatrix} \frac{A}{0} \end{pmatrix} \in V$ is an element of the generic orbit where A is defined as in 4.3. Then $H = G_{A_0} = \left\{\left(\begin{pmatrix} S & \\ & T\end{pmatrix}, S\right) \mid S = \mathrm{diag}(\pm 1, \ldots, \pm 1) \in \mathrm{SO}_m, T \in \mathrm{SO}_{n-m}\right\}$ and $V^H = \left\{\begin{pmatrix} \frac{D}{0} \end{pmatrix} \mid D \in \mathrm{M}_m \text{ is diagonal}\right\}$.

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N°	G	Θ-type	$\mathfrak{h}$	$\dim V/\!/G$	method
	$\mathrm{SL}_n \otimes \mathrm{SL}_m$	(A_{n+m-1},∞)			
1a	$n > m \geq 1$		$\mathfrak{sl}_{n-m} + \mathfrak{sl}_m + \mathfrak{u}_{m(n-m)}$	0	prehom. 5.1
1b	$n = m \geq 1$		$\mathfrak{sl}_n$	1	finite $\bar{N}$, 4.1
	$\mathrm{SL}_n \otimes \mathrm{SO}_m$	(B_{n+m},∞) [1] (D_{n+m},∞) [1]			
2a	$n > m \geq 3$		$\mathfrak{sl}_{n-m} + \mathfrak{so}_m + \mathfrak{u}_{m(n-m)}$	0	prehom. 5.1
2b	$n = m \geq 3$		$\mathfrak{so}_m$	1	finite $\bar{N}$, 4.2
2c	$1 \leq n < m$		$\mathfrak{so}_n + \mathfrak{so}_{m-n}$	1	finite $\bar{N}$, 4.2
	$\mathrm{SL}_n \otimes \mathrm{SP}_{2m}$	(C_{n+m},∞)			
3a	$n > 2m \geq 4$		$\mathfrak{sl}_{n-2m} + \mathfrak{sp}_{2m} + \mathfrak{u}_{2m(n-2m)}$	0	prehom. 5.1
3b	$1 \leq n < 2m$, n odd		$\mathfrak{sp}_{n-1} + \mathfrak{sp}_{2m-n-1} + \mathfrak{u}_{2m-1}$	0	prehom. 5.1
3c	$2 \leq n \leq 2m$, n even		$\mathfrak{sp}_n + \mathfrak{sp}_{2m-n}$	1	restitution, 6.2
	$\mathrm{SO}_n \otimes \mathrm{SP}_{2m}$	$(A_k^{(2)},4)$			
4a	$n > 2m \geq 4$		$\mathfrak{t}_m + \mathfrak{so}_{n-2m}$	m	restitution, 6.1
4b	$2 < n < 2m$, n odd	k odd	$\mathfrak{t}_{\frac{n-1}{2}} + \mathfrak{sp}_{2m-n-1} + \mathfrak{u}_{2m-n}$	$\frac{n-1}{2}$	restitution, 6.1
4c	$2 < n \leq 2m$, n even	k even	$\mathfrak{t}_{\frac{n}{2}} + \mathfrak{sp}_{2m-n}$	$\frac{n}{2}$	restitution, 6.1
5	$\mathrm{SO}_n \otimes \mathrm{SO}_m$	$(B_k^{(1)},2)$ [2] $(D_k^{(1,2)},2)$ [2]			
	$n \geq m > 2$		$\mathfrak{so}_{n-m}$	m	finite $\bar{N}$, 4.4
6	$\mathrm{SP}_{2n} \otimes \mathrm{SP}_{2m}$	$(C_n^{(1)},2)$			
	$n \geq m > 1$		$m\mathfrak{sl}_2 + \mathfrak{sp}_{2n-2m}$	m	finite $\bar{N}$, 4.5
7	$\mathrm{Ad}\,\mathrm{SL}_n$, $n > 2$	$(A_n^{(1)},1)$	$\mathfrak{t}_{n-1}$	$n-1$	adjoint
	$\wedge^2 \mathrm{SL}_n$	(D_n,∞)			
8a	n odd ≥ 3		$\mathfrak{sp}_{n-1} + \mathfrak{u}_{n-1}$	0	prehom. 5.1
8b	n even ≥ 4		$\mathfrak{sp}_n$	1	finite $\bar{N}$, [4]
9	$S^2\,\mathrm{SL}_n$, $n \geq 3$	(C_n,∞)	$\mathfrak{so}_n$	1	finite $\bar{N}$, [4]
	$\wedge^2 \mathrm{SO}_n$				
10a	$n > 3$ odd	$(B_n^{(1)},1)$	$\mathfrak{t}_{\frac{n-1}{2}}$	$\frac{n-1}{2}$	adjoint
10b	$n > 5$ even	$(D_n^{(1)},1)$	$\mathfrak{t}_{\frac{n}{2}}$	$\frac{n}{2}$	adjoint
11	$S^2\,\mathrm{SP}_{2n}$, $n > 1$	$(C_n^{(1)},1)$	$\mathfrak{t}_n$	n	adjoint
	$S_0^2\,\mathrm{SO}_n$	$(A_n^{(2)},4)$			
12a	$n > 4$ odd		$(\mathbb{Z}_2)^{n-1}$	$n-1$	finite $\bar{N}$, 4.3
12b	$n > 4$ even		$(\mathbb{Z}_2)^{n-2}$	$n-1$	finite $\bar{N}$, 4.3
13	$\wedge_0^2\,\mathrm{SP}_{2n}$, $n > 2$	$(A_{2n+1}^{(2)},2)$	nA_1	$n-1$	finite $\bar{N}$, [4]
14	$S^3\,\mathrm{SL}_2$	(G_2,∞)	$\mathbb{Z}_3$	1	6.1
15	$S^4\,\mathrm{SL}_2$	$(A_2^{(2)},4)$	$(\mathbb{Z}_2)^2$	2	finite $\bar{N}$, 4.6
16	$S^3\,\mathrm{SL}_3$	$(D_4^{(3)},3)$	$(\mathbb{Z}_3)^2$	2	finite $\bar{N}$, 4.7
17	$\wedge^3\,\mathrm{SL}_6$	(E_6,∞)	$A_2 + A_2$	1	6.7
18	$\wedge^3\,\mathrm{SL}_7$	(E_7,∞)	G_2	1	finite $\bar{N}$, [4]
19	$\wedge^3\,\mathrm{SL}_8$	(E_8,∞)	A_2	1	finite $\bar{N}$, [4]
20	$\wedge^3\,\mathrm{SL}_9$	$(E_8^{(1)},3)$	$(\mathbb{Z}_3)^4$	4	finite $\bar{N}$, 4.8

TABLE I

[1] In either case if m is odd, B_{n+m} is the Θ-type, and D_{n+m} else.

[2] Depending on the parity of n and m the Θ-type is chosen; so if n and m are odd it is $(D^{(2)}_{\frac{n+m}{2}}, 2)$.

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N°	G	Θ-type	$\mathfrak{h}$	$\dim V/\!/G$	method
21	$\wedge^4 \mathrm{SL}_8$	$(E_7^{(1)}, 2)$	$(\mathbb{Z}_2)^6$	7	finite $\bar{N}$, 4.9
22	$\mathrm{SL}_2 \otimes S^3 \mathrm{SL}_2$	$(G_2^{(1)}, 2)$	$(\mathbb{Z}_2)^2$	2	finite $\bar{N}$, 4.11
23	$\mathrm{SL}_2 \otimes S^2 \mathrm{SL}_3$	(F_4, ∞)	$\mathfrak{N}_4$	1	6.2
24	$\mathrm{SL}_2 \otimes S^2 \mathrm{SL}_4$	$(E_6^{(2)}, 4)$	$(\mathbb{Z}_4)^2$	2	finite $\bar{N}$, 4.12
25	$\mathrm{SL}_2 \otimes \wedge^2 \mathrm{SL}_5$	(E_6, ∞)	$A_1 + \mathfrak{u}_4$	0	prehom. [5]
26	$\mathrm{SL}_2 \otimes \wedge^2 \mathrm{SL}_6$	(E_7, ∞)	$3A_1$	1	6.4
27	$\mathrm{SL}_2 \otimes \wedge^2 \mathrm{SL}_7$	(E_8, ∞)	$A_1 + \mathfrak{u}_6$	0	prehom. [5]
28	$\mathrm{SL}_2 \otimes \wedge^2 \mathrm{SL}_8$	$(E_8^{(1)}, 4)$	$4A_1$	2	6.4
29	$\mathrm{SL}_2 \otimes \wedge^3 \mathrm{SL}_6$	$(E_6^{(1)}, 2)$	$\mathfrak{t}_2$	4	6.3
30	$\mathrm{SL}_2 \otimes \wedge_0^3 \mathrm{SP}_6$	$(F_4^{(1)}, 2)$	$(\mathbb{Z}_2)^3$	4	6.3
31	$\mathrm{SL}_2 \otimes \mathrm{Spin}_7$	$(E_6^{(2)}, 4)$	$A_2 + \mathfrak{t}_1$	1	6.7
32	$\mathrm{SL}_2 \otimes \mathrm{Spin}_{10}$	(E_7, ∞)	$G_2 + A_1$	1	6.7
33	$\mathrm{SL}_2 \otimes \mathrm{Spin}_{12}$	$(E_7^{(1)}, 2)$	$3A_1$	4	6.7
34	$\mathrm{SL}_2 \otimes E_6$	(E_8, ∞)	D_4	1	6.7
35	$\mathrm{SL}_2 \otimes E_7$	$(E_8^{(1)}, 2)$	D_4	4	6.7
36	$\mathrm{SL}_2 \otimes \mathrm{SL}_3 \otimes \mathrm{SL}_3$	(E_6, ∞)	$\mathfrak{t}_2$	1	6.3
37	$\mathrm{SL}_2 \otimes \mathrm{SL}_3 \otimes \mathrm{SL}_4$	(E_7, ∞)	A_1	1	finite $\bar{N}$, 4.14
38	$\mathrm{SL}_2 \otimes \mathrm{SL}_3 \otimes \mathrm{SL}_5$	(E_8, ∞)	$A_1 + \mathfrak{u}_2$	0	prehom. [14, 3.]
39	$\mathrm{SL}_2 \otimes \mathrm{SL}_3 \otimes \mathrm{SL}_6$	$(E_8^{(1)}, 6)$	$A_2 + A_1$	1	finite $\bar{N}$, 4.15
40	$\mathrm{SL}_2 \otimes \mathrm{SL}_4 \otimes \mathrm{SL}_4$	$(E_7^{(1)}, 4)$	$\mathfrak{t}_3$	2	6.3
41	$\mathrm{SL}_3 \otimes S^2 \mathrm{SL}_3$	$(F_4^{(1)}, 3)$	$(\mathbb{Z}_3)^2$	2	finite $\bar{N}$, 4.13
42	$\mathrm{SL}_3 \otimes \wedge^2 \mathrm{SL}_5$	(E_7, ∞)	A_1	1	finite $\bar{N}$, [5]
43	$\mathrm{SL}_3 \otimes \wedge^2 \mathrm{SL}_6$	$(E_7^{(1)}, 3)$	$\mathfrak{t}_1$	3	6.5
44	$\mathrm{SL}_3 \otimes \mathrm{Spin}_{10}$	(E_8, ∞)	$A_1 + A_1$	1	finite $\bar{N}$, [5]
45	$\mathrm{SL}_3 \otimes E_6$	$(E_8^{(1)}, 3)$	A_2	3	6.7
46	$\mathrm{SL}_3 \otimes \mathrm{SL}_3 \otimes \mathrm{SL}_3$	$(E_6^{(1)}, 3)$	$(\mathbb{Z}_3)^2$	3	6.4
47	$\mathrm{SL}_4 \otimes \wedge^2 \mathrm{SL}_5$	(E_8, ∞)	$\mathfrak{N}_5$	1	finite $\bar{N}$, 4.16
48	$\mathrm{SL}_4 \otimes \mathrm{Spin}_{10}$	$(E_8^{(1)}, 4)$	$(\mathbb{Z}_2)^4$	4	6.6
49	$\mathrm{SL}_5 \otimes \wedge^2 \mathrm{SL}_5$	$(E_8^{(1)}, 5)$	$(\mathbb{Z}_5)^2$	2	finite $\bar{N}$, 4.17
50	Spin_7	(F_4, ∞)	G_2	1	finite $\bar{N}$, [4]
51	Spin_9	$(F_4^{(1)}, 2)$	B_3	1	finite $\bar{N}$, [4]
52	Spin_{10}	(E_6, ∞)	$B_3 + \mathfrak{u}_8$	0	prehom. [4]
53	Spin_{12}	(E_7, ∞)	A_5	1	6.7
54	Spin_{14}	(E_8, ∞)	$G_2 + G_2$	1	finite $\bar{N}$, [4]
55	Spin_{16}	$(E_8^{(1)}, 2)$	$(\mathbb{Z}_2)^8$	8	finite $\bar{N}$, 4.10
56	$\wedge_0^3 \mathrm{SP}_6$	(F_4, ∞)	A_2	1	6.7
57	$\wedge_0^4 \mathrm{SP}_8$	$(E_6^{(2)}, 2)$	$(\mathbb{Z}_2)^6$	6	finite $\bar{N}$, 4.9

TABLE I (continued)

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N°	G	Θ-type	$\mathfrak{h}$	$\dim V/\!/G$	method
58	$\mathrm{Ad}\, G_2$	$(G_2^{(1)}, 1)$	$\mathfrak{t}_2$	2	adjoint
59	G_2	$(D_4^{(3)}, 3)$	A_2	1	finite $\bar{N}$, [4]
60	$\mathrm{Ad}\, F_4$	$(F_4^{(1)}, 1)$	$\mathfrak{t}_4$	4	adjoint
61	F_4	$(E_6^{(2)}, 2)$	D_4	2	finite $\bar{N}$, [4]
62	$\mathrm{Ad}\, E_6$	$(E_6^{(1)}, 1)$	$\mathfrak{t}_6$	6	adjoint
63	E_6	(E_7, ∞)	F_4	1	finite $\bar{N}$, [4]
64	$\mathrm{Ad}\, E_7$	$(E_7^{(1)}, 1)$	$\mathfrak{t}_7$	7	adjoint
65	E_7	(E_8, ∞)	E_6	1	6.7
66	$\mathrm{Ad}\, E_8$	$(E_8^{(1)}, 1)$	$\mathfrak{t}_8$	8	adjoint

TABLE I (continued)

4.5. $\mathrm{SP}_{2n} \otimes \mathrm{SP}_{2m}$, $n \geq m > 1$. The representation space V is $\mathrm{M}_{2n\times 2m}$. For $\mu \in \mathbb{C}$ define $D_\mu = \begin{pmatrix} & -\mu \\ \mu & \end{pmatrix}$ and let $J := \mathrm{diag}(D_1, \ldots, D_1)$ be a skew symmetric form of even rank $2k$. Then the symplectic group and Lie algebra are defined by

$$\mathrm{SP}_{2k} := \{S \in \mathrm{GL}_{2k} \mid SJS^t = J\} \text{ and } \mathfrak{sp}_{2k} := \{s \in \mathrm{M}_{2k} \mid sJ + Js^t = 0\}.$$

The stabilizer $\mathfrak{h} := \mathfrak{g}_{A_0}$ of $A_0 := \begin{pmatrix} A \\ 0 \end{pmatrix} \in V$ where $A := \mathrm{diag}(D_1, \ldots, D_m)$ is a generic stabilizer:

$$\mathfrak{h} = \left\{ \left(\left(\begin{array}{c|c} \mathrm{diag}(s_1, \ldots, s_m) & 0 \\ \hline 0 & s' \end{array} \right), \mathrm{diag}(-s_1^t, \ldots, -s_m^t) \right) \;\middle|\; s_i \in \mathfrak{sl}_2, s' \in \mathfrak{sp}_{2n-2m} \right\}$$
$$\cong m\mathfrak{sl}_2 + \mathfrak{sp}_{2n-2m}$$

Then $V^{\mathfrak{h}} = \left\{ \begin{pmatrix} \mathrm{diag}(D_{\lambda_1}, \ldots, D_{\lambda_m}) \\ 0 \end{pmatrix} \;\middle|\; \lambda_1, \ldots, \lambda_m \in \mathbb{C} \right\}$ and so $\dim V^{\mathfrak{h}} = \dim V/\!/G$.

4.6. $S^4\,\mathrm{SL}_2$. The representation space is $R_4 := \mathbb{C}[x, y]_4$. The binary dihedral group $H = G_{x^4+y^4} = \left\langle \begin{pmatrix} i & \\ & -i \end{pmatrix}, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \right\rangle$ is a generic isotropy group and $\dim R_4^H = \dim R_4/\!/G$.

4.7. $S^3\,\mathrm{SL}_3$. Take the the ternary cubics $V := \mathbb{C}[x_1, x_2, x_3]_3$ with the induced natural $G = \mathrm{SL}_3$-representation. Then

$$H = G_{x_1^3+x_2^3+x_3^3} = \left\{ \begin{pmatrix} \zeta_1 & & \\ & \zeta_2 & \\ & & \zeta_3 \end{pmatrix}, \begin{pmatrix} & \zeta_1 & \\ & & \zeta_2 \\ \zeta_3 & & \end{pmatrix}, \begin{pmatrix} & & \zeta_1 \\ \zeta_2 & & \\ & \zeta_3 & \end{pmatrix} \;\middle|\; \begin{array}{c} \zeta_1\zeta_2\zeta_3 = 1 \\ \zeta_i^3 = 1,\; i = 1, 2, 3 \end{array} \right\}$$

is a generic isotropy group. It follows that $V^H = \mathbb{C}(x_1^3 + x_2^3 + x_3^3) \oplus \mathbb{C}x_1x_2x_3$ and therefore $\dim V^H = \dim V/\!/G$.

4.8. $\wedge^3\,\mathrm{SL}_9$. Let $e_1, \ldots, e_9$ be a basis of $\mathbb{C}^9$ and (ijk) denote the skew symmetric tensor $e_i \wedge e_j \wedge e_k \in V := \wedge^3\mathbb{C}^9$. Let us define

$$p_1 := (123) + (456) + (789), \qquad p_2 := (147) + (258) + (369),$$
$$p_3 := (159) + (267) + (348), \qquad p_4 := (168) + (249) + (357).$$

The element $p := \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \lambda_4 p_4$ with $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{C}$ pairwise distinct, is an element of a generic orbit [25]. The stabilizer $H = G_p$ consists of the matrices $\begin{pmatrix} A_1 & & \\ & A_2 & \\ & & A_3 \end{pmatrix}$, $\begin{pmatrix} & & A_3 \\ A_1 & & \\ & A_2 & \end{pmatrix}$, $\begin{pmatrix} & A_2 & \\ & & A_3 \\ A_1 & & \end{pmatrix} \in G$ where the $A_j \in \mathrm{SL}_3$ allow the following shapes:

$$\text{either } A_j = \begin{pmatrix} \xi_{j1} & & \\ & \xi_{j2} & \\ & & \xi_{j3} \end{pmatrix}, \begin{pmatrix} & & \xi_{j3} \\ \xi_{j1} & & \\ & \xi_{j2} & \end{pmatrix}, \text{ or } \begin{pmatrix} & \xi_{j2} & \\ & & \xi_{j3} \\ \xi_{j1} & & \end{pmatrix} \quad \text{for all } j = 1, 2, 3$$

$(\xi_{11}, \xi_{12}, \xi_{13})$	$(\xi_{21}, \xi_{22}, \xi_{23})$	$(\xi_{31}, \xi_{32}, \xi_{33})$
$(1, \zeta, \zeta^2)$	$(1, \zeta, \zeta^2)$	$(1, \zeta, \zeta^2)$
$(\zeta, \zeta^2, 1)$	$(\zeta, \zeta^2, 1)$	$(\zeta, \zeta^2, 1)$
$(\zeta, 1, \zeta^2)$	$(\zeta^2, \zeta, 1)$	$(1, \zeta^2, \zeta)$

The table on the right hand side lists three generators for the group isomorphic to $(\mathbb{Z}_3)^3$ of the entries of A_1, A_2, A_3 where $\zeta = e^{2\pi i/3}$ is a third root of unity. In fact, the entries of A_1 are described by $(\mathbb{Z}_3)^2$ and for any choice for A_1 there are 3 possibilities for A_2 and A_3 is uniquely determined by A_1, A_2. After dividing by the kernel ($\cong \mathbb{Z}_3$) we see that $H \cong (\mathbb{Z}_3)^4$. So one obtains that $V^H = \mathbb{C}p_1 \oplus \mathbb{C}p_2 \oplus \mathbb{C}p_3 \oplus \mathbb{C}p_4$, and $\dim V^H = \dim V/\!/G$.

4.9. $\wedge^4 \mathrm{SL}_8$ *and* $\wedge_0^4 \mathrm{SP}_8$. This is analogous to the computations in 4.8. Let $(ijkl)$ denote the skew symmetric tensor $e_i \wedge e_j \wedge e_k \wedge e_l$ where $e_1, \ldots, e_8$ is a basis of $\mathbb{C}^8$. We define

$$p_1 := (1234) + (5678), \qquad p_2 := (1278) + (3456), \qquad p_3 := (1368) + (2457),$$
$$p_4 := (1467) + (2358),$$
$$p_5 := (1256) + (3478), \qquad p_6 := (1357) + (2468), \qquad p_7 := (1458) + (2367).$$

The generic isotropy group is equal to $H := G_p$ where $p := \sum_{r=1}^{7} r p_r$. It consists of the elements $\begin{pmatrix} A_1 & \\ & A_2 \end{pmatrix}$, $\begin{pmatrix} & A_1 \\ A_2 & \end{pmatrix} \in G$ where $A_1, A_2 \in \mathrm{SL}_4$ have one of the four forms:

$$A_j = \begin{pmatrix} \alpha_{j1} & & & \\ & \alpha_{j2} & & \\ & & \alpha_{j3} & \\ & & & \alpha_{j4} \end{pmatrix}, \begin{pmatrix} & \alpha_{j2} & & \\ \alpha_{j1} & & & \\ & & & \alpha_{j4} \\ & & \alpha_{j3} & \end{pmatrix}, \begin{pmatrix} & & \alpha_{j3} & \\ & & & \alpha_{j4} \\ \alpha_{j1} & & & \\ & \alpha_{j2} & & \end{pmatrix}, \begin{pmatrix} & & & \alpha_{j4} \\ & & \alpha_{j3} & \\ & \alpha_{j2} & & \\ \alpha_{j1} & & & \end{pmatrix}$$

$(\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14})$	$(\alpha_{21}, \alpha_{22}, \alpha_{23}, \alpha_{24})$
$(-1, -1, 1, 1)$	$(-1, -1, 1, 1)$
$(-1, -1, 1, 1)$	$(1, 1, -1, -1)$
$(-1, 1, 1, -1)$	$(1, -1, -1, 1)$
(i, i, i, i)	(i, i, i, i)

The description of the table is similar to 4.8. After dividing with the kernel $H \cong (\mathbb{Z}_2)^6$. Then $V^H = \oplus_{r=1}^{7} \mathbb{C}p_r$ and $\dim V^H = \dim V/\!/G$.

These computations are also useful for $\wedge_0^4 \mathrm{SP}_8$: Consider the $G = \mathrm{SP}_8$-module decomposition $\wedge^4 \mathbb{C}^8 = \wedge_0^4 \mathbb{C}^8 \oplus W \oplus \mathbb{C}_0$ where $W \cong \wedge_0^2 \mathbb{C}^8$ and $\mathbb{C}_0 = \mathbb{C}(p_5 + p_6 + p_7)$ is the trivial

G-module in $\wedge^4\mathbb{C}^8$ (see [2, VI 5.3]). Moreover, it holds $\mathbb{C}p_1 \oplus \mathbb{C}p_2 \oplus \mathbb{C}p_3 \oplus \mathbb{C}p_4 \subset \wedge_0^4\mathbb{C}^8$ and $\mathbb{C}(p_5 - p_6) \oplus \mathbb{C}(p_6 - p_7) \subset \wedge_0^4\mathbb{C}^8$ [2, VI 5.3]. So define $p := \sum_{r=1}^4 rp_r + 5(p_5 - p_6) + 6(p_6 - p_7) \in \wedge_0^4\mathbb{C}^8$ and from above we get that $H := G_p \cong (\mathbb{Z}_2)^6$. These considerations yield that $(\wedge_0^4\mathbb{C}^8)^H = \oplus_{r=1}^4 \mathbb{C}p_r \oplus \mathbb{C}(p_5 - p_6) \oplus \mathbb{C}(p_6 - p_7)$, and therefore $\dim(\wedge_0^4\mathbb{C}^8)^H = \dim \wedge_0^4\mathbb{C}^8 /\!/ G$.

4.10. Spin_{16}. The generic isotropy group $H \cong (\mathbb{Z}_2)^8$ is embedded as follows [21, Table 2]: $H = (\mathbb{Z}_2)^6 \times (\mathbb{Z}_2)^2 \subset \mathrm{SP}_8 / \{\pm \mathrm{id}\} \times \mathrm{SO}_3 \subset G = \mathrm{SO}_{16}$ where $(\mathbb{Z}_2)^6$ is embedded in SP_8 as above in 4.9. The latter inclusion is induced by $(\mathrm{SP}_8 \otimes \mathrm{SL}_2)/\{\pm \mathrm{id}\} \subset G$, which is given by $\left(A, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \mapsto \begin{pmatrix} aA & bA \\ cA & dA \end{pmatrix} \in G$. If SP_8 is given with respect to the skew-symmetric form $J = \begin{pmatrix} & E_4 \\ -E_4 & \end{pmatrix}$, then G is defined by $\left\{ S \in \mathrm{SL}_{16} \mid S^t \begin{pmatrix} & -J \\ J & \end{pmatrix} S = \begin{pmatrix} & -J \\ J & \end{pmatrix} \right\}$. So we obtain that $H = \left\langle \begin{pmatrix} & ig \\ ig & \end{pmatrix}, \begin{pmatrix} ig & \\ & -ig \end{pmatrix} \mid g \in H_{\mathrm{SP}_8} \right\rangle \subset G$, where $H_{\mathrm{SP}_8} \subset \mathrm{SP}_8$ denotes the generic stabilizer of $\wedge_0^4 \mathrm{SP}_8$ (recall that the kernel of the half-spin representation of Spin_{16} is $\mathbb{Z}_2$). Since $\mathrm{Nor}_G(H)^0 = \left(Z_G(H)H\right)^0$ it is enough to show that the centralizer $Z_G(H)$ is finite, which is not difficult to verify by using the finiteness of $Z_{\mathrm{SL}_8}(H_{\mathrm{SP}_8})$ (4.9).

4.11. $\mathrm{SL}_2 \otimes S^3 \mathrm{SL}_2$. Here we argue in a slightly different manner from the previous examples: Let $H \subset G = \mathrm{SL}_2 \times \mathrm{SL}_2$ be the binary dihedral group D_2 which is generated by $\left(\begin{pmatrix} i & \\ & -i \end{pmatrix}, \begin{pmatrix} -i & \\ & i \end{pmatrix} \right), \left(\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \right) \in G$. Notice that the kernel of this representation is $\pm(\mathrm{id}, \mathrm{id})$. The representation space is realized by $V := \mathbb{C}^2 \otimes R_3$, where $R_3 := \mathbb{C}[x, y]_3$. Let e_1, e_2 be the standard basis of $\mathbb{C}^2$. Then $V^H = \mathbb{C}(e_1 \otimes x^3 + e_2 \otimes y^3) \oplus \mathbb{C}(e_1 \otimes xy^2 + e_2 \otimes x^2y)$, and one easily verifies that the normalizer $N := \mathrm{Nor}_G(H)$ is finite. It follows that $GV^H \subset V$ is dense since $\dim G \times^N V^H = \dim G + \dim V^H - \dim N = \dim V$. Hence the generic orbit intersects V^H and the generic stabilizer H' contains H. By Lemma 3.1(b) it exists a Cartan subspace $\mathfrak{c}$ such that $\mathfrak{c} \subset V^{H'} \subset V^H$. But $\dim \mathfrak{c} = 2 = \dim V^H$ which implies that $\mathfrak{c} = V^{H'}$. Furthermore, it is now easy to see that $H' = H$ since $Z_G(\mathfrak{c}) = H$.

4.12. $\mathrm{SL}_2 \otimes S^2 \mathrm{SL}_4$. As usual let e_1, e_2 be the standard basis of $\mathbb{C}^2$ and $V := \mathbb{C}^2 \otimes R_2$ the representation space where $R_2 := \mathbb{C}[u, x, y, z]_2$. The stabilizer $H = G_w$ of an element $w \in W := \mathbb{C}\left(e_1 \otimes (u^2 + x^2) + e_2 \otimes (y^2 + z^2)\right) \oplus \mathbb{C}(e_1 \otimes yz + e_2 \otimes ux)$ in general position is a generic isotropy group. H is generated by the three elements ($\varepsilon = e^{\pi i/4}$)

$$\left(\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} i &&& \\ & i && \\ && i & \\ &&& i \end{pmatrix} \right), \left(\begin{pmatrix} -i & \\ & i \end{pmatrix}, \begin{pmatrix} \varepsilon &&& \\ & \varepsilon^5 && \\ && \varepsilon^3 & \\ &&& \varepsilon^7 \end{pmatrix} \right), \left(\begin{pmatrix} & -i \\ -i & \end{pmatrix}, \begin{pmatrix} &&\varepsilon& \\ &&& \varepsilon \\ & \varepsilon && \\ \varepsilon &&& \end{pmatrix} \right).$$

It is isomorphic (modulo the kernel $\mathbb{Z}_4$) to $(\mathbb{Z}_4)^2$. Hence $V^H = W$ and $\dim V^H = \dim V /\!/ G$.

4.13. $\mathrm{SL}_3 \otimes S^2 \mathrm{SL}_3$. Consider the finite subgroup $H \subset G = \mathrm{SL}_3 \times \mathrm{SL}_3$ generated by the three elements

$$\left(\begin{pmatrix} \zeta && \\ & \zeta & \\ && \zeta \end{pmatrix}, \begin{pmatrix} \zeta^2 && \\ & \zeta^2 & \\ && \zeta^2 \end{pmatrix} \right), \left(\begin{pmatrix} 1 && \\ & \zeta & \\ && \zeta^2 \end{pmatrix}, \begin{pmatrix} 1 && \\ & \zeta & \\ && \zeta^2 \end{pmatrix} \right), \left(\begin{pmatrix} & 1 & \\ && 1 \\ 1 && \end{pmatrix}, \begin{pmatrix} && 1 \\ 1 && \\ & 1 & \end{pmatrix} \right)$$

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where $\zeta = e^{2\pi i/3}$. H is isomorphic to $(\mathbb{Z}_3)^3$ and the kernel of the module is isomorphic to $\mathbb{Z}_3$. The representation space is realized by $V := \mathbb{C}^3 \otimes R_2$ where $R_2 := \mathbb{C}[x, y, z]_2$. Let e_1, e_2, e_3 denote the standard basis of $\mathbb{C}^3$. The space of H-fixed points is

$$V^H = \mathbb{C}(e_1 \otimes x^2 + e_3 \otimes y^2 + e_2 \otimes z^2) \oplus \mathbb{C}(e_2 \otimes xy + e_3 \otimes xz + e_1 \otimes yz).$$

The normalizer $N := \mathrm{Nor}_G(H)$ is easily seen to be finite. Therefore $GV^H \subset V$ is dense because $\dim G \times^N V^H = \dim V$. Now we make use of the same arguments as in 4.11 because $\dim V^H = \dim V/\!/G$, *i.e.*, V^H is a Cartan subspace and H is a generic isotropy group.

4.14. $\mathrm{SL}_2 \otimes \mathrm{SL}_3 \otimes \mathrm{SL}_4$. Consider Lie algebra $\mathfrak{g}$ of $\mathrm{SL}_2 \times \mathrm{SL}_3 \times \mathrm{SL}_4$ acting on $V = \mathrm{M}_{6\times 4} \cong \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^4$ by embedding the $\mathfrak{sl}_2 \times \mathfrak{sl}_3$-action in $\mathfrak{sl}_6$; the embedded Lie algebra is denoted by $\mathfrak{g}_1$. The orbit $\mathfrak{g}m$ with $m = \begin{pmatrix} E_4 \\ \hline & & & 1 \\ 1 & & & \end{pmatrix} \in V$ is generic because $\mathfrak{g}m + V^{\mathfrak{g}_m} = V$. The stabilizer of m is

$$\mathfrak{h} = \left\{ \left(\begin{pmatrix} a & b & & & 2c & \\ c & 3a & & & & 2c \\ & & -3a & b & 2b & \\ & & c & -a & & 2b \\ b & & c & & -a & b \\ & b & & c & c & a \end{pmatrix}, \begin{pmatrix} a & b & & 2c \\ 3c & 3a & & \\ & & -3a & 3b \\ 2b & & c & -a \end{pmatrix} \right) \in \mathfrak{g}_1 \times \mathfrak{sl}_4 \;\middle|\; a, b, c \in \mathbb{C} \right\} \cong \mathfrak{sl}_2.$$

It follows $V^{\mathfrak{h}} = \mathbb{C}m$ and $\dim V^{\mathfrak{h}} = V/\!/G$.

4.15. $\mathrm{SL}_2 \otimes \mathrm{SL}_3 \otimes \mathrm{SL}_6$. This representation is realized by left-action of $G_1 := \mathrm{SL}_2 \times \mathrm{SL}_3 \subset \mathrm{SL}_6$ and right-action of SL_6 on M_6. The stabilizer $H = G_{E_6} = \{(S, T) \in G_1 \times \mathrm{SL}_6 \mid SE_6T^{-1} = E_6\} \cong \mathrm{SL}_2 \times \mathrm{SL}_3$ of the identity matrix $E_6 \in \mathrm{M}_6$ is a generic stabilizer. The H-fixed points are $\mathrm{M}_6^H = \{A \in \mathrm{M}_6 \mid SAS^{-1} = A\} = \mathbb{C}E_6$ and therefore $\dim M_6^H = \dim M_6/\!/G$.

4.16. $\mathrm{SL}_4 \otimes \wedge^2 \mathrm{SL}_5$. Let $e_1, \ldots, e_4$, resp. $f_1, \ldots, f_5$ be the standard basis of $\mathbb{C}^4$, resp. $\mathbb{C}^5$. Then $(i, jk) := e_i \otimes f_j \wedge f_k$ for $1 \leq i \leq 4$, $1 \leq j < k \leq 5$ is a basis of $V := \mathbb{C}^4 \otimes \wedge^2\mathbb{C}^5$. Consider the finite subgroup $H \subset G = \mathrm{SL}_4 \times \mathrm{SL}_5$ generated by the two elements

$$a = \left(\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \right), \quad b = \left(\begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix} \right).$$

The alternating group $\mathfrak{A}_5$ is generated by the permutations $\sigma_1 = (12345)$ and $\sigma_2 = (123)$. The SL_4- (resp. SL_5-) component of a and b are the images of σ_1 and σ_2 of the unique irreducible 4- (resp. 5-) dimensional representation of $\mathfrak{A}_5$. This construction and Schur's Lemma immediately yield that $Z_G(H)$ is contained in the scalar matrices of G, hence

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finite. Since $\mathrm{Nor}_G(H)^0 = (Z_G(H)H)^0$ it follows that $\mathrm{Nor}_G(H)/H$ is finite. The H-fixed point space is $V^H = \mathbb{C}v$ where

$$\begin{aligned} v = {} & (1,12) - (1,15) - (1,24) - (1,25) - (1,45) \\ & + 2(2,12) + 2(2,13) + (2,14) + (2,23) - (2,25) + (2,34) - (2,35) - 2(2,45) \\ & + (3,12) + 2(3,13) + 2(3,14) + (3,23) - 2(3,25) + (3,34) - (3,35) - (3,45) \\ & + (4,12) + (4,13) + 2(4,14) + (4,15) - (4,23) + 2(4,34) + (4,35). \end{aligned}$$

Since $\dim G + \dim V^H - \dim \mathrm{Nor}_G(H) = \dim V$ the finite group H is a generic stabilizer.

4.17. $\mathrm{SL}_5 \otimes \wedge^2 \mathrm{SL}_5$. Take the same notations as in 4.16. Consider the finite subgroup $H \subset G = \mathrm{SL}_5 \times \mathrm{SL}_5$ generated by

$$a = \left(\begin{pmatrix} &&&&1 \\ 1&&&& \\ &1&&& \\ &&1&& \\ &&&1& \end{pmatrix}, \begin{pmatrix} &&&&1 \\ 1&&&& \\ &1&&& \\ &&1&& \\ &&&1& \end{pmatrix} \right), \quad b = \left(\begin{pmatrix} \zeta^4 &&&& \\ &\zeta^2&&& \\ &&1&& \\ &&&\zeta^3& \\ &&&&\zeta \end{pmatrix}, \begin{pmatrix} 1 &&&& \\ &\zeta&&& \\ &&\zeta^2&& \\ &&&\zeta^3& \\ &&&&\zeta^4 \end{pmatrix} \right),$$

$$c = (\zeta^3 E_5, \zeta E_5)$$

where $\zeta = e^{2\pi i/5}$. The H-fixed point space turns out to be

$$\begin{aligned} V^H = {} & \mathbb{C}[(1,12) + (2,23) + (3,34) + (4,45) - (5,15)] \\ & \oplus \mathbb{C}[(1,35) - (2,14) - (3,25) + (4,13) + (5,24)]. \end{aligned}$$

Just like in 4.16 $Z_G(H)$ and therefore $\mathrm{Nor}_G(H)$ are finite. Since $\dim G + \dim V^H - \dim \mathrm{Nor}_G(H) = \dim V$ it is easy to see that $\{g \in G \mid gv = v \ \forall v \in V^H\} = H$ is a generic stabilizer (cf. [18, Lemma 5.1]).

5. **Equivariant automorphisms of prehomogeneous Θ-representations.** For a prehomogeneous module V the embedding of a generic stabilizer H is also the main tool to find the equivariant automorphism group. We determine the dimension of the H-fixed point space V^H. In fact, for every prehomogeneous G-module (G semisimple) it is shown in [14, 2.] that $\dim \mathrm{Aut}_G(V) = \dim V^H = \dim \mathrm{Nor}_G(H)/H$.

PROPOSITION 5.1. *Let V be an irreducible prehomogeneous Θ-representation of a (semisimple) group. Then V^H is one-dimensional. In particular,* $\mathrm{Aut}_G(V) = \mathbb{C}^* \mathrm{id}_V$.

PROOF. For $\mathrm{SL}_n \otimes \mathrm{SL}_m$, $n > m \geq 1$ (N° 1a) consider the representation space V of $n \times m$-matrices. The element $v = \begin{pmatrix} E_m \\ 0 \end{pmatrix}$ is in a generic orbit with stabilizer $H = \left\{ \left(\begin{pmatrix} g & * \\ 0 & s \end{pmatrix}, g \right) \in \mathrm{SL}_n \times \mathrm{SL}_m \mid g \in \mathrm{SL}_m, s \in \mathrm{SL}_{n-m} \right\}$. Clearly, $V^H = \mathbb{C}v$.

The same arguments can also be used for $\mathrm{SL}_n \otimes \mathrm{SO}_m$ (N° 2a), $n > m \geq 3$ as well as for $\mathrm{SL}_n \otimes \mathrm{SP}_{2m}$, $n > 2m \geq 4$ (N° 3a).

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A generic isotropy algebra $\mathfrak{h}$ of $\mathrm{SL}_n \otimes \mathrm{SP}_{2m}$, $2 < n < 2m$, n odd (N° 3a) is given in [20, pp. 101–102]. It is isomorphic to $\mathfrak{sp}_{2m} \oplus \mathfrak{sp}_{2n-m-1} \oplus \mathfrak{u}_{2n-1}$ where $\mathfrak{u}_j$ is a j-dimensional unipotent Lie algebra. It is easy to see that $\dim(\mathbb{C}^{2n} \otimes \mathbb{C}^{2m+1})^{\mathfrak{h}} = 1$.

The module $\wedge^2 \mathrm{SL}_{2m+1}$, $m \geq 1$ (N° 8a) is listed in [4, Table 1]. However, we present this situation explicitly. The skew symmetric matrix M is an element of a generic orbit with stabilizer H:

$$M = \left(\begin{array}{cc|c} 0 & E_m & 0 \\ -E_m & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \qquad H = \left\{ \left(\begin{array}{c|c} A & * \\ \hline 0 & 1 \end{array}\right) \in \mathrm{SL}_{2m+1} \;\middle|\; A \in \mathrm{SP}_{2m} \right\} \cong \mathrm{SP}_{2m} \times U_{2m}$$

We obtain $(\wedge^2 \mathbb{C}^{2m+1})^H = \mathbb{C}M$.

All modules $\mathrm{SL}_2 \otimes \wedge^2 \mathrm{SL}_{2m+1}$, $m \geq 1$ are prehomogeneous and have one-dimensional fixed point space V^H [5, Table 6 N° 1]. These modules handle the cases N° 25 and N° 27 of Table 4.4.

For both modules, $\mathrm{SL}_2 \otimes \mathrm{SL}_3 \otimes \mathrm{SL}_5$ (N° 38) [14, 3.] and Spin_{10} (N° 52) [4, Table 1], the dimension of the fixed point space is one. ∎

REMARK 5.2. For an arbitrary simple prehomogeneous G-module (G semisimple), Proposition 5.1 is not valid. In [14] it is shown that $\mathrm{Aut}_{\mathrm{SL}_3 \times \mathrm{SL}_5 \times \mathrm{SL}_{13}}(\mathbb{C}^3 \otimes \mathbb{C}^5 \otimes \mathbb{C}^{13})$ is two-dimensional.

6**. Other methods.** We briefly introduce the restitution of multilinear invariants which is the main tool to show the triviality of the automorphism group of certain Θ-representations. We keep the notations of the previous sections.

Let G be an algebraic group and $V_1, \ldots, V_m, W$ are defined to be G-modules. We call a G-equivariant morphism $V_1 \oplus \cdots \oplus V_m \longrightarrow W$ a G-covariant (of type W). Any G-covariant can be seen as a sum of multihomogeneous G-covariants (of multi-degree $(d_1, \ldots, d_m)$ with $d_1, \ldots, d_m \in \mathbb{N}$). For a multilinear (*i.e.*, multihomogeneous of multi-degree $(1, \ldots, 1)$) map $f \colon V_1^{d_1} \oplus \cdots \oplus V_m^{d_m} \longrightarrow W$ the multihomogeneous map $R_f \colon V_1 \oplus \cdots \oplus V_m \longrightarrow W$ defined by

$$R_f(v_1, \ldots, v_m) := f(\underbrace{v_1, \ldots, v_1}_{d_1}, \ldots, \underbrace{v_m, \ldots, v_m}_{d_m})$$

is called the restitution of f. Every multihomogeneous G-covariant of multi-degree $(d_1, \ldots, d_m)$ is the restitution of a multilinear G-covariant on $V_1^{d_1} \oplus \cdots \oplus V_m^{d_m}$ with values in W (*cf.* [10, Section 6]).

The vector space of multilinear G-covariants $\mathrm{Mult}(V_1^{d_1} \oplus \cdots \oplus V_m^{d_m}, W)^G$ can be determined by using the canonical G-isomorphism

$$\mathrm{Mult}(V_1^{d_1} \oplus \cdots \oplus V_m^{d_m}, W) \xrightarrow{\sim} \mathrm{Mult}(V_1^{d_1} \oplus \cdots \oplus V_m^{d_m} \oplus W^*, \mathbb{C}).$$

Now, we are able to handle another type of Θ-representations.

PROPOSITION 6.1. $\mathrm{Aut}_{\mathrm{SO}_n \times \mathrm{SP}_{2m}}(\mathbb{C}^n \otimes \mathbb{C}^{2m}) = \mathbb{C}^* \,\mathrm{id}_{\mathbb{C}^n \otimes \mathbb{C}^{2m}}$ *where* $m > 1$ *and* $n > 2$.

PROOF. Distinguish two cases: (a) $2 < n \leq 2m$ and (b) $4 < 2m < n$.

(a) Let $(\,,)$ denote the corresponding SP_{2m}-invariant non-degenerate skew-symmetric bilinear form. By classical invariant theory [26, Theorem 6.1.A] it is known for every $n > 2$, $m > 1$ that

$$\mathbb{C}\Big[(\mathbb{C}^{2m})^n\Big]^{SP_{2m}} = \mathbb{C}\Big[(i|j) \mid 1 \leq i < j \leq n\Big] \tag{1}$$

$$\mathbb{C}\Big[(\mathbb{C}^{2m})^n \oplus (\mathbb{C}^{2m})^*\Big]^{SP_{2m}} = \mathbb{C}\Big[(i|j), \varepsilon_l \mid 1 \leq i < j \leq n,\ 1 \leq l \leq n\Big] \tag{2}$$

where $(i,j)(v_1,\dots,v_n) := (v_i,v_j)$ and $\varepsilon_l(v_1,\dots,v_n,f) := f(v_l)$. Every automorphism $\sigma \in \mathrm{Aut}_{SO_n \times SP_{2m}}(\mathbb{C}^n \otimes \mathbb{C}^{2m})$ can be seen as an n-tuple $(\sigma_1,\dots,\sigma_n)$ of SP_{2m}-covariants (of type $\mathbb{C}^{2m}$) $\sigma_s\colon (\mathbb{C}^{2m})^n \longrightarrow \mathbb{C}^{2m}$, $s = 1,\dots,n$. By determining the restitution of the multilinear invariants of (2) it follows that

$$\sigma_s(v_1,\dots,v_n) = \sum_{r=1}^{n} p_{rs} v_r, \quad s = 1,\dots,n \tag{3}$$

where $p_{rs} \in \mathbb{C}[(\mathbb{C}^{2m})^n]^{SP_{2m}}$ (see above). We claim that all p_{rs} are constant polynomials.

Denoting σ^* the corresponding automorphism on $\mathbb{C}[(\mathbb{C}^{2m})^n]$ we see that $\sigma^*\big((i,j)\big) = \mu(i,j)$ since σ induces an automorphism on $(\mathbb{C}^{2m})^n /\!/ SP_{2m} = \wedge^2\mathbb{C}^n$ (adjoint representation), which is a multiple of the identity (2.5).

Let P denote the $n \times n$-matrix $(p_{ij})_{1 \leq i,j \leq n}$ with $p_{ij} \in \mathbb{C}[(\mathbb{C}^{2m})^n]^{SP_{2m}}$ from equation (3). It was just shown that the $\binom{n}{2} \times \binom{n}{2}$-matrix $\wedge^2 P$ consisting of all 2×2-minors of P is a scalar multiple of the identity matrix $E_{\binom{n}{2}}$. Since the kernel of the canonical homomorphism $GL(V) \longrightarrow GL(\wedge^2 V)$ is $\{\pm \mathrm{id}\}$ ($\dim V > 2$), it follows that $P \in \mathbb{C}^* E_n$, *i.e.*, σ is a scalar multiple of $\mathrm{id}_{(\mathbb{C}^{2m})^n}$ (*cf.* [13, Proof of 3.1])

(b) Exchange the rôles of SP_{2m} and SO_n: Here, $(\,,)$ denotes the corresponding SO_n-invariant non-degenerate symmetric bilinear form. For the SO_n-invariants there is an analogous relation [26, Theorem 2.9.A, 2.17.A]:

$$\mathbb{C}\Big[(\mathbb{C}^n)^{2m}\Big]^{SO_n} = \mathbb{C}\Big[(i,j) \mid 1 \leq i \leq j \leq 2m\Big]$$
$$\mathbb{C}\Big[(\mathbb{C}^n)^{2m} \oplus (\mathbb{C}^n)^*\Big]^{SO_n} = \mathbb{C}\Big[(i,j), \varepsilon_l \mid 1 \leq i \leq j \leq 2m, 1 \leq l \leq 2m\Big]$$

We can make the same conclusions as in (a) since SP_{2m} acts on $(\mathbb{C}^n)^{2m} /\!/ SO_n \cong S^2\mathbb{C}^{2m}$ by the adjoint representation and the kernel of the canonical homomorphism $GL(V) \longrightarrow GL(S^2V)$ is also $\{\pm \mathrm{id}\}$ ($\dim V > 2$). ∎

REMARK 6.2. In the same way as in proof (a) of 6.1 one can show $\mathrm{Aut}_{SL_n \times SP_{2m}}(\mathbb{C}^n \otimes \mathbb{C}^{2m}) = \mathbb{C}^* \mathrm{id}_{\mathbb{C}^n \otimes \mathbb{C}^{2m}}$ for $2 \leq n \leq 2m$, n even. Indeed, $\sigma \in \mathrm{Aut}_{SL_n \times SP_{2m}}(\mathbb{C}^n \otimes \mathbb{C}^{2m})$ induces an SL_n-automorphism $\overline{\sigma} \in \mathrm{Aut}_{SL_n}(\wedge^2\mathbb{C}^n)$ which turns out to be in $\mathbb{C}^* \mathrm{id}_{\wedge^2\mathbb{C}^n}$ (see N° 8b if $n \geq 4$; in case $n = 2$, $\overline{\sigma}$ is linear since $\wedge^2\mathbb{C}^2 \cong \mathbb{C}$).

Analogously, this is also true if n is odd.

In the following an adaptation of the method for finite $\overline{N} = \mathrm{Nor}(H)/H$ works best. The fixed point space V^H of a generic stabilizer H for the following examples no longer

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coincides with a Cartan subspace. However, with the earlier methods we will be able to show that $\mathrm{Aut}_{\overline{N}}(V^H)$ consists of linear automorphisms. Just like in the proof of 2.3 this induces that every $\sigma \in \mathrm{Aut}_G(V)$ is a multiple of id_V by looking at $\sigma \circ \lambda\, \mathrm{id}_V - \lambda\, \mathrm{id}_V \circ \sigma$.

PROPOSITION 6.3. $\mathrm{Aut}_{\mathrm{SL}_2 \times \mathrm{SL}_n \times \mathrm{SL}_n}(\mathbb{C}^2 \otimes \mathbb{C}^n \otimes \mathbb{C}^n) = \mathbb{C}^* \,\mathrm{id}_{\mathbb{C}^2 \otimes \mathbb{C}^n \otimes \mathbb{C}^n}$ *for* $n \geq 3$.

PROOF. Embed $\mathrm{SL}_2 \times \mathrm{SL}_n$ into SL_{2n} and consider the linear $G = \mathrm{SL}_2 \times \mathrm{SL}_n \times \mathrm{SL}_n$-action on the space of $2n \times n$-matrices $V = \mathrm{M}_{2n \times n}$. Let $\mathfrak{t}_{n-1} \subset \mathfrak{sl}_n$ denote the diagonal matrices. The stabilizer $\mathfrak{h} = \mathfrak{g}_A$ of

$$A := \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_n \end{pmatrix} \text{ where } A_j = \begin{pmatrix} a_j \\ b_j \end{pmatrix} \text{ with pairwise distinct } a_i, b_j$$

has the form $\mathfrak{h} = \{(0, t, t) \in \mathfrak{g} \mid t \in \mathfrak{t}_{n-1}\} \cong \mathfrak{t}_{n-1}$. Its fixed point set is

$$V^{\mathfrak{h}} = \left\{ \begin{pmatrix} M_1 & & \\ & \ddots & \\ & & M_n \end{pmatrix} \,\middle|\, M_j = \begin{pmatrix} \lambda_j \\ \mu_j \end{pmatrix} \in \mathbb{C}^2, j = 1, \ldots, n \right\} \cong (\mathbb{C}^2)^n.$$

The normalizer $\mathfrak{n}(\mathfrak{h})$ consists of the elements $(s, t) \in \mathfrak{sl}_{2n} \times \mathfrak{sl}_n$ where

$$s = \begin{pmatrix} s_1 & & \\ & \ddots & \\ & & s_n \end{pmatrix} \text{ with } s_j = \begin{pmatrix} a + d_j & b \\ c & -a + d_j \end{pmatrix}, \quad \sum_{j=1}^{n} d_j = 0$$

and $t \in \mathfrak{t}_{n-1}$. The algebra $\mathfrak{h}$ is a generic stabilizer and $\mathfrak{n}(\mathfrak{h}) \cong \mathfrak{sl}_2 \times \mathfrak{t}_{n-1} \times \mathfrak{t}_{n-1} \subset \mathfrak{g}$. Here we cannot make use of Lemma 3.1. So take a closer look at the $\mathrm{Nor}_G(H)/H$-action on $V^{\mathfrak{h}}$ which is equivalent to the $\Gamma := \mathrm{SL}_2 \times S_n \ltimes T_{n-1}$-action on $(\mathbb{C}^2)^n$ defined as follows:

$$\big(s, \mathrm{diag}(t_1, \ldots, t_n), \tau\big) \cdot (v_1, \ldots, v_n) = (t_1 s v_{\tau(1)}, \ldots, t_n s v_{\tau(n)})$$

It is shown in [13, 3.1.] that $\mathrm{Aut}_\Gamma\big((\mathbb{C}^2)^n\big) = \mathbb{C}^* \,\mathrm{id}_{(\mathbb{C}^2)^n}$ which induces $\mathrm{Aut}_G(V) = \mathbb{C}^* \,\mathrm{id}_V$. ∎

PROPOSITION 6.4. $\mathrm{Aut}_{\mathrm{SL}_2 \times \mathrm{SL}_{2n}}(\mathbb{C}^2 \otimes \wedge^2 \mathbb{C}^{2n}) = \mathbb{C}^* \,\mathrm{id}_{\mathbb{C}^2 \otimes \wedge^2 \mathbb{C}^{2n}}$ *for* $n \geq 3$.

PROOF. Let e_1, e_2, resp. $f_1, \ldots, f_{2n}$ be the standard basis of $\mathbb{C}^2$, resp. of $\mathbb{C}^{2n}$. Define $v_{i,j,k} := e_i \otimes (f_j \wedge f_k) \in V := \mathbb{C}^2 \otimes \wedge^2 \mathbb{C}^{2n}$ for $1 \leq i \leq 2$, $1 \leq j < k \leq 2n$. Consider the $G = \mathrm{SL}_2 \times \mathrm{SL}_{2n}$-orbit through

$$v = \sum_{i=1}^{2} \sum_{j=1}^{n} v_{i,2j-1,2j} \in V \quad \text{where} \quad H = \left\{ \left(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_n \end{pmatrix} \right) \,\middle|\, A_j \in \mathrm{SL}_2 \right\} \cong (\mathrm{SL}_2)^n$$

is the stabilizer of v. The H-fixed points are $V^H = \oplus_{i=1}^{2} \oplus_{j=1}^{n} \mathbb{C} v_{i,2j-1,2j}$. The group $\overline{N} = \mathrm{Nor}_G(H)/H$ is isomorphic to $\Gamma := \mathrm{SL}_2 \times S_n \ltimes T_{n-1}$. It follows that H is a generic isotropy group since $\overline{GV^H} = V$. The $\overline{N}$-action on V^H is equivalent to the Γ-module $(\mathbb{C}^2)^n$ as described in the proof of 6.3. We have $\mathrm{Aut}_\Gamma\big((\mathbb{C}^2)^n\big) = \mathbb{C}^* \,\mathrm{id}_{(\mathbb{C}^2)^n}$ as shown in [13, 3.1.] which induces $\mathrm{Aut}_G(V) = \mathbb{C}^* \,\mathrm{id}_V$. ∎

6.1. $S^3\,\mathrm{SL}_2$. This module is isomorphic to the SL_2-representation on the binary forms $V = \mathbb{C}[x,y]_3$. A generic isotropy group is given by $H = G_{x^3+y^3} = \left\{ \begin{pmatrix} \zeta & \\ & \zeta^{-1} \end{pmatrix} \,\middle|\, \zeta^3 = 1 \right\} \cong \mathbb{Z}_3$. Every $\sigma \in \mathrm{Aut}_G(V)$ induces a $\overline{\sigma} \in \mathrm{Aut}_{\mathrm{Nor}_G(H)}(V^H)$ which must be linear, for $\overline{\sigma}$ preserves $\mathbb{C}x^3 = V^U$ where $U := \left\{ \begin{pmatrix} 1 & \\ a & 1 \end{pmatrix} \,\middle|\, a \in \mathbb{C} \right\}$, and analogously $\overline{\sigma}$ also preserves $\mathbb{C}y^3$ (Lemma 2.1).

6.2. $\mathrm{SL}_2 \otimes S^2\,\mathrm{SL}_3$. This module is realized by the $G = \mathrm{SL}_2 \times \mathrm{SL}_3$-action on $V = \mathbb{C}^2 \otimes R_2$ where $R_2 := \mathbb{C}[x,y,z]_2$ are the tenary forms of degree 2. Let e_1, e_2 be the standard basis of $\mathbb{C}^2$ and define $v_1 := e_1 \otimes (x^2+yz)$, $v_2 := e_2 \otimes (y^2+xz)$, $v := v_1+v_2 \in V$. A generic stabilizer H is equal to G_v (*cf.* [18, p. 243]); it is generated by the three elements ($\zeta = e^{2\pi i/3}$)

$$g_1 := \left(\begin{pmatrix} \zeta & \\ & \zeta^2 \end{pmatrix}, \begin{pmatrix} \zeta & & \\ & \zeta^2 & \\ & & 1 \end{pmatrix} \right), \quad g_2 := \left(\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \frac{1}{3} \begin{pmatrix} -1 & 2 & 1 \\ 2 & -1 & 1 \\ 4 & 4 & -1 \end{pmatrix} \right),$$

$$g_3 := \left(\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \frac{1}{3} \begin{pmatrix} -1 & 2\zeta & \zeta^2 \\ 2\zeta^2 & -1 & \zeta \\ 4\zeta & 4\zeta^2 & -1 \end{pmatrix} \right).$$

The finite group H is isomorphic to $\mathfrak{A}_4$, the alternating group of 4 elements (the isomorphism is given by $g_1 \longmapsto (234)$, $g_2 \longmapsto (12)(34)$, $g_3 \longmapsto (14)(23)$). As usual we determine the H-fixed points in V which turn out to be $V^H = \mathbb{C}v_1 \oplus \mathbb{C}v_2$. Since $T_1 \times \{E_3\} \subset N := \mathrm{Nor}_G(H)$ one easily sees that every $\varphi \in \mathrm{Aut}_N(V^H)$ is linear by using Lemma 2.1.

6.3. $\mathrm{SL}_2 \otimes \wedge^3\,\mathrm{SL}_6$ *and* $\mathrm{SL}_2 \otimes \wedge_0^3\,\mathrm{SP}_6$. Let e_1, e_2, resp. $f_1, \ldots, f_6$ be the standard basis of $\mathbb{C}^2$, resp. of $\mathbb{C}^6$. Then $(ijk) := f_i \wedge f_j \wedge f_k$ for $1 \le i < j < k \le 6$ is a basis of $\wedge^3\mathbb{C}^6$. Consider the element

$$\begin{aligned} v := \sum_{j=1}^{2} \big(& j\, e_j \otimes (123) + 2j\, e_j \otimes (126) + 3j\, e_j \otimes (135) + 4j\, e_j \otimes (156) \\ & + 5j\, e_j \otimes (234) + 6j\, e_j \otimes (246) + 7j\, e_j \otimes (345) + 8j\, e_j \otimes (456) \big). \end{aligned}$$

The stabilizer $H = G_v \subset G = \mathrm{SL}_2 \times \mathrm{SL}_6$ of $v \in V = \mathbb{C}^2 \otimes \wedge^3\mathbb{C}^6$ has the following shape:

$$H = \left\{ \left(\begin{pmatrix} \varepsilon & \\ & \varepsilon \end{pmatrix}, \begin{pmatrix} S & \\ & S \end{pmatrix} \right) \in G \,\middle|\, S = \begin{pmatrix} \lambda & & \\ & \mu & \\ & & (\lambda\mu)^{-1} \end{pmatrix}, \lambda, \mu \in \mathbb{C}^*, \det S = \varepsilon = \pm 1 \right\} \cong T_2 \times \mathbb{Z}_2$$

For the space of H-fixed points one obtains

$$\begin{aligned} V^H = \bigoplus_{j=1}^{2} \big(& \mathbb{C}e_j \otimes (123) \oplus \mathbb{C}e_j \otimes (126) \oplus \mathbb{C}e_j \otimes (135) \oplus \mathbb{C}e_j \otimes (156) \\ & \oplus \mathbb{C}e_j \otimes (234) \oplus \mathbb{C}e_j \otimes (246) \oplus \mathbb{C}e_j \otimes (345) \oplus \mathbb{C}e_j \otimes (456) \big). \end{aligned}$$

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The normalizer $N := \mathrm{Nor}_G(H)$ is the following semidirect product:

$$N = \mathrm{SL}_2 \times \left\{ A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in \mathrm{SL}_6 \;\middle|\; A_j = \mathrm{diag}(a_{j1}, a_{j2}, a_{j2}), \det A = 1 \right\} \rtimes S_3$$

It follows that Gv is a generic orbit. The identity component of N/H is isomorphic to $(\mathrm{SL}_2)^4$ and therefore the N-module V^H is equivalent to the $\mathrm{SO}_4 \times \mathrm{SO}_4$-module $\mathbb{C}^4 \otimes \mathbb{C}^4$ (because $\mathbb{C}[\mathbb{C}^2 \otimes \mathbb{C}^2]^{\mathrm{SL}_2 \times \mathrm{SL}_2} = \mathbb{C}[q]$ where q is a quadratic form). It follows with 4.4 that $\mathrm{Aut}_N(V^H) = \mathbb{C}^* \,\mathrm{id}_{V^H}$.

To examine the automorphism group of $\mathrm{SL}_2 \otimes \wedge_0^3 \mathrm{SP}_6$ take the above notations. By using the methods in [2, VI 5.3] the skew-symmetric tensors (123), (126), (135), (156), (234), (246), (345), (456) are elements of $\wedge_0^3 \mathbb{C}^6$. Therefore the element v from above is also an element of the generic orbit of the simple $G = \mathrm{SL}_2 \times \mathrm{SP}_6$-module $V = \mathbb{C}^2 \otimes \wedge_0^3 \mathbb{C}^6$. The stabilizer $H = G_v$ is of the following shape:

$$H = \left\{ \left(\begin{pmatrix} \varepsilon & \\ & \varepsilon \end{pmatrix}, \begin{pmatrix} S & \\ & S \end{pmatrix} \right) \in G \;\middle|\; S = \begin{pmatrix} \pm 1 & & \\ & \pm 1 & \\ & & \pm 1 \end{pmatrix}, \det S = \varepsilon = \pm 1 \right\} \cong (\mathbb{Z}_2)^4$$

The H-fixed point space as well as $\mathrm{Nor}_G(H)^0$ are the same as for $\mathrm{SL}_2 \otimes \wedge^3 \mathrm{SL}_6$ above. So the same arguments lead to $\mathrm{Aut}_G(V) = \mathbb{C}^* \,\mathrm{id}_V$.

6.4. $\mathrm{SL}_3 \otimes \mathrm{SL}_3 \otimes \mathrm{SL}_3$. Let e_1, e_2, e_3 be the standard basis of $\mathbb{C}^3$ and define $(ijk) := e_i \otimes e_j \otimes e_k \in V = \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ for $i, j, k = 1, 2, 3$. The isotropy group H of

$$v := (111) + 2(222) + 3(333) + 4(123) + 5(132) + 6(213) + 7(231) + 8(312) + 9(321)$$

is the finite group generated by the three elements ($\zeta = e^{2\pi i/3}$)

$$\left(\begin{pmatrix} \zeta & & \\ & \zeta^2 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} \zeta^2 & & \\ & 1 & \\ & & \zeta \end{pmatrix}, \begin{pmatrix} 1 & & \\ & \zeta & \\ & & \zeta^2 \end{pmatrix} \right), \quad \left(\begin{pmatrix} \zeta & & \\ & 1 & \\ & & \zeta^2 \end{pmatrix}, \begin{pmatrix} \zeta^2 & & \\ & \zeta & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & \zeta^2 & \\ & & \zeta \end{pmatrix} \right),$$
$$\left(\begin{pmatrix} \zeta & & \\ & \zeta & \\ & & \zeta \end{pmatrix}, \begin{pmatrix} \zeta & & \\ & \zeta & \\ & & \zeta \end{pmatrix}, \begin{pmatrix} \zeta & & \\ & \zeta & \\ & & \zeta \end{pmatrix} \right).$$

The space of H-fixed points is easily computed:

$$V^H = \mathbb{C}(111) \oplus \mathbb{C}(222) \oplus \mathbb{C}(333) \oplus \mathbb{C}(123) \oplus \mathbb{C}(132) \oplus \mathbb{C}(213) \oplus \mathbb{C}(231) \oplus \mathbb{C}(312) \oplus \mathbb{C}(321)$$

The connected component of $N := \mathrm{Nor}_G(H)$ has the shape

$$N^0 = \left\{ (S_1, S_2, S_3) \in G \;\middle|\; S_j = \begin{pmatrix} \lambda_j & & \\ & \mu_j & \\ & & (\lambda_j \mu_j)^{-1} \end{pmatrix}, \lambda_j, \mu_j \in \mathbb{C}^*, j = 1, 2, 3 \right\} \cong (T_2)^3.$$

Since $\dim G + \dim V^H - \dim N = \dim V$ the finite group H is a generic stabilizer. Let $V^H_{(ijk)} \subset V^H$ be the hyperplane spanned by all standard basis elements except

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$(ijk) \in V^H$ and consider the element $s_t := (S, S, S) \in N$ with $S = \operatorname{diag}(t, t, t^{-2}), t \in \mathbb{C}^*$. Then $\{w \in V^H \mid \lim_{t\to 0} s_t w \text{ exists}\} = V^H_{(333)}$, and this hyperplane is stabilized by every $\varphi \in \operatorname{Aut}_{N^0}(V^H)$. Analogously, $V^H_{(123)}$ is $\operatorname{Aut}_{N^0}(V^H)$-stable by taking $s_t := \big(\operatorname{diag}(t^{-2}, t, t), \operatorname{diag}(t, t^{-2}, t), \operatorname{diag}(t, t, t^{-2})\big) \in N^0$. In total one obtains 9 hyperplanes in general position which are $\operatorname{Aut}_{N^0}(V^H)$-stable. By Lemma 2.1 $\operatorname{Aut}_{N^0}(V^H)$ only consists of linear automorphisms.

6.5. $\mathrm{SL}_3 \otimes \wedge^2 \mathrm{SL}_6$. Let e_1, e_2, e_3, resp. $f_1, \ldots, f_6$ be the standard basis of $\mathbb{C}^3$, resp. $\mathbb{C}^6$. Then $v_{i,jk} := e_i \otimes (f_j \wedge f_k)$, $1 \le i \le 3$, $1 \le j < k \le 6$ is a basis of $V = \mathbb{C}^3 \otimes \wedge^2\mathbb{C}^6$. The isotropy group of the element

$$\begin{aligned} v := {} & v_{1,14} + 2v_{1,25} + 3v_{1,36} + 4v_{2,14} + 5v_{2,25} + 6v_{2,36} + 7v_{3,14} + 8v_{3,25} + 9v_{3,36} \\ & + 10v_{1,15} + 11v_{1,16} + 12v_{1,24} + 13v_{1,26} + 14v_{1,34} + 15v_{1,35} \end{aligned}$$

turns out to be a generic stabilizer and has the form

$$H := \left\{ \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} \lambda E_3 & \\ & \lambda^{-1} E_3 \end{pmatrix} \right) \in G \;\middle|\; \lambda \in \mathbb{C}^* \right\} \cong \mathbb{C}^*.$$

The space of H-fixed points looks as follows:

$$V^H = \bigoplus_{i=1}^{3} (\mathbb{C}v_{i,14} \oplus \mathbb{C}v_{i,15} \oplus \mathbb{C}v_{i,16} \oplus \mathbb{C}v_{i,24} \oplus \mathbb{C}v_{i,25} \oplus \mathbb{C}v_{i,26} \oplus \mathbb{C}v_{i,34} \oplus \mathbb{C}v_{i,35} \oplus \mathbb{C}v_{i,36}).$$

Since $\bar{N}^0 := (\operatorname{Nor}_G(H)/H)^0 = \mathrm{SL}_3 \times (\mathrm{SL}_3)^2$ and the $\bar{N}^0$-action on V^H is equivalent to the natural $(\mathrm{SL}_3)^3$-action on $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ it holds that $\operatorname{Aut}_{\bar{N}^0}(V^H) = \mathbb{C}^* \operatorname{id}_{V^H}$ (6.4).

6.6. $\mathrm{SL}_4 \otimes \mathrm{Spin}_{10}$. Consider the finite subgroup $H \subset G := \mathrm{SL}_4 \times \mathrm{Spin}_{10}$ generated by the two elements:

$$\begin{aligned} h_1 &:= \big(\operatorname{diag}(1, 1, -1, -1), \operatorname{diag}(1, -i, 1, i, 1; -1, i, -1, -i, -1)\big) \\ h_2 &:= \big(\operatorname{diag}(-1, 1, -1, 1), \operatorname{diag}(i, i, 1, 1, 1; -i, -i, -1, -1, -1)\big). \end{aligned}$$

The Spin_{10}-part of h_1 acts as $\operatorname{diag}(E_8, -E_8)$ on $\mathbb{C}^{16}$ (see [20, 5.28, 5.38]). For a short outline of the spin-representation of Spin_{10} we refer to [20, p. 110 ff. and 5.38].

The representation space of $\mathrm{SL}_4 \otimes \mathrm{Spin}_{10}$ is defined to be the space of 4×16-matrices $V = \mathrm{M}_{4\times 16}$. The space of H-fixed points turns out to be:

$$V^H = \left\{ \left(\begin{array}{cccc|cccc|cccc|cccc} u_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_2 & 0 & u_3 & 0 & u_4 & 0 & 0 \\ 0 & 0 & u_5 & 0 & u_6 & 0 & u_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_8 \\ 0 & u_9 & 0 & 0 & 0 & 0 & 0 & 0 & u_{10} & 0 & u_{11} & 0 & u_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & u_{13} & 0 & u_{14} & 0 & u_{15} & 0 & 0 & 0 & 0 & 0 & 0 & u_{16} & 0 \end{array} \right) \;\middle|\; u_i \in \mathbb{C} \right\}$$

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The Lie algebra $\mathfrak{n}$ of $N := \mathrm{Nor}_G(H)$ consists of the elements

$$\left(\begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_3 & \\ & & & -t_1-t_2-t_3 \end{pmatrix}, \left(\begin{array}{ccccc|ccccc} a_1 & & & & & & & & & \\ & a_2 & & & & & & & & \\ & & a_3 & & a_{35} & & & & & b_{35} \\ & & & a_4 & & & & & & \\ & & a_{53} & & a_5 & & & -b_{35} & & \\ \hline & & & & & -a_1 & & & & \\ & & & & & & -a_2 & & & \\ & & & & c_{35} & & & -a_3 & & -a_{53} \\ & & & & & & & & -a_4 & \\ & & -c_{35} & & & & & -a_{35} & & -a_5 \end{array}\right) \right)$$

where all variables are complex numbers. The algebra $\mathfrak{n}$ is isomorphic to $\mathfrak{k}_3 \oplus \mathfrak{k}_3 \oplus \mathfrak{so}_4\left(\begin{pmatrix} & E_2 \\ E_2 & \end{pmatrix}\right)$, where $\mathfrak{k} := \mathfrak{k}_3 \oplus \mathfrak{k}_3$ commutes with $\mathfrak{so}_4$ (*cf.* [20, 5.38]); the second copy of $\mathfrak{k}_3$ in $\mathfrak{k}$ consists of the elements $(a_1, a_2, a_4) \in \mathfrak{so}_{10}$. For a generic element $v \in V^H$, Gv is a generic orbit and $GV^H \subset V$ is dense since $\dim Gv = 60$ and $\dim(G \times^N V^H) = 64 = \dim V$. Therefore it suffices to show that $\mathrm{Aut}_N(V^H)$ consists of linear elements. Notice that H is not a generic isotropy group, one can only say that H is contained in it. A generic stabilizer is isomorphic to $(\mathbb{Z}_2)^4$ [18, Table 1].

Up to an outer isomorphism the $\mathfrak{so}_4$-module V^H corresponds to the $\mathrm{SL}_2 \times \mathrm{SL}_2$-module $(\mathbb{C}^2)^4 \oplus (\mathbb{C}^2)^4$ where the first (second) copy of SL_2 naturally acts on the first (second) four copies of $\mathbb{C}^2$ (consider the $\mathfrak{so}_4$-part in [20, 5.38] acting on $V^H \cong (\mathbb{C}^2)^8$). Its ring of invariant functions is

$$\mathbb{C}\left[(\mathbb{C}^2)^4 \oplus (\mathbb{C}^2)^4\right]^{\mathrm{SL}_2 \times \mathrm{SL}_2} = \mathbb{C}\left[(\mathbb{C}^2)^4\right]^{\mathrm{SL}_2} \otimes \mathbb{C}\left[(\mathbb{C}^2)^4\right]^{\mathrm{SL}_2} = \mathbb{C}\left[[i,j] \mid \begin{smallmatrix} 1 \le i < j \le 4 \text{ or} \\ 5 \le i < j \le 8 \end{smallmatrix}\right]$$

where $[i,j](v_1, \ldots, v_8) = \det(v_i, v_j)$. The ideal of the relations among the $[i,j]$ is generated by the Plücker relations $[1,2][3,4]-[1,3][2,4]+[1,4][2,3]$ and $[5,6][7,8]-[5,7][6,8]+[5,8][6,7]$. Using the fact $\mathrm{Aut}_{\mathrm{SL}_2 \times S_4 \ltimes T_3}\left((\mathbb{C}^2)^4\right) = \mathbb{C}^* \, \mathrm{id}_{(\mathbb{C}^2)^4}$ [13, Prop. 3.1] and the $\mathfrak{k}_3$-equivariance of the copy $\mathfrak{k}_3 \subset \mathfrak{so}_{10}$ every N-automorphism of V^H is linear. Since $GV^H \subset V$ is dense $\mathrm{Aut}_G(V) = \mathbb{C}^* \, \mathrm{id}_V$.

6.7. . For the last few cases of Table 4.4 where $\mathrm{Nor}_G(H)/H$ is not finite, we are going to use Élashvili's tables [5, Table 6] and [4, Table 1]. Let (G, V) denote a G-module V. As usual $H \subset G$ is a generic stabilizer and $\overline{N} := \mathrm{Nor}_G(H)/H$. In all following examples we use the fact that if $\mathrm{Aut}_{\overline{N}}(V^H) = \mathbb{C}^* \, \mathrm{id}_{V^H}$, then also $\mathrm{Aut}_G(V) = \mathbb{C}^* \, \mathrm{id}_V$ (see proof of 2.3).

For $(G, V) = \mathrm{SL}_2 \otimes \mathrm{Spin}_{10}$ (N° 32) it is $(\overline{N}^0, V^H) \cong (T_3 \subset \mathrm{SL}_4, \mathbb{C}^4)$. This representation does not admit any nonlinear automorphisms: Take $t_u = \mathrm{diag}(u^{-3}, u, u, u) \in T_3$, $u \in \mathbb{C}^*$. Let $v \in \mathbb{C}^4$, then $\lim_{u \to 0} t_u v$ exists if and only if v lies in a hyperplane. This hyperplane is stabilized by any T_3-equivariant automorphism (*cf.* 6.3). By changing the spot of the entry u^{-3} one obtains four hyperplanes in total which are in general position. Now Lemma 2.1 finishes this example.

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Concerning $\mathrm{SL}_2 \otimes \mathrm{Spin}_{12}$ (N° 33) there is a mistake in [5, Table 6, No. 7]. A generic stabilizer is isomorphic to $3A_1$ embedded in D_6 [8] (also cf. [20, Section 5, Proposition 38]). Its normalizing Lie algebra in $A_1 + D_6$ is then isomorphic to $7A_1$. Hence $(\bar{N}^0, V^H)$ is isomorphic to $\big((\mathrm{SL}_2)^4, (\mathbb{C}^2)^{\otimes 4}\big) \cong \big((\mathrm{SO}_4)^2, (\mathbb{C}^4)^{\otimes 2}\big)$. This module is without nonlinear automorphisms (4.4).

For $\mathrm{SL}_2 \otimes E_6$ (N° 34) we have $(\bar{N}, V^H) \cong \big(\mathrm{SL}_2 \times S_3 \ltimes T_2, (\mathbb{C}^2)^3\big)$ whose equivariant automorphism are linear [13, 3.1].

The module $\mathrm{SL}_2 \otimes E_7$ (N° 35) yields $(\bar{N}, V^H) \cong \big((\mathrm{SL}_2)^4, (\mathbb{C}^2)^{\otimes 4}\big) \cong \big((\mathrm{SO}_4)^2, (\mathbb{C}^4)^{\otimes 2}\big)$. By 4.4 there are no nonlinear automorphisms.

For $(G, V) = \mathrm{SL}_3 \otimes E_6$ (N° 45) one obtains $(\bar{N}, V^H) \cong \big((\mathrm{SL}_3)^3, (\mathbb{C}^3)^{\otimes 3}\big)$; in 6.4 all equivariant automorphisms are proved to be linear.

The modules $\wedge^3 \mathrm{SL}_6$ (N° 17), $\mathrm{SL}_2 \otimes \mathrm{Spin}_7$ (N° 31), Spin_{12} (N° 53), $\wedge_0^3 \mathrm{SP}_6$ (N° 56) and E_7 (N° 65) are all of the same type: Using the tables [5, Table 6], [4, Table 1] all these modules fulfil $(\bar{N}^0, V^H) \cong (\mathbb{C}^*, \mathbb{C}^2)$ and $\dim V /\!/ G = \dim V^H /\!/ \bar{N}^0 = 1$. $\mathbb{C}^*$ acts on $\mathbb{C}^2$ by a positive and a negative weight. By a limit consideration either line through the weight vector is preserved by every $\sigma \in \mathrm{Aut}_{\bar{N}^0}(V^H)$ implying that σ is linear (see Lemma 2.1).

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