NON-NEGATIVE VALUES OF QUADRATIC FORMS

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(Received 5 February 1969)

Communicated by E. S. Barnes

1. Introduction

In a paper [1] of the same title Barnes considered the problem of finding an upper bound for the infimum $m_+(f)$ of the non-negative values[1] of an indefinite quadratic form f in n variables, of given determinant $\det(f) \neq 0$ and of signature s. In particular it was announced (and later proved – see [2]) that $m_+(f) \leqq (16/5)^{\frac{1}{3}}$ for ternary quadratic forms of determinant 1 and signature -1. A simple consequence of this result is that $m_+(f) \leqq (256/135)^{\frac{1}{4}}$ for quaternary quadratic forms of determinant -1 and signature -2.

In this paper it will be shown that one can do considerably better than $(16/5)^{\frac{1}{3}}$ for most ternary quadratic forms f of signature -1, and that consequently $m_+(f) < (128/81)^{\frac{1}{4}}$ for quaternary quadratic forms of signature -2. It should be pointed out that the restriction that $|\det(f)| = 1$ is really no restriction at all as multiplication of a form of this type by $d^{\frac{1}{4}}$ gives a form f with $|\det(f)| = d$ and it plainly follows by the results that $m_+(f) < (128d/81)^{\frac{1}{4}}$ for all quaternary quadratic forms f with $|\det(f)| = d$ and of signature -2.

2. Statement of results

The following are the results proved. For convenience the signature has been changed to $+1$ and $m_-(f) = m_+(-f)$ has been considered.

THEOREM 1. *Let $f(x, y, z)$ be a ternary quadratic form of signature* 1 *and let* $|\det(f)| = d \neq 0$. *Then $m_-(f) < (8d/3)^{\frac{1}{3}}$ unless f is equivalent to a multiple of one of the following forms*:

$$f_1(x, y, z) = x^2+xy+y^2+15yz-15z^2$$
$$f_2(x, y, z) = x^2+xy+y^2+xz+32yz-29z^2$$
$$f_3(x, y, z) = x^2+y^2+8yz-8z^2.$$

[1] A form $f(x, y, \cdots, z)$ is said to take the value v if there exist integers $x, y, \cdots, z$ not all zero such that $f(x, y, \cdots, z) = v$.

https://doi.org/10.1017/S1446788700009484 Published online by Cambridge University Press

Furthermore $m_-(f_1) = 6 = (16d/5)^{\frac{1}{3}}$, $m_-(f_2) = 9 = (27d/10)^{\frac{1}{3}}$ *and* $m_-(f_3) = 4 = (8d/3)^{\frac{1}{3}}$.

THEOREM 2. *Let* $g(t, x, y, z)$ *be a quaternary quadratic form of signature* 2 *and let* $|\det(g)| = d \neq 0$. *Then* $m_-(g) < (128d/81)^{\frac{1}{4}}$.

3. Deduction of theorem 2

Let $g(t, x, y, z)$ be a quaternary quadratic form of signature 2 and let $|\det(g)| = d \neq 0$. If $m_+(g) = 0$ we have $m_-(g) = 0$ by Oppenheim [3] and so g satisfies the conclusion of Theorem 2. If $m_+(g) > 0$ we may take $m_+(g) = 1$; if this does not hold multiply g by $(m_+(g))^{-1}$. Let $m_-(g) = a$; we assume $a > 1$, else the symmetric minimum result of Oppenheim [4] yields $d \geqq \frac{7}{4} > \frac{81}{128}a^4$.

As $m_+(g) = 1$, g takes, for any $n > 1$, a value v_n satisfying $1 \leqq v_n < 1\frac{1}{n}$. By applying a suitable integral unimodular transformation to g we obtain a form g_n, equivalent to g, of the shape

$$(1) \qquad g_n(t, x, y, z) = v_n(t+\lambda_n x+\mu_n y+\delta_n z)^2 + v_n^{-1} f_n^*(x, y, z),$$

where f_n^* is a ternary quadratic form of signature 1. If f_n^* were to take a value $u < 0$ at $(x, y, z) = (X, Y, Z)$ then setting $(x, y, z) = (Xt, Yt, Zt)$ gives a binary section of g_n of determinant $-u$, and this section cannot take a value in the open interval $(-a, 1)$. Thus $u \leqq -a-\frac{1}{4}a^2$ by Segre [5], so $m_-(f_n^*) \geqq a+\frac{1}{4}a^2$. But $|\det(f_n^*)| = d$ and theorem 1 gives f_n^* a multiple of either f_1, f_2 or f_3, or $(8d/3)^{\frac{1}{3}} > m_-(f_n^*)$. The latter possibility yields $(8d/3)^{\frac{1}{3}} > a+\frac{1}{4}a^2$, which implies that $m_-(g) = a < (128d/81)^{\frac{1}{4}}$ since $(1+\frac{1}{4}a)^3 a^{-1}$ has a minimum of 27/16 attained at $a = 2$.

It now remains to consider the possibility that, for each n, $f_n^* = m_n f_{j_n}(x, y, z)$ for $j_n = 1, 2$ or 3. If $v_n \neq 1$ for any n we may choose a sequence $n_1, n_2, \cdots$ such that as $n_i \to \infty$ we have $v_{n_i} \to 1$, $\lambda_{n_i} \to \lambda$, $\mu_{n_i} \to \mu$, $\delta_{n_i} \to \delta$ and $m_{n_i} \to m$ for some λ, μ, δ and m, and such that j_n remains fixed (say at j). Denoting $(t+\lambda x+\mu y+\delta z)^2 + mf_j(x, y, z)$ by $g^*(t, x, y, z)$ it is clear that by choosing n_i large enough we can get values of g_{n_i}, and thus g, arbitrarily close to any specified value of g^*. Hence $m_+(g^*) = 1$ and $m_-(g) \leqq m_-(g^*)$, and we have reduced this case to the special case where $v_n = 1$. Hence it remains only to show that if

$$g = (t+\lambda x+\mu y+\delta z)^2 + mf_j(x, y, z) = g_j(t, x, y, z)$$

for $j = 1, 2$ or 3 then $m_-(g) < (128d/81)^{\frac{1}{4}}$.

(a) Let $g = g_1(t, x, y, z)$ and suppose that $m_-(g) = a \geqq (218d/81)^{\frac{1}{4}} = (320m^3/3)^{\frac{1}{4}}$. As $m_-(f_1) = 6$ and we require $m_-(mf_1) \geqq a+\frac{1}{4}a^2$, we must have $a^4 \geqq 40(a+\frac{1}{4}a^2)^3/81$ which is possible (for $a > 1$) only if $a < 4 \cdot 1$. Hence $m < 1 . 3837$. As $||\lambda-\frac{1}{2}|| < \frac{1}{6}$, $||\lambda-\mu-\frac{1}{2}|| < \frac{1}{6}$ and $||\mu-\frac{1}{2}|| < \frac{1}{6}$ are not simultaneously possible,[2] consideration of $g(t, 1, 0, 0)$, $g(t, 1, -1, 0)$ and $g(t, 0, 1, 0)$

[2] $||x||$ is used to denote the distance from x to the nearest integer.

https://doi.org/10.1017/S1446788700009484 Published online by Cambridge University Press

yields $m \geqq 8/9$. Hence $a > 2.94$. As f_1 takes the value -6, g has a section of the form $(t+\gamma)^2-6m$, and as $5\frac{1}{3} \leqq 6m < 8.31$ choosing $4 \leqq (t+\gamma)^2 \leqq 6.25$ yields a contradiction to either $m_+(g) = 1$ or $m_-(g) = a$ unless $6m \geqq 4+a$. A number of iterations on this and $a \geqq (320m^3/3)^{\frac{1}{4}}$ yields $m > 1.31$ and $a > 3.9$. As f_1 takes the value -9 (at $(4,1,-1)$), g has a section of the form $(t+\rho)^2-9m$. But $11.7 < 9m < 12.5$ and so choosing $9 \leqq (t+\rho)^2 \leqq 12.25$ yields a contradiction to either $m_+(g) = 1$ or $m_-(g) = a > 3.9$. This shows that $m_-(g_1) < (128d/81)^{\frac{1}{4}}$.

(b) Let $g = g_2(t, x, y, z)$ and suppose that $m_-(g) = a \geqq (128d/81)^{\frac{1}{4}} = (1280m^3/3)^{\frac{1}{4}}$. Then from $m_-(f_2) = 9$ we get $a^4 \geqq 1280(a+\frac{1}{4}a^2)^3/2187$ which can hold only for $a < 2.5$. Hence $m < \frac{3}{4}$. However we then have a value $(t+\lambda)^2+m$ of g which contradicts $m_+(g) = 1$ if $0 \leqq (t+\lambda)^2 \leqq \frac{1}{4}$. Hence $m_-(g_2) < (128d/81)^{\frac{1}{4}}$.

(c) Let $g = g_3(t, x, y, z)$ and suppose that $m_-(g) = a \geqq (128d/81)^{\frac{1}{4}} = (1024m^3/27)^{\frac{1}{4}}$. Then from $m_-(f_3) = 4$ we get $a^4 \geqq 1024m^3/27 \geqq 16(a+\frac{1}{4}a^2)^3/27$ which is possible only for $a = 2$ and $m = \frac{3}{4}$. Considering $g(t, 1, 0, 0)$, $g(t, 0, 1, 0)$ and $g(t, 3, 0, 1)$ yields that $\lambda = \mu = \frac{1}{2}$, $\delta = 0$ in order that $m_+(g) = 1$. But then $g(3, 1, -1, 1) = -1\frac{1}{2}$ contradicting $m_-(g) = a = 2$. This completes the deduction of Theorem 2.

At this stage it should be pointed out that the deduction of Theorem 2 only requires theorem 1 for $d < 435$, for from this theorem we have that excluding the three critical forms every ternary form of signature 1 takes a value in the interval $(-(8d/3)^{\frac{1}{3}}, (d/435)^{\frac{1}{3}}]$ by the method used in [6]. But where $f_n^*(x, y, z)$ is as in (1), we have $m_-(f_n^*) \geqq (a+\frac{1}{4}a^2)$ and $m_+(f_n^*) \geqq \frac{3}{4}$ (else choosing the square in (1) suitably gives a value v of g satisfying $0 \leqq v < \frac{1}{4}v_n+\frac{3}{4}v_n^{-1} < 1$ for $v_n \leqq 1$, contradicting $m_+(g) = 1$). Hence, neglecting the initial forms which may be treated as above, either $(a+\frac{1}{4}a^2)^3 < 8d/3$ which yields $a < (128d/81)^{\frac{1}{4}}$ as before or $d/435 \geqq 27/64$. Then the assumption $a^4 \geqq 128d/81$ yields $a > 4.1266$. But by [2] $m_-(f_n^*) \leqq (16d/5)^{\frac{1}{3}}$ which yields $(a+\frac{1}{4}a^2)^3 \leqq 81/40a$ which is false for $a > 4.1$. This contradiction is sufficient to complete the deduction of Theorem 2.

4. Proof of theorem 1

By a result of Oppenheim [3], $m_+(f) = 0$ implies that

$$m_-(f) = 0 < (8|\det(f)|/3)^{\frac{1}{3}}$$

for indefinite ternary forms. Hence in proving theorem 1 we may assume $m_+(f) > 0$ and indeed $m_+(f) = 1$ after multiplication by $(m_+(f))^{-1}$. Furthermore we may also assume, by virtue of theorem 3.1 of [6], that f actually takes the value 1. Thus it is only necessary to prove:

THEOREM 3. *Let $f(x, y, z)$ be a ternary quadratic form of signature* 1, *let* $|\det(f)| = d \neq 0$, *and let $m_+(f) = 1$ be attained by f. Then $m_-(f) < (8d/3)^{\frac{1}{3}}$ unless f is equivalent to one of the forms f_1, f_2 or f_3 as listed in theorem* 1. *Furthermore each of these forms has $m_+(f) = 1$, while $m_-(f_1) = 6$, $m_-(f_2) = 9$ and $m_-(f_3) = 4$.*

We first show that it is necessary only to consider $d \leqq 823\frac{7}{8}$. In order to avoid cluttering the proof of this we have a few lemmas.

LEMMA 1. *Let $k \geqq 9$ be an integer, define*

$$K = k^2+6k+1, \quad t(S) = K^2(1+4/S)/64,$$
$$d_1 = K(K+12)/64 \quad \text{and} \quad d_2 = \max(\min\{t(S), 9(S+\sqrt{5})^2/64\})$$

where the maximum is taken over all positive integers S, and let this maximum be taken at S^*. Then $S^* = [K/3]+1$ and $d_2 = t(S^*) < d_1$.

LEMMA 2. *Let $k \geqq 13$ be integral and let*

$$d_k(r, s) = (k^2+4k)^2\{(r+2)^2s^2+4(r+2)s(rs+r+s)\}/64(rs+r+s)^2.$$

Then $k^{-3}d_k(r, s) \geqq k^{-3}d_k(S^*, S^*) > \frac{3}{8}$ for $k \geqq 14$ and $r \leqq s \leqq S^*$.

LEMMA 3. *Let $k \geqq 13$ be integral, let d_1 be as in lemma 1 and let l satisfy $0 < l < 1$. Then $F(k, l) = (k+l)^3/(d_1+\frac{1}{8}Kl)$ has its supremum at $k = 13$, $l = 1$ and this supremum is less than $\frac{8}{3}$.*

PROOF OF LEMMA 1. Plainly $t(S) < d_1 < 9(\frac{1}{3}K+\sqrt{5})^2/64$ for $S > \frac{1}{3}K$, so $t(S) < d_1 < 9(S+\sqrt{5})^2/64$ for $S > \frac{1}{3}K$. It is also clear that $t(S) > d_1$ for $S < \frac{1}{3}K$. But as $K \not\equiv 0 \pmod 3$ it follows that $S < \frac{1}{3}K$ implies that $3S \leqq K-1$, and then

$$9(S+\sqrt{5})^2/64 \leqq (K+3\sqrt{5}-1)^2/64 < (K^2+12K)/64$$

for $K > 75$. Now for $K > 120$ we have

$$9(\tfrac{1}{3}(K-1)+\sqrt{5})^2/64 < (K+5\,.\,75)^2/64 < K^2(1+12(K+2)^{-1})/64$$

and so

$$9([K/3]+\sqrt{5})^2/64 \leqq 9(\tfrac{1}{3}(K-1)+\sqrt{5})^2/64 < t([K/3]+1).$$

Thus as $t(S)$ is a decreasing function of S and $9(S+\sqrt{5})^2/64$ an increasing one it follows that for $K > 120$ we have $S^* = [K/3]+1$ and $d_2 = t(S^*) < d_1$. The lemma now follows on observing that $K > 120$ for $k \geqq 9$.

PROOF OF LEMMA 2. Since $d_k(s, s) = (k^2+4k)^2(1+4/s)/64$ which is a decreasing function of s, since $s \leqq S^*$ and since $3S^* \leqq K+2$ the lemma simply reduces to showing that $d_k(r, s)$ has negative derivative with respect to r, that $k^{-1}(k+4)^2(1+12/(k^2+6k+3))$ has positive derivative with respect to k for $k \geqq 14$ and that for $k = 14$, $d_k(S^*, S^*) > 1029$.

PROOF OF LEMMA 3. This is a consequence of the fact that $F(k, l)$ positive derivative with respect to l and that $F(k, 1)$ has negative derivative with respect to k.

We are now in a position to prove the claim that it is only necessary to consider $d \leqq 823\frac{7}{8}$ in proving theorem 3.

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LEMMA 4. *Let f satisfy the condition of theorem* 3 *and let* $d > 823\frac{7}{8}$. *Then* $m_-(f) < (8d/3)^{\frac{1}{3}}$.

PROOF. Suppose to the contrary that $m_-(f) \geqq (8d/3)^{\frac{1}{3}}$. Then $m_-(f) > (2197)^{\frac{1}{3}} = 13$. Let $k = [m_-(f)] \geqq 13$ and let $l = m_-(f) - k$. Firstly if $l = 0$ then $k \geqq 14$ and by theorem 2 of [7] it follows that either $d = d_k(r, s)$ for some appropriate $r \leqq s \leqq S^*$, or $d \geqq \min(d_1, d_2)$. But $\min(d_1, d_2) = d_2 = t(S^*) > d_k(S^*, S^*)$ by lemma 1, so by lemma 2 we have $k^{-3}d > \frac{3}{8}$, i.e. $m_-(f) < (8d/3)^{\frac{1}{3}}$. Secondly if $l > 0$ we write f as $(x + \lambda y + \mu z)^2 + q(y, z)$, by choosing a suitable equivalent form, where q is an indefinite binary form, and let $m_-(q) = e$. Since q can take no values in $(-e, \frac{3}{4})$ we have by Segré [5] that $|\det(q)| \geqq \frac{3}{4}e + \frac{1}{4}e^2$, i.e. $d \geqq \frac{3}{4}e + \frac{1}{4}e^2$. As q takes values $-e(1+\delta)$ for arbitrarily small $\delta \geqq 0$, f has a section of the form $(x + \rho t)^2 - e(1+\delta)t^2$ for arbitrarily small $\delta \geqq 0$. Because these sections can take no values in the interval $(-m_-(f), 1)$ we have by the corollary to theorem 1 of [7] that $e(1+\delta) \geqq \frac{1}{4}K + l$. Hence $e \geqq \frac{1}{4}K + l$, so $d \geqq d_1 + \frac{1}{8}Kl$. Hence by lemma 3 we have $m_-(f) < (\frac{8}{3}d)^{\frac{1}{3}}$. This contradiction is sufficient to prove the lemma.

To complete the proof of theorem 3 we consider various sub-intervals of $(0, 823\frac{7}{8}]$ in turn.

LEMMA 5. *Let f satisfy the conditions of theorem* 3 *and let* $d \leqq 67.5$. *Then either* $m_-(f) < (8d/3)^{\frac{1}{3}}$ *or f is equivalent to either* f_1 *or* f_3. *Furthermore*

$$m_+(f_1) = m_+(f_3) = 1, \; m_-(f_1) = 6 \;\textit{ and }\; m_-(f_3) = 4.$$

PROOF. This is theorem C_8 combined with lemmas 2.8 and 2.9 of [6].

LEMMA 6. *Let f satisfy the conditions of theorem* 3 *and let* $67.5 < d \leqq 81$. *Then* $m_-(f) < (8d/3)^{\frac{1}{3}}$.

PROOF. Suppose $m_-(f) \geqq (8d/3)^{\frac{1}{3}}$. Since f takes the value 1 we may choose an equivalent form $g = (x + \lambda y + \mu z)^2 + q(y, z)$ where q is an indefinite binary form. Applying transformations which turn q into elements of the chain (q_i) of reduced forms equivalent to q, and applying suitable parallel transformations to x we obtain a chain of forms

$$g_i = (x + \lambda_i y + \mu_i z)^2 + (-1)^{i+1} a_{i+1}(z - F_i y)(z + S_i y),$$

each equivalent to f, with the following property. There exists a chain of positive integers p_i, $-\infty < i < \infty$, such that F_i and S_i are given by the simple continued fractions $(p_i, p_{i+1}, p_{i+2}, \cdots)$ and $(0, p_{i-1}, p_{i-2}, \cdots)$ respectively. Furthermore if $\Delta^2 = 4d$ denotes the discriminant of q then $a_{i+1}K_i = \Delta$ where $K_i = F_i + S_i$. In addition it is plain that $a_i \geqq \frac{3}{4}$ for even i to ensure $m_+(f) = 1$.

If k denotes the integer part of $m_-(f)$ and if $m_-(f) > k$ then by the corollary to theorem 1 of [7] applied to $(x + \mu_i z)^2 + (-1)^{i+1} a_{i+1} z^2$ for i odd we have that $a_{i+1} \geqq \frac{1}{4}(k+1)^2 + m_-(f)$. This yields $K_i \leqq \Delta(\frac{1}{4}(k+1)^2 + (\frac{2}{3}\Delta^2)^{\frac{1}{3}})$ and this expression is a maximum for maximum Δ. Now $d > 67\frac{1}{2}$ implies $m_-(f) > 5.6462$,

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so $a_{i+1} > 14.6462$ for even i and $k = 5$. Since $d \leqq 81$ implies $\Delta \leqq 18$ we have $K_i \leqq 1.2$ (i even), $K_i \leqq 24$ (i odd). These bounds imply that $p_i = 1$ (i even) and $6 \leqq p_i \leqq 22$ (i odd), so for i even we have $K_i > 1+2\,(0, 22, 1, 23) = 599/551$, which implies that $a_{i+1} < 16.6$ (i even) in order that $d \leqq 81$.

For the remainder of the proof of this lemma i shall denote any even integer, and since the chain (p_i) is reversible at any point by the transformation $y' = -y$ we shall assume $F_i \leqq 1+S_i$. The suffix i shall be dropped from K_i, F_i, S_i, λ_i and μ_i, and the suffix $i+1$ from a_{i+1} unless ambiguity would result. $m_-(f)$ and $m_+(f)$ will be abbreviated to m_- and m_+ respectively.

In the section $(x+\mu)^2-a$, in order not to contradict $m_+ = 1$ or the definition of m_- we need $(4-||\mu||)^2-a \geqq 1$, $a \leqq 15$ and $(3+||\mu||)^2-a \leqq -m_-$. Hence

$$(2) \qquad 14||\mu|| < 6-m_-$$

and so $||\mu|| < .0253$. The bound on a now yields, as $aK = \Delta > \surd 270$, that $K > 1.0954$. Thus $1.0435 < F \leqq 1.1$ and $F-1 \leqq S < 1.1565$. We now eliminate various ranges of S in turn.

(a) $S = (0, 6, 1, \cdots) > (0, 6, 1, 23) > .1437$. This yields $K > 1.1872$, and iteration of $m_- > (\frac{2}{3}\Delta^2)^{\frac{1}{4}}$, $a \geqq 9+m_-$ gives $m_- > 5.94$, $a > 14.94$. Then $25.7 < a(1+F)(1-S) < 26.42$, so choosing x with $20.25 \leqq (x+\lambda-\mu)^2 \leqq 25$ yields a contradiction (to $m_+ = 1$ or $m_- > 5.94$) unless $(x+\lambda-\mu)^2 < 20.48$. Thus $||\lambda-\mu-\frac{1}{2}|| < .03$, so $100 \leqq (x+2\lambda-2\mu)^2 < 101.3$ for some x. As $102.8 < T(2, -2) < 105.7$ this yields a contradiction[3]. Hence we must have $S < (0, 7, 1, 7) < 0.127$.

(b) $0.1 < S < 0.127$. Analysis as in (a) yields $m_- > 5.73$, $a > 14.73$ and that if $F > 1.084$ then $m_- > 5.92$. We have $27.83 < T(1, 2) < 30.53$ where the lower bound may be increased to 28.33 if $F \leqq 1.084$. Furthermore if $F > 1.084$ we have $38.19 < T(2, 3) < 40.61$. Considering $25 \leqq (x+\lambda+2\mu)^2 \leqq 30.25$ yields a contradiction in $g(x, 1, 2)$ unless $||\lambda+2\mu-\frac{1}{2}|| \leqq .132$($||\lambda+2\mu-\frac{1}{2}|| < .09$ if $F \leqq 1.084$).

If $F > 1.084$ we have $||\mu|| < .006$ from (2) and so $||2\lambda+3\mu|| < .27$. Then in $g(x, 2, 3)$, $32.83 < (x+2\lambda+3\mu)^2 \leqq 36$ yields a contradiction unless $T(2, 3) > 38.75$, when $36 \leqq (x+2\lambda+3\mu)^2 < 39.4$ yields a contradiction. Hence $F \leqq 1.084$ and so $||\lambda+2\mu-\frac{1}{2}|| < .09$ from the above.

Now from (2) we have $||\mu|| < .02$, so $||2\lambda-\mu|| < .28$, hence $32.71 < (x+2\lambda-\mu)^2 \leqq 36$ for some x. But $33.9 < T(2, -1) < 38.02$, so in order to avoid a contradiction we must have $T(2, -1) \leqq 35$ and $||2\lambda-\mu|| < 0.1$. These imply $S > .115$, so $S > .125$ as $(0, 8, 1, \cdots) < .113$, and hence $F < 1.075$. Then $K < 1.1685$, $m_- > 5.85$, $a > 14.85$. Furthermore $(0, 12, 1, \cdots) > .076$, so $F < (1, 13, 1, 13) < 1.072$, so $29.25 < T(1, 2) < 30.53$. Then with $25 \leqq (x+\lambda+2\mu)^2$

[3] For brevity we have denoted $a(z-Fy)(z+Sy)$ by $T(y, z)$ throughout the remainder of this paper.

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$\leqq 30.25$ we obtain a contradiction completing the elimination of this range for S. Hence $S \leqq 0.1$, and as $(0, 9, 1, \cdots) > 0.1$ we must therefore have $S < (0, 10, 1{,}10) < .0917$.

(c) $.077 < S < 0917$. This possibility may also be eliminated by reference to $g(x, 1, 2)$, $g(x, 2, 3)$ and $g(x, 2, -1)$. We have $27.82 < T(1, 2) < 30.02$, so considering $25 \leqq (x+\lambda+2\mu)^2 \leqq 30.25$ yields $||\lambda+2\mu-\frac{1}{2}|| < .14$. Thus $||2\lambda+3\mu||$ $< .3053$, so $36 \leqq (x+2\lambda+3\mu) < 39.76$ for suitable x. But $38.07 < T(2, 3) <$ 43.6, so either (i) $T(2, 3) < 38.76$ or (ii) $T(2, 3) \geqq 36+m_-$.

The first possibility yields $F > 1.082$, $K > 1.164$, $m_- > 5.83$, $a > 14.83$, $T(1, 2) > 28.17$, $||\lambda+2\mu-\frac{1}{2}|| < .11$ and $||2\lambda+3\mu|| < .2453$ in turn. But now choosing x with $36 \leqq (x+2\lambda+3\mu)^2 < 39.1$ yields a contradiction since the improved bound on a yields $T(2, 3) > 38.55$.

Considering the second possibility we note that $36.92 < T(2, -1) < 40.03$, so $36 \leqq (x+2\lambda-\mu)^2 \leqq 42.25$ yields $||2\lambda-\mu-\frac{1}{2}|| < .35$ in order to avoid a contradiction. Hence $||2\lambda+3\mu-\frac{1}{2}|| < .4512$, so $T(2, 3) > (6.0488)^2+m_- > 42.26$. This yields $F < 1.0579$, $T(1, 2) > 28.658$, $||\lambda+2\mu-\frac{1}{2}|| < .055$ and $||2\lambda-\mu|| <$ $.2365$ in turn. Then either $33.21 < (x+2\lambda-\mu)^2 \leqq 36$ or $36 \leqq (x+2\lambda-\mu)^2 < 39$ will yield a contradiction. This eliminates this range for S, so $S \leqq .077$. As $(0, 12, 1, 23) > .077$ we must therefore have $S < (0, 13, 1, 13) < .0718$.

(d) $.054 < S < .0718$. This case is easily eliminated, for $27.78 < T(1, -1)$ < 29.3 which implies that $||\lambda-\mu-\frac{1}{2}|| < .136$. Thus $||2\lambda-\mu|| < .298$ and choosing x with $36 \leqq (x+2\lambda-\mu)^2 < 39.67$ yields a contradiction as $38.72 < T(2, -1) <$ 41.6. Hence $S \leqq .054$, and as $(0, 17, 1, 23) > .055$ we must have $S < (0, 18, 1, 18)$ $< .0528$.

(e) $.0527 < S < .0528$. This case yields $||2\lambda-\mu|| < .298$ as above, and since $38.72 < T(2, -1) < 41.672$ we obtain a contradiction unless $a > 14.99$ and $F > 1.0517$. This yields $K > 1.1044$, $m_- > 5.674$ and so our value $g(x, 2, -1)$ still yields a contradiction. Thus $S \leqq .0527$, and as $(0, 18, 1, 23) > .0527$ we must have $S < (0, 19, 1, 19) < .0502$.

(f) $.05 < S < .0502$. This implies that $aFS < .791$, so $||\lambda-\frac{1}{2}|| < .05$ in order to avoid a contradiction. Hence $||2\lambda-\mu|| < .126$, so we can choose x with $36 \leqq$ $(x+2\lambda-\mu)^2 < 37.6$. As $40 < T(2, -1) < 41.85$ this gives a contradiction unless $||2\lambda-\mu|| < .018$ and $a < 14.92$. Then $||8\lambda-\mu|| < .149$, so $81 \leqq (x+8\lambda-\mu)^2 < 83.8$ for some x. But $F > 1.0474$ in order that $T(1, 1) \geqq \frac{3}{4}$, so $83.7 < T(8, -1) < \underline{84.6}$, yielding a contradiction. Hence as $(0, 19, 1, \cdots) > .05$ we must have $S \leqq \overline{(0, 20, 1)}$ $= S'$. But then $FS < \frac{1}{20}$ unless $F-1 = S = S'$, so $aFS < \frac{3}{4}$, yielding a contradiction, unless $a = 15$ and $F-1 = S = S'$. But this implies $\Delta^2 = 270$, contradicting the initial assumption that $d > 67.5$.

LEMMA 7. *Let f satisfy the conditions of theorem* 3 *and let* $81 < d \leqq 128\frac{5}{8}$. *Then* $m_-(f) < (8d/3)^{\frac{1}{2}}$.

PROOF. Suppose $m_-(f) \geqq (8d/3)^{\frac{1}{2}}$. We first observe that theorem 2 of [7],

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together with its associated tables 1 and 2, yield $d > 96.7$, and consequently $m_- > 6.364$. Analysis as at the beginning of the proof of lemma 6 yields that $K_i < 30.244$ (i odd), $K_i < 1.17834$ (i even), $p_i = 1$ (i even), $6 \leqq p_i \leqq 29$ (i odd), $18.614 < a_{i+1} < 21.265$ (i even), $F_i > 1.0333$ (i even) and $S_i < .1451$ (i even). Once again we drop the suffixes i, $i+1$ for even i, and take $F \leqq 1+S$.

In the section $(x+\mu)^2 - a$, in order not to contradict $m_+ = 1$ or the definition of m_- we need

$$(4\tfrac{1}{2} - ||\mu - \tfrac{1}{2}||)^2 - a \geqq 1, \; a \leqq 19.25 \text{ and } (3\tfrac{1}{2} + ||\mu - \tfrac{1}{2}||)^2 - a \leqq -m_-.$$

Hence

$$||\mu - \tfrac{1}{2}|| \leqq (7 - m_-)/16 \tag{3}$$

and so $||\mu - \frac{1}{2}|| < .04$. We now proceed to exhaust all the possibilities for S.

(a) $S < .048$. This yields $36 < T(1, -1) < 37.84$ (bearing in mind that $F - 1 \leqq S$), hence with $30.25 \leqq (x+\lambda-\mu)^2 \leqq 36$ we require $||\lambda - \mu - \frac{1}{2}|| < .12$ in order to avoid a contradiction. Thus $||\lambda|| < .16$, so g takes a value at most $(.16)^2 + 19.25(1.048)(.048) < 1$, contradicting $m_+ = 1$. Hence $S \geqq .048$. But $(0, 20, 1, 6) < .048$, so $S > (0, 19, 1, 20) > .0501$.

(b) A similar argument to the above, using $g(x, 1, 2)$ and $g(x, 1, 1)$, yields that $F > 1.04$, and repetition yields $F > 1.0415$ (which gives $F > (1, 23, 1, 24) > 1.0417$) and so $S < .137$ and $p_i \leqq 23$ for all odd i. As $(0, 6, 1, \cdots) > .14$ we must therefore have $S < (0, 7, 1, 7) < .127$.

(c) $0.10 < S < .127$. This yields $K > 1.141$, $m_- > 6.79$, $a > 19.04$, and hence $||\mu - \frac{1}{2}|| < .014$ from (3). Now $33.93 < T(1, -1) < 36.02$, so we need $||\lambda - \mu|| < .09$ in order to avoid a contradiction. Hence $||2\lambda - \mu - \frac{1}{2}|| < .194$, so $39.76 < (x_1 + 2\lambda - \mu)^2 \leqq 42.25$ and $42.25 \leqq (x_2 + 2\lambda - \mu)^2 < 44.81$ for suitable x_1, x_2. One of these choices will give a contradiction as $43.7 < T(2, -1) < 48.64$. Hence $S \leqq .10$, so $S < .09167$ as in (b) of the proof of the previous lemma.

(d) $.09 < S < .09167$. In this case $K > 1.131$, $m_- > 6.74$, $a > 18.99$, and so $35.2 < T(1, -1) < 36.6$. Choosing $30.25 \leqq (x+\lambda-\mu)^2 \leqq 36$ now yields a contradiction. Hence $S \leqq .09$, which implies that $S < (0, 11, 1, 11) < .08392$.

(e) $.05 < S < .08392$. In this case we have, observing that $S \geqq .0787$ implies that $K > 1.12$, $m_- > 6.69$ and $a > 18.94$, that $35 < T(1, -1) < 37.49$. Hence choosing $30.25 \leqq (x+\lambda-\mu)^2 \leqq 36$ yields $||\lambda - \mu - \frac{1}{2}|| < .08$ in order to avoid a contradiction. Thus $||\lambda + 2\mu|| < .20$, so $33 < (x+\lambda+2\mu)^2 \leqq 36$ for suitable x. This yields a contradiction since $35 < T(1, 2) < 39$, completing the proof of the lemma.

LEMMA 8. *Let f satisfy the conditions of theorem* 3 *and let* $128\frac{5}{8} < d \leqq 192$. *Then* $m_-(f) < (8d/3)^{\frac{1}{4}}$.

PROOF. Suppose $m_-(f) \geqq (8d/3)^{\frac{1}{4}}$. By a method similar to that used in proving lemma 7 the results of [7] yield $d > 149.3$ and $m_- > 7.3565$. Again analysis as in lemma 6 yields $K_i < 37.051$ (i odd), $K_i < 1.15471$ (i even), $p_i = 1$ (i even), $7 \leqq p_i \leqq 35$ (i odd), $23.3565 < a_{i+1} < 26.254$ (i even), $F_i > 1.02779$ (i even) and $S_i < \cdot 127$ (i even). As usual we drop suffixes for even i and take $F \leqq S+1$. Treatment of the section $(x+\mu)^2 - a$ as in earlier lemmas yields that $a \leqq 24$ and

$$(4) \qquad ||\mu|| \leqq (8-m_-)/18,$$

and so $||\mu|| < .03575$. We now proceed to exhaust all possibilities for S.

(a) $0.1 < S < 0.127$. In this case $K > 1.12779$, $m_- > 7.83$, $a > 23.83$ and $42.17 < T(1, -1) < 44.384$. Hence choosing $36 \leqq (x+\lambda-\mu)^2 \leqq 42.25$ we get a contradiction unless $||\lambda-\mu|| < .046$ and

$$(5) \qquad T(1, -1) \geqq 36+m_-.$$

Now $47.75 < T(1, 2) < 49.63$, and our bounds on $||\lambda-\mu||$ and $||\mu||$ imply that $||\lambda+2\mu|| < .16$ so with $46 < (x+\lambda+2\mu)^2 \leqq 49$ we obtain a contradiction unless $||\lambda+2\mu|| < .02$ and $T(1, 2) \leqq 48$. The latter yields $F > 1.0408$, $K > 1.1408$, $m_- > 7.915$, $a > 23.915$, and so from (5) we obtain $S < .11$. Thus $S < (0, 9, 1, 9) < .1011$. But then $68 < T(2, 3) < 71$, while as $||2\lambda+3\mu|| < .08$ we have $64 \leqq (x+2\lambda+3\mu)^2 < 66$ for some x. This contradiction yields $S \leqq 0.1$, so $S < .09167$.

(b) $.0769 < S < .09167$. This implies that $K > 1.0996$, $m_- > 7.67$ and $a > 23.67$, while if $F \geqq 1.05$ we obtain $K > 1.1269$, $m_- > 7.82$ and $a > 23.82$. Now $43.567 < T(1, -1) < 46.01$. Considering $36 \leqq (x+\lambda-\mu)^2 \leqq 42.25$ if $T(1, -1) < 45.7$ and $42.25 \leqq (x+\lambda-\mu)^2 \leqq 49$ if $T(1, -1) \geqq 45.7$ yields a contradiction unless $||\lambda-\mu|| < .17$. Now $60.03 < T(2, -1) < 64.06$ if $F \geqq 1.05$: but $||2\lambda-\mu|| < .36$, so $58.3 < (x+2\lambda-\mu)^2 \leqq 64$ for some x, yielding a contradiction unless $||2\lambda-\mu|| < .18$. Hence if $F \geqq 1.05$ we have $63.1 < T(2, 3) < 69$ and $||2\lambda+3\mu|| < .22$ (as $||\mu|| < .01$ from (4)). Then either $60.5 < (x+2\lambda+3\mu)^2 \leqq 64$ or $64 \leqq (x+2\lambda+3\mu)^2 < 68$ yields a contradiction. Hence $F < 1.05$.

We now have $47.61 < T(1, 2) < 48.81$, so $42.25 \leqq (x+\lambda+2\mu)^2 \leqq 49$ yields a contradiction unless $||\lambda+2\mu|| < .12$ and $T(1, 2) \leqq 48$. But $a(5-4F+4S-3FS) < 23.1$, so $T(2, 3) < 48+23.1 = 71.1$. As $T(2, 3) > 63.1$ and $||2\lambda+3\mu|| < .26$, choosing either $59.9 < (x+2\lambda+3\mu)^2 \leqq 64$ or $64 \leqq (x+2\lambda+3\mu)^2 < 68.3$ yields a contradiction. Hence $S \leqq .0769$ which implies that $S < (0, 13, 1, 13) < .0718$.

(c) $.05 < S < .0718$. This yields $K > 1.07779$, $m_- > 7.54$, $a > 23.54$ and $||\mu|| < .028$. Now $44 < T(1, -1) < 47$ and $45 < T(1, 2) < 48.4$, so splitting up these ranges at 45.622 yields $||\lambda-\mu|| < .172$ and $||\lambda+2\mu|| < .172$ by a method similar to that which gave $||\lambda-\mu|| < .17$ in (b) above. Furthermore as $63.67 < T(2, 3) < 71.26$, working similar to that used at the end of (b) will give a contradiction unless $||2\lambda+3\mu-\frac{1}{2}|| < .2332$. Now $||\lambda+3\mu|| < .2$, while $||2\lambda+3\mu-\frac{1}{2}|| < .2332$ implies that $||\lambda+3\mu|| > .0914$, so $139 < (x+\lambda+3\mu)^2 < 141.815$ for suitable x. As $139.428 < T(1, 3) < 145.4$ we must have, in order to avoid a contradiction, $||\lambda+3\mu|| < .15$ and $T(1,3) < 140.815$. This latter implies that $a < 23.775$, so $61.5 < T(2,-1) < 66.34$. However $||2\lambda+3\mu-\frac{1}{2}|| < .2332$ yields $||2\lambda-\mu-\frac{1}{2}|| < .3452$, while $||\lambda-\mu|| < .172$, $||\lambda+2\mu|| < .172$ and $||\lambda+3\mu|| < .15$ combine to yield $||2\lambda-\mu|| < .3308$, so $58.816 < (x+2\lambda-\mu)^2 < 61.6$ for suitable x. This $g(x, 2, -1)$ contradicts either $m_+ = 1$ or $m_- > 7.54$. Hence $S \leqq .05$, so $S < (0, 20, 1, 20) < .04773$.

(d) $.02779 < S < .04773$. In this case $45 < T(1,-1) < 48$ and $45 < T(1,2) < 48$, so $||\lambda-\mu|| < .18$ and $||\lambda+2\mu|| < .18$ by a method similar to that used in (c). These imply $||\lambda+\mu|| < .18$, so $(x+\lambda+\mu)^2 < .033$ for suitable x. Hence $T(1,1) > .969$ to avoid contradicting $m_+ = 1$. This implies $F > 1.03844$, so $S > .03844$, $K > 1.0768$, $m_- > 7.536$ and $a > 23.536$. Then $140 < T(1,3) < 144.63$ and $139.61 < T(1,-2) < 144$, where the lower bound can be raised to 140 in the latter case unless both $a < 23.61$ and $S > .042$, in which case $T(2,-1) < 66.7$.

Suppose firstly that $T(1,-2) > 140$. Then as $||\lambda+3\mu|| < .21$ and $||\lambda-2\mu|| < .21$ we can choose corresponding squares between 139 and 144. These give a contradiction unless $||\lambda+3\mu|| < .13$ and $||\lambda-2\mu|| < .13$. Combining these, since $||\mu|| < .03$, yields that $||2\lambda-\mu|| < .26$, so $59.9 < (x_1+2\lambda-\mu)^2 \leqq 64$ and $64 \leqq (x_2+2\lambda-\mu)^2 < 68.3$ for suitable x_1, x_2. One of these choices gives a contradiction as $64 < T(2,-1) < 70$.

The second case is dealt with similarly – we obtain $||\lambda+3\mu|| < .13$, $||\lambda-2\mu|| < .143$, $||2\lambda-\mu|| < .286$, so $59.5 < (x+2\lambda-\mu)^2 \leqq 64$ for suitable x. This gives a contradiction since $64 < T(2,-1) < 66.7$. This completes the proof of lemma 8.

Lemma 9. *Let f satisfy the conditions of theorem* 3 *and let* $192 < d \leqq 273\frac{3}{8}$. *Then either f is equivalent to $f_2(x,y,z)$ or $m_-(f) < (8d/3)^{\frac{1}{3}}$.*

Proof. Suppose $m_-(f) \geqq (8d/3)^{\frac{1}{3}}$. By a method similar to that used in earlier lemmas we have $d > 220.5$, $m_- > 8.377$, $K_i < 44.0906$ (i odd), $K_i < 1.13054$ (i even), $p_i = 1$ (i even), $9 \leqq p_i \leqq 43$ (i odd), $28.627 < a_{i+1} < 31.633$ (i even), $F_i > 1.0227$ (i even) and $S_i < .101$ (i even). As usual we drop the suffixes for even i and take $F \leqq S+1$. Then treatment of the section $(x+\mu)^2-a$ as in earlier lemmas yields $a \leqq 29.25$ and $||\mu-\frac{1}{2}|| \leqq (9-m_-)/20$, from which we have $||\mu-\frac{1}{2}|| < .032$. We now proceed to eliminate all possibilities for S except that giving f_2.

(a) $S < .0457$. We have $55.88 < T(1,2) < 60.06$ and $55.25 < T(1,-1) < 57.821$. Choosing corresponding squares between 49 and 56.25 yields a contradiction unless $||\lambda+2\mu|| < .19$ and $||\lambda-\mu|| < .032$. However these combine to give $||3\mu|| < .222$, plainly contradicting $||\mu-\frac{1}{2}|| < .032$. Hence $S \geqq .0457$, so $S > (0, 20, 1, 21) > .0477$.

(b) $0.477 < S < .101$. In this case $K > 1.0704$, so $m_- > 8.55$ and $a > 28.8$. We have $55.26 < T(1,2) < 60$, so $||\lambda+2\mu|| < .19$ as above. Also $52.573 < T(1,-1) < 57.05$, so with $49 \leqq (x+\lambda-\mu)^2 \leqq 56.25$ we see that (i) $T(1,-1) \leqq 55.25$ to avoid a contradiction similar to that in (a), and (ii) $||\lambda-\mu-\frac{1}{2}|| < .181$.

Now $T(2, -1) = 2T(1, -1) - a(1+2FS)$ and $a(1+2FS) > 31.49$, so $T(2, -1) < 79.01$. Suppose that $T(2, -1) > 71.25$. Then $\|2\lambda-\mu\| < .225$ else either $72.25 \leqq (x+2\lambda-\mu)^2 < 77$ or $67.65 < (x+2\lambda-\mu)^2 \leqq 72.25$ will yield a contradiction. This implies that $\|\lambda-\mu-\frac{1}{2}\| > .121$, so we can replace (i) above by $T(1, -1) < 53.45$, yielding $T(2, -1) < 75.41$. Repeating this cycle eventually leads to $\|\lambda-\mu-\frac{1}{2}\| > .182$, contradicting an earlier bound. We therefore have $T(2, -1) \leqq 71.25$, $S > .0938$, so $S > (0, 9, 1.0227) > .10022$. Then $F < 1.03032$. $K > 1.12292$, $m_- > 8.948$, $a > 29.198$ and $\|\lambda-\frac{1}{2}\| < .0026$. Now $70.957 < T(2, -1) \leqq 71.25$, so $64 \leqq (x+2\lambda-\mu)^2 \leqq 72.25$ yields $\|2\lambda-\mu\| < .018$, which in conjunction with the bounds on $\|\mu-\frac{1}{2}\|$ and $\|\lambda+2\mu\|$ yields $\|\lambda\| < .0103$. Then $\|5\lambda+6\mu\| < .07$, so $167 < (x+5\lambda+6\mu)^2 \leqq 169$ for suitable x, giving a contradiction, as $161 < T(5, 6) < 168.7$, unless $T(5, 6) < 168$. Hence $F > 1.02298$, so $F \geqq (\overline{1, 42, 1, 9}) > 1.0233$ (as $(1, 43, 1, 9, 1, 8) < 1.0229$, and $p_{i+3} = 9$ on applying the results so far to the point $i+2$ with the chain reversed). In addition $181.53 < T(6, 7) < 191.38$, and as $\|6\lambda+7\mu-\frac{1}{2}\| < .08$ suitable choice of x yields a contradiction unless $T(6, 7) > 191.198$. This implies $F < 1.0235$, and as $(1, 41, 1, 10) > 1.0238$ we must have $F = (\overline{1, 42, 1, 9})$. Reversing the chain about $i-2$ and applying these results gives $S = (0, \overline{9, 1, 42, 1})$.

That $a = 29.25$ and $\|\lambda+\mu-\frac{1}{2}\| = 0$ follows on observing that $0 < g(x, 1, 1) < 1$ unless equality holds in $(x+\lambda+\mu)^2 \leqq \frac{1}{4}$ and $T(1, -1) = a/39 \leqq \frac{3}{4}$. Similarly $\|10\lambda-\mu-\frac{1}{2}\| = 0$, which when added to $\|\lambda+\mu-\frac{1}{2}\| = 0$ and compared with $\|\lambda\| < .0103$ yields $\|\lambda\| = 0$. Then $\|\mu-\frac{1}{2}\| = 0$ and the form is f_2, as desired. The proof that $m_+(f_2) = 1$ and $m_-(f_2) = 9$ is left till later.

LEMMA 10. *Let f satisfy the conditions of theorem 3 and let $273\frac{3}{8} < d \leqq 375$. Then $m_-(f) < (8d/3)^{\frac{1}{3}}$.*

PROOF. Suppose $m_-(f) \geqq (8d/3)^{\frac{1}{3}}$. Then by the usual method we get $d > 314.1$, $m_- > 9.4263$, $K_i < 51.64$ (i odd), $K_i < 1.1066$ (i even), $p_i = 1$ (i even), $11 \leqq p_i \leqq 49$ (i odd), $34.4263 < a_{i+1} < 37.241$ (i even), $F_i > 1.02$ (i even) and $S_i < .0866$ (i even). As usual we drop the suffixes for even i and take $F \leqq S+1$. Then treatment of the section $(x+\mu)^2 - a$ as in earlier lemmas yields $a \leqq 35$ and $\|\mu\| < (10-m_-)/22$ from which we get $\|\mu\| < .0261$. We now proceed to eliminate all possibilities for S.

(a) $S > .0621$. In this case we have $K > 1.0821$, $m_- > 9.816$, $\|\mu\| < .01$ and $a > 34.816$. Then $64.18 < T(1, -1) < 67.14$, so with $56.25 \leqq (x+\lambda-\mu^2) \leqq 64$ we must have $\|\lambda-\mu-\frac{1}{2}\| < .075$ to avoid a contradiction. This gives $\|2\lambda+3\mu\| < .20$, and so either $96 < (x+2\lambda+3\mu)^2 \leqq 100$ or $100 \leqq (x+2\lambda+3\mu)^2 < 104.1$ yields a contradiction as $99 < T(2, 3) < 107$. Hence $S \leqq .0621$, so $S \leqq (0, \overline{16, 1})$, $< .05903$.

(b) $0.05 < S < .0591$. Analysis as in (a) yields $K > 1.07$, $m_- > 9.725$ and $\|\mu\| < .013$. If $F < 1.03$ we have $63 < T(1, -1) > 67.5$ and so with $56.25 \leqq$

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$(x+\lambda-\mu)^2 \leqq 64$ we require $\|\lambda-\mu-\frac{1}{2}\| < .101$ to avoid a contradiction. Then $\|2\lambda+3\mu\| < .267$ and as $100 < T(2, 3) < 105$ we obtain a contradiction as in (a). Hence $F \geqq 1.03$. Then $93.7199 < T(2, -1) < 98$, and with $90.25 \leqq (x+2\lambda-\mu)^2 \leqq 100$ we require $\|2\lambda-\mu\| < .268$ to avoid a contradiction. Thus $\|2\lambda+3\mu\| < .32$, so $93.7 < (x+2\lambda+3\mu)^2 \leqq 100$ for suitable x, and as $96.17 < T(2, 3) < 102.6$ we get a contradiction unless $T(2, 3) \leqq 99$. Because of the relation between F, K, m_- and a this last inequality yields $F > 1.042$, $m_- > 9.89$ and $a > 34.89$, $T(2, 3) > 96.834$, $\|2\lambda+3\mu\| < .11$ and $\|\mu\| < .005$. That $\|\lambda-\frac{1}{2}\|$ is small comes from consideration of $g(x, 1, -1)$, so we must have $\|\lambda-\frac{1}{2}\| < .063$. Then $\|3\lambda-\mu-\frac{1}{2}\| < .194$, and as $118.24 < T(3, -1) < 123.4$ we get a contradiction unless $T(3, -1) > 120.14$. This is true only if $S < .0583$, so $S \leqq (0, \overline{17, 1}) < .0558$ as $(0, 16, 1, 50) > .0588$.

From the above we can deduce that $138.8 < T(4, -1) < 146$ and that $137 < (x+4\lambda-\mu)^2 \leqq 144$ for suitable x, so we get a contradiction unless $T(4, -1) \leqq 143$. This implies that $S > .051$, so $S > (0, 18, 1, 50) > .05268$, giving $143 < T(4, 5) < 153.8$. But the bounds on $\|2\lambda+3\mu\|$ and $\|\mu\|$ imply that $\|4\lambda+5\mu\| < .225$, so $138.65 < (x_1+4\lambda+5\mu)^2 \leqq 144$ and $144 \leqq (x_2+4\lambda+5\mu)^2 < 149.46$ for suitable choices of x_1, x_2. One of these choices gives a contradiction. Thus $S \leqq .05$, so $S < (0, 20, 1, 20) < .04773$.

(c) $.04 < S < .04773$. This yields $m_- > 9.652$, $a > 34.652$, $\|\mu\| < .016$ and $66.4 < T(1, -1) < 68.55$, so $\|\lambda-\mu-\frac{1}{2}\| < .175$ to avoid a contradiction. Now $95.28 < T(2, -1) < 99.2$ so with $90.25 \leqq (x+2\lambda-\mu)^2 \leqq 100$ we deduce $\|2\lambda-\mu\| < .2$. Thus $\|2\lambda+3\mu\| < .264$, so $94.78 < (x+2\lambda+3\mu)^2 \leqq 100$ for some x. As $97.02 < T(2, 3) < 104.01$ we get a contradiction unless $\|2\lambda+3\mu\| < .1$ and $T(2, 3) \leqq 99$. Then $F > 1.0362$, so $144 < T(4, -1) < 151.8$. One of the values $(12-\delta)^2-T(4, -1)$, $(12+\delta)^2-T(4, -1)$ yields a contradiction if $\|4\lambda-\mu\| = \delta < (1+m_-)/48$, so $\|4\lambda-\mu\| > .221$. Hence $\|\lambda-\frac{1}{2}\| > .05$ and $\|2\lambda+3\mu\| > .054$. This decreases our upper bound on $T(2, 3)$ to 98.0 yielding $F > 1.04$. This gives $K > 1.08$, $m_- > 9.78$, $\|\mu\| < .01$, $\|4\lambda-\mu\| > .224$, $\|2\lambda+3\mu\| > .077$ and so on — this iteration eventually yields $F > 1.048$ which is impossible as $F \leqq S+1$ and $S < .048$. Hence $S \leqq .04$, so $S < (0, 25, 1, 25) < .0386$.

(d) $.03 < S < .0386$. Following the method of (c) we obtain $m_- > 9.575$, $a > 34.575$, $\|\mu\| < .02$, $67.14 < T(1, -1) < 68.919$, $\|\lambda-\mu-\frac{1}{2}\| < .204$ and $96.99 < T(2, -1) < 100.68$. But $\|2\lambda-\mu\| < .428$, so $91.6 < (x+2\lambda-\mu)^2 \leqq 100$ for suitable x Hence $T(2, -1) \leqq 99$ and $\|2\lambda-\mu\| <$ 102. Following (c) again we have $\|2\lambda+3\mu\| < .182$, $98.18 < T(2, 3) < 103.5$, $\|2\lambda+3\mu\| < .042$, $T(2, 3) \leqq 99$, $F > 1.0321$, $S > .0321$, $a(3+2S) > 105.944$ and after a couple of iterations $F > 1.033$. Thus $K > 1.066$, $m_- > 9.697$, $a > 34.697$, $a(3+2S) > 106.381$, $F > 1.0346$. Then $F > (1, 27, 1, 50) > 1.0357$, so $K > 1.0714$ and $m_- > 9.725$. Noting that $a > 34.8$ implies that $F > 1.0368$ to keep $T(2, 3) \leqq 99$ we have $147 < T(4, -1) < 153.64$. But $\|2\lambda-\mu\| < .102$ and $\|\mu\| < (10-m_-)/22 < .013$ combine to yield $\|4\lambda-\mu\| < .217$. As $.217 < (1+m_-)/48$ we can now obtain a contradiction

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as in (c). Hence $S < .03$, so $S(0, 33, 1, 33) < .0295$.

(e) $.02 < S < .0295$. As $99 < T(2, 3) < 102.79$ we obtain a contradiction by choosing x such that $90.25 \leqq (x+2\lambda+3\mu)^2 \leqq 100$ unless $||2\lambda+3\mu-\frac{1}{2}|| < .163$. This implies that $||2\lambda-\mu-\frac{1}{2}|| < .27$, so $90.25 \leqq (x+2\lambda-\mu)^2 < 96$ for some x. As $98.376 < T(2, -1) < 102.144$ we obtain a contradiction unless $||2\lambda-\mu-\frac{1}{2}|| < .1291$. Hence $||\lambda-\frac{1}{2}|| > .1724$, so $(x+\lambda)^2 < .1074$ for suitable x. Then $aFS > .8926$, yielding $S > .0248$. A similar treatment yields $||\lambda-\mu-\frac{1}{2}|| > .1549$, $a(F-1)(S+1) > .8809$ and $F > 1.0244$. Hence $K > 1.0492$, $m_- > 9.57$ and $||\mu|| < .02$. Now $T(2, -1) < 101.45$ and analysis as above gives $||2\lambda-\mu-\frac{1}{2}|| < .086$. This on combining with $||\mu|| < .02$ yields $||3\lambda-\mu-l/4|| < .14$ for $l = 1$ or $l = -1$, so $123.4 < (x+3\lambda-\mu)^2 < 129.7$ for suitable x. But $128.3 < T(3, -1) < 132.01$ so we obtain a contradiction unless $T(3, -1) < 128.7$. Thus $a < 34.7$, $T(2, -1) < 99.88$, $||2\lambda-\mu|| < .01$, $||3\lambda-\mu-l/4|| < .03$ and $125.8 < (x+3\lambda-\mu)^2 < 127.23$ for some x, yielding a contradiction as required.

LEMMA 11. *Let f satisfy the conditions of theorem* 3 *and let* $375 < d \leqq 499\frac{1}{8}$. *Then* $m(f) < (8d/3)^{\frac{1}{2}}$.

PROOF. Suppose $m_-(f) \geqq (8d/3)^{\frac{1}{2}}$. Then by the usual method we have $d > 435.06$, $m_- > 10.507$, $K_i < 59.578$ (i odd), $K_i < 1.08322$ (i even), $p_i = 1$ (i even), $12 \leqq p_i \leqq 57$ (i odd), $40.757 < a_{i+1} < 43.25$ (i even), $F_i > 1.01724$ (i even) and $S_i < .066$ (i even). As usual we drop the suffixes for even i and take $F \leqq S+1$. The usual treatment of $(x+\mu)^2-a$ yields $a \leqq 41.25$ and $||\mu-\frac{1}{2}|| \leqq (11-m_-)/24$, so $||\mu-\frac{1}{2}|| < .021$. We now proceed to exhaust all possibilities for S.

(a) $.05 < S < .066$. In this case we have $K > 1.06724$, $m_- > 10.87$, $a > 41.12$ and $||\mu-\frac{1}{2}|| < .006$. As $81.4 < T(1, 2) < 83.754$ we obtain, with $72.25 \leqq (x+\lambda+2\mu)^2 \leqq 81$, a contradiction unless $||\lambda+2\mu-\frac{1}{2}|| < .038$. Then $||2\lambda-\mu-\frac{1}{2}|| < .106$, so $107 < (x+2\lambda-\mu)^2 \leqq 110.25$, which yields a contradiction, as $108 < T(2, -1) < 114$, unless $T(2, -1) \leqq 109.25$. This is true only if $S > .06222$, so $F < 1.021$ by our bound on K. Then we have a contradiction as $a(F-1)(S+1) < .93$ while $||\lambda+\mu|| < .05$ implies that $(x+\lambda+\mu)^2 < .003$ for suitable x. Hence $S \leqq .05$, so $S < .04773$.

(b) $.04 < S < .04773$. Analysis as in (a) yields $m_- > 10.77$, $a > 41.02$, $||\mu-\frac{1}{2}|| < .01$ and $80.1 < T(1, 2) < 83.03$, the lower bound being obtained by observing that if $F > 1.035$ then $a > 41.12$ as in (a). Then choosing x with $72.25 \leqq (x+\lambda+2\mu)^2 \leqq 81$ yields a contradiction as $T(1, 2) > 83.01$ only if $a > 41.24$ which implies that $m_- > 10.8$. Hence $S \leqq .04$, so $S < .0386$.

(c) $S < .0386$. In this case $80 < T(1, 2) < 82.66$, where the lower bound is obtained by observing that if $F > 1.038$ then $a > 41.12$ as in (b). Then choosing x with $72.25 \leqq (x+\lambda+2\mu)^2 \leqq 81$ yields a contradiction to either $m_+ = 1$ or $m_- > 10.5$.

LEMMA 12. *Let f satisfy the conditions of theorem* 3 *and let* $499\frac{1}{8} < d \leqq 648$. *Then* $m_-(f) < (8d/3)^{\frac{1}{2}}$.

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PROOF. Suppose $m_-(f) \geqq (8d/3)^{\frac{1}{2}}$. Then by the usual method we have $d > 587.313$, $m_- > 11.613$, $K_i < 64.613$ (i odd), $K_i < 1.0607$ (i even), $p_i = 1$ (i even), $21 \leqq p_i \leqq 62$ (i odd), $47.613 < a_{i+1} < 49.36$ (i even), $F_i > 1.0158$ (i even) and $S_i < .045$ (i even). As usual we drop the suffixes for even i and take $F \leqq S+1$. Since $(0, 21, 1) > .045$ we have $S < (0, 22, 1, 22) < .0436$. Furthermore $K > 1.0316$ implies that $m_- > 11.733$ and $a > 47.733$. Treatment of the section $(x+\mu)^2 - a$ as in earlier lemmas yields $a \leqq 48$ and $||\mu|| \leqq (12-m_-)/26$, so $||\mu|| < .011$. We proceed to eliminate various ranges for S.

(a) $.032 < S < .0436$. Then $m_- > 11.88$ and $92.02 < T(1, -1) < 94.262$, so with $81 \leqq (x+\lambda-\mu)^2 \leqq 90.25$ we obtain a contradiction unless $||\lambda-\mu|| < .077$. As $||\mu|| < .005$ we have $||\lambda+\mu|| < .087$, so $(x+\lambda+\mu)^2 < .008$ for some x. Hence $a(F-1)(S+1) > .992$, so $F > 1.0198$, implying that $S < .041$. Now $||3\lambda-\mu|| < .241$, so $162.79 < (x_1+3\lambda-\mu)^2 \leqq 169$ and $169 \leqq (x_2+3\lambda-\mu)^2 < 175.33$ for suitable x_1, x_2. However $170.4 < T(3, -1) < 177.31$, so one of the values $g(x_1, 3, -1)$, $g(x_2, 3, -1)$ yields a contradiction. Thus $S \leqq .032$, so $S < (0, 31, 1, 31) < .0313$.

(b) $.0158 < S < .0313$. Following the method of (a) we have $93.1 < T(1, -1) < 95.23$, $||\lambda-\mu|| < .14$, $||\lambda+\mu|| < .162$, $a(F-1)(S+1) > .973$ and $F > 1.0196$. Similarly $||\lambda|| < .151$, $aFS > .977$ and $S > .0199$. Hence $K > 1.0395$ and $m_- > 11.8$. Now if $S \geqq .0253$ we have $175.7 < T(3, -1) < 180.8$ where the lower bound may be increased to 177.12 if $S < .029$ and the upper bound decreased to 179.11 if $S \geqq .029$. If $S \geqq .029$ we have $K > 1.048$, $m_- > 11.88$, $T(1, -1) < 94.57$ and $||\lambda-\mu|| < .095$. In this case $||3\lambda-\mu|| < .295$, so $169 \leqq (x+3\lambda-\mu)^2 < 176.76$ for suitable x, giving a contradiction. If $.0253 \leqq S < .029$ we have $T(1, -1) < 94.77$ and $||\lambda-\mu|| < .11$. Then $||3\lambda-\mu|| < .346$, so $169 \leqq (x+3\lambda-\mu)^2 < 178.12$ for suitable x, giving a contradiction.

Hence $S < .0253$, so $S < (0, 39, 1, 39) < .02502$. Then $179.2 < T(3, -1) < 183.24$, while as $T(1, -1) < 95.03$ we have $||\lambda-\mu|| < .124$, $||3\lambda-\mu|| < .388$ and so $169 \leqq (x+3\lambda-\mu)^2 < 179.3$ for suitable x. To avoid a contradiction we must have $||3\lambda-\mu|| < .095$, and this yields $||\lambda|| < .035$. Considering $g(x, 1, 0)$ as above now yields $F > 1.0202$, so $255.1 < T(5, -1) < 264$. But $||5\lambda-\mu|| < \frac{1}{3}(5(.095)+2(.008)) < .166$ since λ and μ are small, so for suitable choices of x_1 and x_2 we have $250 < (x_1+5\lambda-\mu)^2 \leqq 256$ and $256 \leqq (x_2+5\lambda-\mu)^2 < 262$. One of these choices gives a contradiction.

LEMMA 13. *Let f satisfy the conditions of theorem 3 and let $648 < d \leqq 823\frac{7}{8}$. Then $m_-(f) < (8d/3)^{\frac{1}{2}}$.*

PROOF. Suppose $m_-(f) \geqq (8d/3)^{\frac{1}{2}}$. By the usual method we have $d > 776.08$, $m_- > 12.74$, $K_i < 76.55$ (i odd), $K_i < 1.0391$ (i even), $p_i = 1$ (i even), $32 \leqq p_i \leqq 74$ (i odd), $54.99 < a_{i+1} < 55.923$ (i even), $F_i > 1.0133$ (i even) and $S_i < .0258$ (i even). As usual we drop the suffixes for even i and take $F \leqq S+1$. Then treatment of the section $(x+\mu)^2 - a$ in the usual manner yields $a \leqq 55.25$ and

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$||\mu-\frac{1}{2}|| \leqq (13-m_-)/28$. As $K > 1.0266$ we have $m_- > 12.86$, $a > 55.11$ and $||\mu-\frac{1}{2}|| < .005$. Now $108 < T(1, -1) < 109.76$ so with $100 \leqq (x+\lambda-\mu)^2 \leqq 110.25$ we obtain a contradiction unless $||\lambda-\mu-\frac{1}{2}|| < .06$. But $109 < T(2, 1) < 111$, so with $100 \leqq (x+\lambda+2\mu)^2 \leqq 110.25$ we obtain a contradiction unless $||\lambda+2\mu-\frac{1}{2}|| < .01$. Then $||3\mu|| < .06+.01 = .07$, contradicting $||\mu-\frac{1}{2}|| < .005$.

This now completes the proof of theorem 3 apart from showing that $m_+(f_2) = 1$ and $m_-(f_2) = 9$.

LEMMA 14. *Let f_2 be defined as in theorem 1. Then $m_+(f_2) = 1$ and $m_-(f_2) = 9$.*

PROOF. As $f_2(x, y, z) = x^2+xy+y^2+xz+32yz-29z^2$ it is only necessary to show that f_2 cannot take any of the values $0, -1, -2, -3, -4, -5, -6, -7$ and -8, since $f_2(4, 0, 1) = -9$. The values -1, -3, -4 and -7 are eliminated by observing that $f_2 \equiv (x-4y-4z)^2+3y^2 \pmod 9$. As $f_2 \equiv x^2+xz+z^2 \pmod 5$ after replacing z by $z-y$ it follows that $f_2 \equiv 0 \pmod 5$ iff $x = 5X$ and $z = 5Z$ for some integers, X, Z. Then $\frac{1}{5}f_2 \equiv 3y^2 \pmod 5$, which implies that f_2 does not take the value -5, whilst f_2 can take the value zero only at points $(x, y, z) = 5(X, Y, Z)$, which are not primitive. This implies f_2 cannot take the value 0 at all.

The remaining even values are eliminated by considering congruencees modulo powers of 2 as follows. We have $4f_2 \equiv (x+2y)^2+3(x+2z)^2 \pmod 8$ so f_2 is even only if x is even. Writing $x = 2X$ yields $f_2 \equiv (X+y)^2+3(X+z)^2+4Xz \pmod{32}$, so $f_2 \equiv 2 \pmod 4$ is impossible. This eliminates the values -2 and -6. Plainly $f_2 \equiv 0 \pmod 8$ only if y and z have the same parity. For y, z both even, say $y = 2Y$, $z = 2Z$, f_2 cannot take the value -8 at (x, y, z) else f_2 would take the value -2 at (X, Y, Z), which we know is impossible. Hence if $f_2 = -8$ then y and z are both odd. It is now clear that we must have $y-z \equiv 2 \pmod 4$ and X odd to ensure $f_2 = -8$ as otherwise $f_2 \equiv 4 \pmod 8$. Substituting $x = 2m+1$, $y = 2n+1$, $z = 2n+3+4s$ yields

$$f_2 = 16\,(m^2+3mn+n^2+5m-4n-29s+5ms-13ns-33s-8),$$

showing that f_2 cannot take the value -8.

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