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Abstract

We investigate the distribution of real algebraic numbers of a fixed degree that have a close conjugate number, with the distance between the conjugate numbers being given as a function of their height. The main result establishes the ubiquity of such algebraic numbers in the real line and implies a sharp quantitative bound on their number. Although the main result is rather general, it implies new estimates on the least possible distance between conjugate algebraic numbers, which improve recent bounds obtained by Bugeaud and Mignotte. So far, the results à la Bugeaud and Mignotte have relied on finding explicit families of polynomials with clusters of roots. Here we suggest a different approach in which irreducible polynomials are implicitly tailored so that their derivatives assume certain values. We consider some applications of our main theorem, including generalisations of a theorem of Baker and Schmidt and a theorem of Bernik, Kleinbock and Margulis in the metric theory of Diophantine approximation.

1. Introduction

1.1 Separation of conjugate algebraic numbers

The question 'How close to each other can two conjugate algebraic numbers of degree n be?' crops up in a variety of problems in number theory as well as in some applications. Over the past 50 years or so, a number of upper and lower bounds have been found for this distance. However, exact answers are known only in the degree-two and degree-three cases. In order to set the scene for our discussion, we first introduce some quantities.

Throughout this paper we deal with algebraic numbers in $\mathbb{C}$, the set of complex numbers. Let $n \geqslant 2$. Recall that two complex algebraic numbers are said to be *conjugate* (over $\mathbb{Q}$) if they are roots of the same irreducible (over $\mathbb{Q}$) polynomial with rational integer coefficients. Define κ_n (respectively, κ_n^*) to be the infimum of κ such that the inequality

$$|\alpha_1 - \alpha_2| > H(\alpha_1)^{-\kappa}$$

holds for arbitrary conjugate algebraic numbers (respectively, algebraic integers) $\alpha_1 \neq \alpha_2$ of degree n with sufficiently large height $H(\alpha_1)$. Here and elsewhere $H(\alpha)$ denotes the height of an algebraic number α, which is the absolute height of the minimal polynomial of α over $\mathbb{Z}$. Clearly, $\kappa_n^* \leqslant \kappa_n$ for all n.

In [Mah64], Mahler established the upper bound $\kappa_n \leqslant n-1$, which is apparently the best estimate to date. It is an easy exercise to show that $\kappa_2 = 1$ (see, e.g., [BM09]). Furthermore, Evertse proved in [Eve04] that $\kappa_3 = 2$. In the case of algebraic integers, $\kappa_2^* = 0$ and $\kappa_3^* \geqslant 3/2$.

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The bound $\kappa_3^* \geqslant 3/2$ has been proved by Bugeaud and Mignotte [BM09], who also showed that the equality $\kappa_3^* = 3/2$ is equivalent to Hall's conjecture on the difference between integers x^3 and y^2. The latter is known to be a special case of the *abc*-conjecture of Masser and Oesterlé; see [BM09] for further details and references.

For $n > 3$, estimates for κ_n are less satisfactory. At first, Mignotte showed in [Mig83] that $\kappa_n, \kappa_n^* \geqslant n/4$ for all $n \geqslant 3$. Recently, Bugeaud and Mignotte [BM04, BM09] have shown that

$$\begin{array}{lll} \kappa_n \geqslant n/2 & \text{when } n \geqslant 4 & \text{is even,} \\ \kappa_n^* \geqslant (n-1)/2 & \text{when } n \geqslant 4 & \text{is even,} \\ \kappa_n \geqslant (n+2)/4 & \text{when } n \geqslant 5 & \text{is odd,} \\ \kappa_n^* \geqslant (n+2)/4 & \text{when } n \geqslant 5 & \text{is odd.} \end{array}$$

The above results are obtained by presenting explicit families of irreducible polynomials of degree n whose roots are close enough. Bugeaud and Mignotte [BM09] pointed out that 'at present there is no general theory for constructing integer polynomials of degree at least four with two roots close to each other'. In this paper we shall make an attempt to address this issue. One particular consequence of our results is the following theorem that improves the lower bounds of Bugeaud and Mignotte in the apparently more difficult case of odd n.

Theorem 1. *For any $n \geqslant 2$ we have* $\min\{\kappa_n, \kappa_{n+1}^*\} \geqslant (n+1)/3$.

Theorem 1 will follow from a more general counting result, namely Corollary 2 below. In fact, a lot more is established. We show that algebraic numbers of degree n (or algebraic integers of degree $n+1$) with a close conjugate form a 'highly dense' (ubiquitous) subset in the real line; see Theorem 2.

1.2 The distribution of close conjugate algebraic numbers

First, let us introduce some notation. Throughout this paper, $\#S$ will stand for the cardinality of S and λ will denote Lebesgue measure in $\mathbb{R}$. Given an interval $J \subset \mathbb{R}$, $|J|$ will denote the length of J. Also, $B(x, \rho)$ denotes the interval in $\mathbb{R}$ centred at x of radius ρ. By $\ll$ and $\gg$ we will mean the Vinogradov symbols with implicit constant depending on n only. We shall write $a \asymp b$ when the inequalities $a \ll b$ and $a \gg b$ hold simultaneously.

Let $n \geqslant 2$ be an integer, and let $\mu \geqslant 0$, $0 < \nu < 1$ and $Q > 1$. Let $\mathbb{A}_{n,\nu}(Q, \mu)$ be the set of algebraic numbers $\alpha_1 \in \mathbb{R}$ of degree n and height $H(\alpha_1)$ which satisfy

$$\nu Q \leqslant H(\alpha_1) \leqslant \nu^{-1} Q \tag{1}$$

and

$$\nu\, Q^{-\mu} \leqslant |\alpha_1 - \alpha_2| \leqslant \nu^{-1} Q^{-\mu} \quad \text{for some } \alpha_2 \in \mathbb{R} \text{ that is conjugate to } \alpha_1. \tag{2}$$

Similarly, we define $\mathbb{A}^*_{n,\nu}(Q, \mu)$ to be the set of algebraic integers $\alpha_1 \in \mathbb{R}$ of degree $n+1$ and height $H(\alpha_1)$ that satisfy (1) and (2). Before stating our main result, let us agree that $\mathbb{A}^\circ_{n,\nu}(Q, \mu)$ will refer to either of the sets $\mathbb{A}_{n,\nu}(Q, \mu)$ and $\mathbb{A}^*_{n,\nu}(Q, \mu)$.

Theorem 2. *For any $n \geqslant 2$ there is a constant $\nu > 0$ which depends only on n and has the following property: for any μ satisfying*

$$0 < \mu \leqslant \frac{n+1}{3} \tag{3}$$

and any interval $J \subset [-1/2, 1/2]$,

$$\lambda\Bigg(\bigcup_{\alpha_1\in\mathbb{A}^\circ_{n,\nu}(Q,\mu)} B(\alpha_1, Q^{-n-1+2\mu}) \cap J\Bigg) \geqslant \frac{3}{4}|J| \tag{4}$$

for all sufficiently large Q.

Remark 1. The constant 3/4 in the right-hand side of (4) is not 'critical' and can be replaced by any positive number less than 1.

Remark 2. In principle, the above theorem holds for $\mu = 0$, although in this case ensuring that α_2 is real becomes more delicate. Also note that in the case of $\mu = 0$, the optimal distribution of real algebraic numbers of degree n (respectively, algebraic integers of degree $n+1$) was first established in [Ber99] (respectively, in [Bug02]). The above result is stated for the unit symmetric interval $[-1/2, 1/2]$. However, using shifts by an integer, it can be extended to an arbitrary interval in $\mathbb{R}$; see [Ber99] for appropriate techniques.

Corollary 1. *For any* $n \geqslant 2$ *there is a positive constant* ν *which depends only on* n *and is such that for any* μ *satisfying (3) and any interval* $J \subset [-1/2, 1/2]$,

$$\#(\mathbb{A}^\circ_{n,\nu}(Q,\mu) \cap J) \geqslant \tfrac{1}{2} Q^{n+1-2\mu}|J| \tag{5}$$

for all sufficiently large Q.

Proof. Obviously, if $B(\alpha_1, Q^{-n-1+2\mu}) \cap J/2 \neq \varnothing$, then $\alpha_1 \in J$ provided that Q is sufficiently large. Then, using (4), we obtain

$$\#(\mathbb{A}^\circ_{n,\nu}(Q,\mu) \cap J) 2Q^{-n-1+2\mu} \geqslant \lambda\Bigg(\bigcup_{\alpha_1\in\mathbb{A}^\circ_{n,\nu}(Q,\mu)} B(\alpha_1, Q^{-n-1+2\mu}) \cap \frac{1}{2}J\Bigg) \overset{(4)}{\geqslant} \frac{1}{4}|J|,$$

from which (5) readily follows. □

Corollary 2. *Let* $n \geqslant 2$. *Then for all sufficiently large* $Q > 1$ *there are* $\gg Q^{(n+1)/3}$ *real algebraic numbers* α_1 *of degree* n *(or real algebraic integers* α_1 *of degree* $n+1$*) with height* $H(\alpha_1) \asymp Q$ *such that*

$$|\alpha_1 - \alpha_2| \asymp Q^{-(n+1)/3} \quad \text{for some } \alpha_2 \in \mathbb{R} \text{ that is conjugate to } \alpha_1. \tag{6}$$

Corollary 2 follows from Corollary 1 upon taking μ to be $(n+1)/3$. As a consequence of Corollary 2, we obtain Theorem 1.

2. Auxiliary lemmas

The following statement, established in [Ber09, Theorem 5.8], is the crucial ingredient of the proofs of all the results in this paper.

Lemma 1. *Let* $f_0, \dots, f_n$ *be real analytic functions defined on an interval* $I \subset \mathbb{R}$ *which are linearly independent over* $\mathbb{R}$. *Let* $x_0 \in I$ *be a point where the Wronskian* $W(f_0, \dots, f_n)(x_0)$ *is non-zero. Then there exist an interval* $I_0 \subset I$ *centred at* x_0 *and positive constants* C *and* α *that satisfy the following property: for any interval* $J \subset I_0$ *there is a constant* $\delta = \delta_J$ *such that for any positive* $\theta_0, \dots, \theta_n$,

$$\lambda\left\{x \in J : \begin{array}{l} \exists (a_0, \dots, a_n) \in \mathbb{Z}^{n+1} \setminus \{0\} \text{ satisfying} \\ |a_0 f_0^{(i)}(x) + \dots + a_n f_n^{(i)}(x)| < \theta_i \ \forall i = 0, \dots, n \end{array}\right\} \leqslant C\left(1 + \left(\frac{\Theta}{\delta}\right)^{\alpha}\right)\theta^{\alpha}|J| \tag{7}$$

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where

$$\theta = (\theta_0 \cdots \theta_n)^{1/(n+1)} \quad \textit{and} \quad \Theta := \max_{1 \leqslant r \leqslant n} \frac{\theta_0 \cdots \theta_{r-1}}{\theta^r}. \tag{8}$$

With a view to the applications we have in mind, we now estimate Θ.

Lemma 2. *Assume that $\theta \leqslant 1$ and that for some index $m \leqslant n$ we have*

$$\theta_0, \ldots, \theta_{m-1} \leqslant k \quad \textit{and} \quad \theta_m, \ldots, \theta_n \geqslant k^{-1} \quad \textit{for some real } k \geqslant 1. \tag{9}$$

Then

$$\Theta \leqslant k^{n-1} \max\left\{\frac{\theta_0}{\theta_0 \cdots \theta_n}, \frac{1}{\theta_n}\right\}. \tag{10}$$

Proof. By the assumption that $\theta \leqslant 1$, for all $r \in \{1, \ldots, n\}$ we have $\theta^r \geqslant \theta^{n+1} = \theta_0 \cdots \theta_n$. Therefore Θ satisfies

$$\Theta \leqslant \frac{1}{\theta_0 \cdots \theta_n} \max_{1 \leqslant r \leqslant n} \theta_0 \cdots \theta_{r-1}. \tag{11}$$

In view of (9), it is readily seen that

$$\max_{1 \leqslant r \leqslant m} \theta_0 \cdots \theta_{r-1} \leqslant k^{n-1} \max_{1 \leqslant r \leqslant m} \theta_0 \frac{\theta_1}{k} \cdots \frac{\theta_{r-1}}{k} \overset{(9)}{=} k^{n-1}\theta_0 \tag{12}$$

and

$$\max_{m < r \leqslant n} \theta_0 \cdots \theta_{r-1} \leqslant \max_{m < r \leqslant n} \prod_{i=0}^{m-1} \theta_i \prod_{i=m}^{r-1} k\theta_i \overset{(9)}{=} \prod_{i=0}^{m-1} \theta_i \prod_{i=m}^{n-1} k\theta_i \leqslant k^{n-1}\theta_0 \cdots \theta_{n-1}. \tag{13}$$

Combining (12) and (13) with (11) gives (10). □

We will be using Lemma 1 with $f_i(x) = x^i$ for $0 \leqslant i \leqslant n$. In this case, the Wronskian $W(f_0, \ldots, f_n)$ is identically equal to $n!$ and Lemma 1 is applicable to a neighbourhood of any point $x_0 \in \mathbb{R}$. The system of inequalities in (7) becomes

$$|P(x)| < \theta_0, \quad |P'(x)| < \theta_1, \quad \cdots \quad |P^{(n)}(x)| < \theta_n, \tag{14}$$

where $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ is a non-zero integral polynomial of degree at most n and the set in the left-hand side of (7) is simply

$$A_n(J; \theta_0, \ldots, \theta_n) := \{x \in J : (14) \text{ holds for some } P \in \mathbb{Z}[x] \setminus \{0\} \text{ with } \deg P \leqslant n\}. \tag{15}$$

Then, upon combining Lemmas 1 and 2 and using a standard compactness argument (for instance, [BDV07, proof of Lemma 6]) we get the following lemma.

Lemma 3. *There exist constants $C > 0$ and $\alpha > 0$ which depend only on n and are such that for any interval $J \subset [-1/2, 1/2]$ there is a constant $\delta_J > 0$ such that for any positive numbers $\theta_0, \ldots, \theta_n$ satisfying $\theta = (\theta_0 \cdots \theta_n)^{1/(n+1)} \leqslant 1$ and (9), we have*

$$\lambda(A_n(J; \theta_0, \ldots, \theta_n)) \leqslant C\left(1 + \frac{k^{\alpha(n-1)}}{\delta_J^\alpha} \max\left\{\frac{\theta_0}{\theta_0 \cdots \theta_n}, \frac{1}{\theta_n}\right\}^\alpha\right)\theta^\alpha |J|. \tag{16}$$

3. Tailored polynomials

Let $\xi_0,\dots,\xi_n\in\mathbb{R}^+$ satisfy the conditions

$$\begin{aligned} &\xi_i\ll 1 \quad \text{when } 0\leqslant i\leqslant m-1,\\ &\xi_i\gg 1 \quad \text{when } m\leqslant i\leqslant n,\\ &\xi_0<\varepsilon, \qquad \xi_n>\varepsilon^{-1} \end{aligned} \tag{17}$$

for some $0<m\leqslant n$ and $\varepsilon>0$, where the implied constants depend on n only. Assume also that

$$\prod_{i=0}^{n}\xi_i=1. \tag{18}$$

The following lemma lies at the heart of the proof of Theorem 2. It enables us to construct 'tailor-made' irreducible polynomials whose derivatives assume certain values. Of course, there is a connection with Taylor's formula, too. Therefore, we shall call these *tailored polynomials*.

Lemma 4. *For any $n\geqslant 2$, there are positive constants δ_0 and c_0 which depend on n only and have the following property: for any interval $J\subset[-1/2,1/2]$ there is a sufficiently small $\varepsilon=\varepsilon(n,J)>0$ such that for any $\xi_0,\dots,\xi_n$ satisfying (17) and (18), there exists a measurable set $G_J\subset J$ satisfying*

$$\lambda(G_J)\geqslant \tfrac{3}{4}|J| \tag{19}$$

such that for every $x\in G_J$ there are $n+1$ linearly independent primitive irreducible polynomials $P\in\mathbb{Z}[x]$ of degree exactly n such that

$$\delta_0\xi_i\leqslant|P^{(i)}(x)|\leqslant c_0\xi_i \quad \text{for all } i=0,\dots,n. \tag{20}$$

Proof. Let $n\geqslant 2$, and suppose that $\xi_0,\dots,\xi_n$ are given and satisfy (17) and (18) for some m and ε. Let $J\subset[-1/2,1/2]$ be any interval and let $x\in J$. Consider the system of inequalities

$$|P(x)|\leqslant\xi_i \quad \text{for } 0\leqslant i\leqslant n, \tag{21}$$

where $P(x)=a_nx^n+\cdots+a_1x+a_0$. Let B_x be the set of $(a_0,\dots,a_n)\in\mathbb{R}^{n+1}$ satisfying (21). Clearly, B_x is a convex body in $\mathbb{R}^{n+1}$ that is symmetric about the origin. In view of (18), the volume of this body equals $2^{n+1}\prod_{i=1}^{n}i!^{-1}$. Let $\lambda_0\leqslant\lambda_1\leqslant\cdots\leqslant\lambda_n$ be the successive minima of B_x. Clearly, $\lambda_i=\lambda_i(x)$ is a function of x. By Minkowski's theorem for successive minima,

$$\frac{2^{n+1}}{(n+1)!}\leqslant\lambda_0\cdots\lambda_n\,\mathrm{Vol}\,B_x\leqslant 2^{n+1}.$$

Substituting the value of $\mathrm{Vol}\,B_x$ gives $\lambda_0\cdots\lambda_n\leqslant\prod_{i=1}^{n}i!$. Therefore, since $\lambda_0\leqslant\cdots\leqslant\lambda_n$, we get

$$\lambda_n\leqslant\lambda_0^{-n}\prod_{i=1}^{n}i!. \tag{22}$$

Our next goal is to show that λ_0 is bounded below by a constant unless x belongs to a small subset of J. Let $E_\infty(J,\delta_1)$ be the set of $x\in J$ such that $\lambda_0=\lambda_0(x)\leqslant\delta_1$, where $\delta_1<1$. By the definition of λ_0, there is a non-zero polynomial $P\in\mathbb{Z}[x]$, with $\deg P\leqslant n$, satisfying

$$|P^{(i)}(x)|\leqslant\delta_1\xi_i \quad \text{for } 0\leqslant i\leqslant n. \tag{23}$$

Let $\theta_0=\delta_1\xi_0$ and $\theta_i=\xi_i$ for $1\leqslant i\leqslant n$. Then $E_\infty(J,\delta_1)\subset A_n(J;\theta_0,\dots,\theta_n)$, where $A_n(\cdot)$ is as defined in (15). In view of (17) and (18), Lemma 3 is applicable. For this choice of $\theta_0,\dots,\theta_n$

we have $\theta = \delta_1^{1/(n+1)}$. Then

$$\lambda(E_\infty(J,\delta_1)) \leqslant \lambda(A_n(J;\theta_0,\dots,\theta_n)) \ll \left(1+\frac{1}{\delta_J^\alpha}\max\left\{\frac{\delta_1\xi_0}{\delta_1},\frac{1}{\xi_n}\right\}^\alpha\right)\delta_1^{\alpha/(n+1)}|J|.$$

By (17), $\max\{\xi_0,\xi_n^{-1}\} < \varepsilon$. Therefore $\mu(E_\infty(J,\delta_1)) \ll \delta_1^{\alpha/(n+1)}|J|$ provided that $\varepsilon < \delta_J$. Thus there is a sufficiently small δ_1, depending on n only, such that

$$\lambda(E_\infty(J,\delta_1)) \leqslant \frac{1}{4n+8}|J|. \tag{24}$$

By construction, for any $x \in J\setminus E_\infty(J,\delta_1)$ we have

$$\lambda_0 \geqslant \delta_1. \tag{25}$$

Combining (22) and (25) gives

$$\lambda_n \leqslant c_1 := \delta_1^{-n}\prod_{i=1}^{n} i!, \tag{26}$$

where c_1 depends on n only. By the definition of λ_n, there are $n+1$ linearly independent integer points $\mathbf{a}_j = (a_{0,j},\dots,a_{n,j})$, $0 \leqslant j \leqslant n$, lying in the body $\lambda_n B_x \subset c_1 B_x$. In other words, the polynomials $P_j(x) = a_{n,j}x^n + \dots + a_{0,j}$, $0 \leqslant j \leqslant n$, satisfy the system of inequalities

$$|P_j^{(i)}(x)| \leqslant c_1\xi_i \quad \text{for } 0 \leqslant i \leqslant n. \tag{27}$$

Let $A = (a_{i,j})_{0\leqslant i,j\leqslant n}$ be the integer matrix composed of the integer points $\mathbf{a}_j$, $0 \leqslant j \leqslant n$. Since all these points are contained in the body $c_1 B_x$, we have that $|\det A| \ll \mathrm{Vol}(B_x) \ll 1$; that is, $|\det A| < c_2$ for some constant c_2 that depends on n only. By Bertrand's postulate, choose a prime number p which satisfies

$$c_2 \leqslant p \leqslant 2c_2. \tag{28}$$

Then $|\det A| < p$. Since $\mathbf{a}_0,\dots,\mathbf{a}_n$ are linearly independent integer points, $|\det A| \geqslant 1$. Therefore, $\det A \not\equiv 0 \pmod p$ and the system

$$A\bar{t} \equiv \bar{b} \pmod p \tag{29}$$

has a unique non-zero integer solution $\bar{t} = {}^t(t_0,\dots,t_n) \in [0,p-1]^{n+1}$, where $\bar{b} := {}^t(0,\dots,0,1)$ and ^{t} denotes transposition. For $l = 0,\dots,n$, define $\bar{r}_l = {}^t(1,\dots,1,0,\dots,0) \in \mathbb{Z}^{n+1}$ where the number of zeros is l. Since $\det A \not\equiv 0 \pmod p$, for every $l = 0,\dots,n$ the system

$$A\bar{\gamma} \equiv -\frac{A\bar{t}-\bar{b}}{p} + \bar{r}_l \pmod p \tag{30}$$

has a unique non-zero integer solution $\bar{\gamma} = \bar{\gamma}_l \in [0,p-1]^{n+1}$. Define $\bar{\eta} = \bar{\eta}_l := \bar{t} + p\bar{\gamma}_l$ for $0 \leqslant l \leqslant n$. Consider the $n+1$ polynomials of the form

$$P(x) = a_n x^n + \dots + a_0 := \sum_{i=0}^{n} \eta_i P_i(x) \in \mathbb{Z}[x] \tag{31}$$

where $(\eta_0,\dots,\eta_n) = \bar{\eta}$, which of course depends on the parameter $l \in \{0,\dots,n\}$. Since $\bar{r}_0,\dots,\bar{r}_n$ are linearly independent, it is easily seen that $\bar{\eta}_0,\dots,\bar{\eta}_n$ are linearly independent. Hence the polynomials given by (31) are linearly independent and thus non-zero.

Observe that $A\bar{\eta}$ is actually the column ${}^t(a_0,\dots,a_n)$ of coefficients of P. By construction, $\bar{\eta} \equiv \bar{t} \pmod p$, and therefore $\bar{\eta}$ is also a solution of (29). Then, since $\bar{b} = {}^t(0,\dots,0,1)$ and $A\bar{\eta} \equiv \bar{b} \pmod p$, we have that $a_n \not\equiv 0 \pmod p$ and $a_i \equiv 0 \pmod p$ for $i = 0,\dots,n-1$. Furthermore,

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by (30) we have that $A\overline{\eta} \equiv \overline{b} + p\overline{r}_l \pmod{p^2}$. So, upon substituting the values of $\overline{b}$ and $\overline{r}_l$ into this congruence, one can readily verify that $a_0 \equiv p \pmod{p^2}$ and hence that $a_0 \not\equiv 0 \pmod{p^2}$. By Eisenstein's criterion, P is irreducible.

Since both $\overline{t}$ and $\overline{\gamma}_l$ lie in $[0, p-1]^{n+1}$ and we have $\overline{\eta} = \overline{t} + p\overline{\gamma}_l$, it is readily seen that $|\eta_i| \leqslant p^2$ for all i. Therefore, using (27) and (28) we obtain that

$$|P^{(i)}(x)| \leqslant c_0\xi_i \quad \text{for } 0 \leqslant i \leqslant n, \tag{32}$$

with $c_0 = 4(n+1)c_1c_2^2$. Without loss of generality, we may assume that the $n+1$ linearly independent polynomials P constructed above are primitive (that is, the coefficients of P are coprime), as otherwise we could just divide the coefficients of P by their greatest common multiple and, clearly, such division would not affect the validity of (32). Thus, $P \in \mathbb{Z}[x]$ are primitive irreducible polynomials of degree n which satisfy the right-hand side of (20). The final part of the proof is aimed at establishing the left-hand side of (20). The arguments are applied to each of the polynomials P that we have constructed.

Let $\delta_0 > 0$ be a sufficiently small parameter depending on n. For each $j = 0, \ldots, n$, let $E_j(J, \delta_0)$ be the set of $x \in J$ such that there is a non-zero polynomial $R \in \mathbb{Z}[x]$, with $\deg R \leqslant n$, satisfying

$$|R^{(i)}(x)| \leqslant \delta_0^{\delta_{i,j}} c_0^{1-\delta_{i,j}} \xi_i \tag{33}$$

where $\delta_{i,j}$ equals 1 if $i = j$ and equals 0 otherwise. Let $\theta_i = \delta_0^{\delta_{i,j}} c_0^{1-\delta_{i,j}} \xi_i$. Then $E_j(J, \delta_0) \subset A_n(J; \theta_0, \ldots, \theta_n)$. In view of (17) and (18), Lemma 3 is applicable provided that $\varepsilon < \min\{c_0^{-1}, c_0\delta_0\}$. Then, by Lemma 3,

$$\lambda(E_j(J, \delta_0)) \ll \left(1 + \frac{1}{\delta_J^\alpha} \max\left\{\frac{c_0\xi_0}{c_0^n\delta_0}, \frac{1}{\delta_0 c_0 \xi_n}\right\}^\alpha\right)(\delta_0 c_0^n)^{1/(n+1)}|J|.$$

It is readily seen that the above maximum is less than or equal to δ_J if $\varepsilon < \delta_J\delta_0 c_0$. So

$$\lambda(E_j(J, \delta_0)) \leqslant \frac{1}{4n+8}|J| \tag{34}$$

provided that $\varepsilon < \min\{\delta_J\delta_0 c_0, c_0^{-1}, c_0\delta_0\}$ and $\delta_0 = \delta_0(n)$ is sufficiently small. By construction, for any x in the set G_J defined by

$$G_J := J \Big\backslash \left(\bigcup_{j=0}^n E_j(J, \delta_0) \cup E_\infty(J, \delta_1)\right),$$

we necessarily have that $|P^{(i)}(x)| \geqslant \delta_0\xi_i$ for all $i = 0, \ldots, n$, where P is the same as in (32). Therefore, the left-hand side of (20) holds for all i. Finally, observe that

$$\lambda(G_J) \geqslant |J| - \sum_{i=0}^n \lambda(E_i(J, \delta_0)) - \lambda(E_\infty(J, \delta_1)) \overset{(24) \text{ and } (34)}{\geqslant} |J| - (n+2)\,\frac{1}{(4n+8)}|J| = \frac{3}{4}|J|.$$

This verifies (19) and completes the proof. □

4. Tailored monic polynomials

The following result is the analogue of Lemma 4 for monic polynomials.

Lemma 5. *For any $n \geqslant 2$, there are positive constants δ_0 and c_0 which depend on n only and have the following property: for any interval $J \subset [-1/2, 1/2]$ there is a sufficiently small $\varepsilon = \varepsilon(n, J) > 0$*

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such that for any positive $\xi_0, \ldots, \xi_n$ *satisfying (17) and (18), there exists a measurable set* $G_J \subset J$ *satisfying*

$$\lambda(G_J) \geqslant \tfrac{3}{4}|J| \tag{35}$$

such that for every $x \in G_J$ *there is an irreducible monic polynomial* $P \in \mathbb{Z}[x]$ *of degree* $n+1$ *which satisfies (20).*

Proof. We will essentially follow the proof of Lemma 4 but replace the construction of P with a different procedure that makes use of ideas from [Bug02]. Let $G_J = J \smallsetminus E_\infty(J, \delta_1)$, where δ_1 is defined in the same way as in the proof of Lemma 4. Then we have (24), which implies (19). Take any $x \in G_J$. Arguing the same way as in Lemma 4, we obtain $n+1$ linearly independent polynomials $P_j(x) = a_{n,j}x^n + \cdots + a_{0,j} \in \mathbb{Z}[x]$, $0 \leqslant j \leqslant n$, that satisfy (27). The matrix $A = (a_{i,j})_{0 \leqslant i,j \leqslant n}$ satisfies $|\det A| < c_2$ for some constant c_2 that depends on n only. Again we choose a prime p satisfying (28) so that $\det A \not\equiv 0 \pmod p$. It is readily verified that

$$\det(P_j^{(i)}(x))_{0 \leqslant i,j \leqslant n} = \det A \prod_{i=0}^{n} i! \neq 0.$$

Therefore, there is a unique solution $(t_0, \ldots, t_n) \in \mathbb{R}^{n+1}$ to the following system of linear equations:

$$\frac{(n+1)!}{(n+1-i)!} x^{n+1-i} + p \sum_{j=0}^{n} t_j P_j^{(i)}(x) = 2(n+1)pc_1\xi_i \quad \text{for } 0 \leqslant i \leqslant n. \tag{36}$$

Since $\det A \not\equiv 0 \pmod p$, at least one of $a_{0,0}, \ldots, a_{0,n}$ is not divisible by p. Without loss of generality we will assume that $a_{0,0} \not\equiv 0 \pmod p$. For $j = 1, \ldots, n$ define $\eta_j = [t_j]$, where $[\,\cdot\,]$ denotes the integer part. Further, define η_0 to be either $[t_0]$ or $[t_0]+1$ so that

$$\eta_0 a_{0,0} + \cdots + \eta_n a_{0,n} \not\equiv 0 \pmod p. \tag{37}$$

This is possible because $a_{0,0} \not\equiv 0 \pmod p$. Define

$$P(x) = x^{n+1} + a_n x^n + \cdots + a_0 := x^{n+1} + p \sum_{i=0}^{n} \eta_i P_i(x) \in \mathbb{Z}[x].$$

Obviously, $\deg P = n+1$. The leading coefficient of P is 1 and so is not divisible by p^2. By (37), $a_0 \not\equiv 0 \pmod{p^2}$. However, by the construction of P, we have that $a_i \equiv 0 \pmod p$ for all $i = 1, \ldots, n$. Therefore, by Eisenstein's criterion, P is irreducible over $\mathbb{Q}$.

Finally, it follows from the definition of η_j that $|t_j - \eta_j| \leqslant 1$ for all $j = 0, \ldots, n$. Therefore, using the definition of P and (27), we can verify that

$$\left| \frac{(n+1)!}{(n+1-i)!} x^{n+1-i} + p \sum_{j=0}^{n} t_j P_j^{(i)}(x) - P^{(i)}(x) \right| \leqslant (n+1)pc_1\xi_i \quad \text{for } 0 \leqslant i \leqslant n.$$

Combining this with (36) gives

$$(n+1)pc_1\xi_i \leqslant |P^{(i)}(x)| \leqslant 3(n+1)pc_1\xi_i \quad \text{for } 0 \leqslant i \leqslant n.$$

Thus, taking $\delta_0 = (n+1)pc_1$ and $c_0 = 3(n+1)pc_1$ gives (20). The proof is complete. □

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5. Proof of Theorem 2

We now give a complete proof of the theorem in the case where $\mathbb{A}^{\circ}_{n,\nu}(Q,\mu)=\mathbb{A}_{n,\nu}(Q,\mu)$. At the end of the section we will say in what way the proof has to be modified in order to establish the theorem in the case where $\mathbb{A}^{\circ}_{n,\nu}(Q,\mu)=\mathbb{A}^{*}_{n,\nu}(Q,\mu)$.

Fix $n\geqslant 2$ and let μ satisfy (3). Let δ_0 and c_0 be the same as in Lemma 4. Define the following parameters:

$$\xi_0=\eta Q^{-n+\mu},\quad \xi_1=\eta^{-n}Q^{1-\mu},\quad \xi_i=\eta Q\quad\text{for } 2\leqslant i\leqslant n, \tag{38}$$

where $0<\eta<1$ is a sufficiently small fixed parameter, depending on n only, which will be specified later. Fix any interval $J\subset[-1/2,1/2]$, and let $\varepsilon=\varepsilon(n,J)$ be the same as in Lemma 4. Then, (17) is satisfied with $m\in\{1,2\}$ for sufficiently large Q. Also, the validity of (18) easily follows from (38). Let G_J be the set arising from Lemma 4 and take $x\in G_J$. Then, by Lemma 4, there is a primitive irreducible polynomial $P\in\mathbb{Z}[x]$ of degree n that satisfies (20).

Finding α_1. Let $y\in\mathbb{R}$ be such that $|y-x|=Q^{-n-1+2\mu}$. By (3), we have $|y-x|<1$. Further, by Taylor's formula,

$$P(y)=\sum_{i=0}^{n}\frac{1}{i!}P^{(i)}(x)(y-x)^i. \tag{39}$$

Using the inequality $|x-y|<1$ together with (3), (20) and (38), we verify that

$$|P^{(i)}(x)(y-x)^i|<\eta c_0Q^{-n+\mu}\quad\text{for } i\geqslant 2. \tag{40}$$

Also, by (20) and (38), $|P(x)|\leqslant\eta c_0Q^{-n+\mu}$. Therefore,

$$\sum_{i\neq 1}\left|\frac{1}{i!}P^{(i)}(x)(y-x)^i\right|\leqslant\eta c_0Q^{-n+\mu}\sum_{i=0}^{n}\frac{1}{i!}<3\eta c_0Q^{-n+\mu}. \tag{41}$$

On the other hand,

$$|P'(x)(y-x)|\overset{(20)\text{ and }(38)}{\geqslant}\delta_0\eta^{-n}Q^{-\mu+1}Q^{-n-1+2\mu}\geqslant\delta_0\eta^{-2}Q^{-n+\mu}. \tag{42}$$

It follows from (41) and (42) that $P(y)$ takes different signs at the endpoints of the interval $|y-x|\leqslant Q^{-n-1+2\mu}$ provided that $\eta\leqslant\delta_0/(3c_0)$. By the continuity of P, there must be a root α_1 of P in this interval; that is,

$$|x-\alpha_1|<Q^{-n-1+2\mu}. \tag{43}$$

Finding α_2. Let $y_\rho=x+\rho Q^{-\mu}$, where $2\leqslant|\rho|<Q^{\mu/2}$. In what follows we will again use (39), this time with $y=y_\rho$. Using $|x-y|<1$, $|\rho|\leqslant Q^{\mu/2}$, (20) and (38), we verify that

$$|P^{(i)}(x)(y_\rho-x)^i|<\eta|\rho|c_0Q^{1-2\mu}\quad\text{for } i\geqslant 3. \tag{44}$$

By (3), (20), (38) and the fact that $|\rho|\geqslant 2$, we have that

$$|P(x)|\leqslant\eta c_0Q^{-n+\mu}\leqslant|\rho|\eta c_0Q^{1-2\mu}$$

and

$$|P'(x)(y_\rho-x)|\leqslant\eta^{-n}c_0Q^{1-\mu}|\rho|Q^{-\mu}=\eta^{-n}c_0|\rho|Q^{1-2\mu}.$$

The preceding two estimates, together with (44), give

$$\sum_{i\neq 2}\left|\frac{1}{i!}P^{(i)}(x)(y_\rho-x)^i\right|\leqslant\eta^{-n}|\rho|c_0Q^{1-2\mu}\sum_{i=0}^{n}\frac{1}{i!}<3\eta^{-n}|\rho|c_0Q^{1-2\mu}. \tag{45}$$

On the other hand,

$$\left|\frac{1}{2!}P''(x)(y_\rho - x)^2\right| \overset{(20)\text{ and }(38)}{\geqslant} \frac{1}{2}\delta_0\eta Q|\rho|^2Q^{-2\mu} = \frac{1}{2}\delta_0\eta Q^{1-2\mu}\rho^2. \tag{46}$$

It follows from (45) and (46) that $P(y)$ has the same sign at the points $y_{\pm\rho_0}$ (the same as $P''(x)$) with $\rho_0 = 8c_0\eta^{-n-1}\delta_0^{-1}$.

On the other hand, using (3) and arguing the same way as in 'Finding α_1', one readily verifies that $P(y_2)$ and $P(y_{-2})$ have different signs. Therefore, $P(y)$ changes sign on one of the intervals

$$[-\rho_0 Q^{-\mu}, -2Q^{-\mu}], \quad [2Q^{-\mu}, \rho_0 Q^{-\mu}].$$

By the continuity of P, there must be a root α_2 of P in that interval, i.e.

$$2Q^{-\mu} \leqslant |x - \alpha_2| < \rho_0 Q^{-\mu}. \tag{47}$$

Combining (3), (43) and (47) gives $Q^{-\mu} \leqslant |\alpha_1 - \alpha_2| \leqslant (\rho_0 + 1)Q^{-\mu}$, thus establishing (2).

Estimates for the height. Using the fact that $|x| \leqslant 1/2$ along with (20), (3) and (38), we verify that

$$\begin{aligned}
&|a_n| \asymp Q,\\
&|a_{n-1}| = |P^{(n-1)}(x) - \frac{n!}{1!(n-1)!}\,a_n x| \ll Q,\\
&|a_{n-2}| = |P^{(n-2)}(x) - \frac{n!}{2!(n-2)!}\,a_n x^2 - \frac{(n-1)!}{1!(n-2)!}\,a_{n-1}x| \ll Q,\\
&\vdots
\end{aligned}$$

The upshot is that $H(\alpha_1) \asymp Q$. This establishes (1) and completes the proof of Theorem 2 in the case where $\mathbb{A}^\circ_{n,\nu}(Q,\mu) = \mathbb{A}^*_{n,\nu}(Q,\mu)$.

In the case of $\mathbb{A}^\circ_{n,\nu}(Q,\mu) = \mathbb{A}^*_{n,\nu}(Q,\mu)$, the proof remains essentially the same. The only necessary modification arises from taking into account the $(n+1)$st derivative of P. This derivative is identically equal to $(n+1)!$ and will cause no trouble in establishing the estimates (41), (42), (45) and (46), which are key to finding α_1 and α_2. The height is estimated in exactly the same way.

Remark 3. From the above proof we have that $|a_n| \asymp Q$. This condition can readily be used to show that any α_i conjugate to α_1 is bounded by a constant which depends on n only. This follows from the well-known property that $|\alpha_i| \ll H(\alpha_i)/|a_n|$; see [Spr69].

6. Applications to metric Diophantine approximation

We begin by recalling a result due to Bernik *et al.* In order to state their theorem, we first introduce the set

$$\mathcal{P}_n(\mu, w) = \left\{x \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right] : \begin{cases}|P(x)| < H(P)^{-w-\mu}\\ |P'(x)| < H(P)^{1-\mu}\end{cases} \text{hold for i.m. } P \in \mathbb{Z}[x] \text{ with } \deg P \leqslant n\right\},$$

where $n \geqslant 2$, $\mu \geqslant 0$, $H(P)$ denotes the (absolute) height of P and 'i.m.' stands for 'infinitely many'. Applying Dirichlet's pigeonhole principle readily gives that $\mathcal{P}_n(\mu, w) = [-1/2, 1/2]$ if $w \leqslant n - 2\mu$. However, when $w > n - 2\mu$, the makeup of the set $\mathcal{P}_n(\mu, w)$ changes completely. The following is a consequence of the theorem from [BKM01, § 8.3].

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THEOREM BKM (Bernik, Kleinbock and Margulis). *Let $n \geqslant 2$ and $\mu \geqslant 0$. Then for any $w > n - 2\mu$ the set $\mathcal{P}_n(\mu, w)$ is of Lebesgue measure zero.*

Theorem BKM is a delicate generalisation of Mahler's problem [Mah32], which corresponds to the $\mu = 0$ case of Theorem BKM, although Mahler's problem was settled by Sprindžuk [Spr65]. It is counterintuitive that for a fixed μ the set $\mathcal{P}_n(\mu, w)$ must get smaller as w increases. Hausdorff dimension is traditionally used to study questions of this sort in metric number theory. Using Theorem 2, we are able to obtain the following lower bound on the size of $\mathcal{P}_n(\mu, w)$. In what follows, 'dim' will mean Hausdorff dimension.

THEOREM 3. *Let $n \geqslant 2$ be an integer and let $0 < \mu < (n + 1)/3$. Then for any $w > n - 2\mu$,*

$$\dim \mathcal{P}_n(\mu, w) \geqslant \frac{n + 1 - 2\mu}{w + 1}. \tag{48}$$

In the case where $\mu = 0$, inequality (48) was first established by Baker and Schmidt [BS70], who also conjectured that (48) when $\mu = 0$ is actually an equality. This conjecture was proved in [Ber83]. In view of Theorem BKM and, indeed, Theorem 3, it is natural to consider the following generalisation of the Baker–Schmidt conjecture.

CONJECTURE 1. Let n, μ and w be as in Theorem 3. Then (48) is an equality.

Another consequence of this work in the spirit of Theorem BKM is the following.

THEOREM 4. *Let $v_0, \dots, v_{m-1} \geqslant 0$ and $v_m, \dots, v_n \leqslant 0$ be such that $v_0 > 0$, $v_n < 0$ and $v_0 + \cdots + v_n > 0$. Then for almost every $x \in \mathbb{R}$ there are only finitely many $Q \in \mathbb{N}$ such that*

$$|P^{(i)}(x)| < Q^{-v_i} \quad \textit{for all } 0 \leqslant i \leqslant n, \quad \textit{for some } P \in \mathbb{Z}[x] \setminus \{0\} \textit{ with } \deg P \leqslant n. \tag{49}$$

Proof. Let $v_0, \dots, v_n$ be given. Without loss of generality we can assume that $x \in [-1/2, 1/2]$. Let

$$S_t := \{x \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right] : |P^{(i)}(x)| \ll 2^{-v_i t} \ \forall 0 \leqslant i \leqslant n, \text{ for some } P \in \mathbb{Z}[x] \setminus \{0\} \text{ with } \deg P \leqslant n\}.$$

It is readily seen that our goal is to prove that $\limsup_{t \to \infty} S_t$ has measure zero. By the Borel–Cantelli lemma, this will follow once we show that $\sum_{t=1}^{\infty} \lambda(S_t) < \infty$. The latter is easily verified by applying Lemma 3. □

Mahler's problem corresponds to Theorem 4 with $v_1 = \cdots = v_n = -1$. Theorem BKM follows from Theorem 4 upon taking $v_2 = \cdots = v_n = -1$. We have a fair amount of flexibility in choosing the exponents v_i in Theorem 4, and there are some restrictions which we believe can be safely removed. We now state a unifying conjecture as follows.

CONJECTURE 2. Let $\varepsilon > 0$. Then for almost every $x \in \mathbb{R}$ the inequality

$$\prod_{i=0}^{n} |P^{(i)}(x)| < H(P)^{-\varepsilon} \tag{50}$$

has only finitely many solutions $P \in \mathbb{Z}[x]$ with $\deg P \leqslant n$.

It is likely that in (50) the height $H(P)$ can be replaced with $\Pi_+(P) := \prod_{i=1}^{n} \max\{1, |a_i|\}$, where $P(x) = a_n x^n + \cdots + a_1 x + a_0$. Also, by using the inhomogeneous transference principle of [BV08], one should be able to establish an inhomogeneous version of Conjecture 2 modulo the homogeneous statement.

6.1 Proof of Theorem 3

We shall use the ubiquitous systems technique, which we now briefly recall in a simplified form (see [BDV06] for more details and [BBD02] for the related notion of regular systems). Let I be an interval in $\mathbb{R}$ and let $\mathcal{R} := (r_\alpha)_{\alpha\in J}$ be a family of points r_α in I indexed by a countable set $\mathcal{J}$. Let $\beta : \mathcal{J} \to \mathbb{R}^+ : \alpha \mapsto \beta_\alpha$ be a function on $\mathcal{J}$ that attaches a 'weight' β_α to the point r_α. For $t \in \mathbb{N}$, let $\mathcal{J}(t) := \{\alpha \in \mathcal{J} : \beta_\alpha \leqslant 2^t\}$ and assume that $\mathcal{J}(t)$ is always finite.

Let $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that $\lim_{t\to\infty} \rho(t) = 0$, which we refer to as the *ubiquity function*. The system $(\mathcal{R}; \beta)$ is said to be *locally ubiquitous in I relative to ρ* if there is an absolute constant $k_0 > 0$ such that for any interval $J \subset I$,

$$\liminf_{t\to\infty} \lambda\bigg(\bigcup_{\alpha\in\mathcal{J}(t)} B(r_\alpha, \rho(2^t)) \cap J\bigg) \geqslant k_0|J|. \tag{51}$$

Given a function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$, let

$$\Lambda_{\mathcal{R}}(\Psi) := \{x \in I : |x - r_\alpha| < \Psi(\beta_\alpha) \text{ holds for infinitely many } \alpha \in \mathcal{J}\}.$$

The following lemma is [BDV07, Theorem 10]; alternatively, it follows from the more general [BDV06, p. 20, Corollary 4].

Lemma 6. *Let $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a monotonic function such that for some $\phi < 1$, $\Psi(2^{t+1}) \leqslant \phi\Psi(2^t)$ holds for t sufficiently large. Let $(\mathcal{R}, \beta)$ be a locally ubiquitous system in B_0 relative to ρ. Then for any $s \in (0, 1)$,*

$$\mathcal{H}^s(\Lambda_{\mathcal{R}}(\Psi)) = \infty \quad \text{if} \sum_{t=1}^{\infty} \frac{\Psi(2^t)^s}{\rho(2^t)} = \infty. \tag{52}$$

The ubiquitous system. Let $n \geqslant 2$ and suppose that μ satisfies $0 < \mu < (n+1)/3$. Choose $\mu' = \mu + \delta < (n+1)/3$ with $\delta > 0$. Let $\mathcal{R}$ be the set of algebraic numbers $\alpha_1 \in \mathbb{R}$ of degree n such that

$$|\alpha_1 - \alpha_2| \leqslant \nu^{-1} H(\alpha_1)^{-\mu'} \quad \text{for some } \alpha_2 \in \mathbb{R} \text{ conjugate to } \alpha_1 \tag{53}$$

and

$$|\alpha_i| \ll \nu^{-1} \quad \text{for any } \alpha_i \in \mathbb{C} \text{ conjugate to } \alpha_1, \tag{54}$$

where the constant implied by the Vinogradov symbol depends on n only. We will identify $\mathcal{J}$ with $\mathcal{R}$, so that formally $r_\alpha = \alpha$. Further, let $\beta_\alpha = \nu H(\alpha)$ and $\rho(q) := q^{-n-1+2\mu'}$. Then, by Theorem 2 together with Remark 3, there is a constant ν such that $(\mathcal{R}, \beta)$ is locally ubiquitous in $I := [-1/2, 1/2]$ with respect to the above ρ. Given $w > 0$, let $\Psi(q) = q^{-w-1}$. Clearly, $\Psi(2^{t+1}) \leqslant \Psi(2^t)/2$, and so Lemma 6 is applicable to this Ψ. Let $s = (n + 1 - 2\mu')/(w + 1)$. Since $w > n - 2\mu'$, we have $s < 1$. Then $\Psi(q)^s/\rho(q)$ is identically 1 and therefore the sum in (52) diverges. By Lemma 6, we have that $\mathcal{H}^s(\Lambda_{\mathcal{R}}(\Psi)) = \infty$. By the definition of Hausdorff dimension, $\dim \Lambda_{\mathcal{R}}(\Psi) \geqslant s = (n + 1 - 2\mu - 2\delta)/(w + 1)$. Since $\delta > 0$ is arbitrary, it remains to show that

$$\Lambda_{\mathcal{R}}(\Psi) \subset \mathcal{P}_n(\mu, w). \tag{55}$$

By definition, for every $x \in \Lambda_{\mathcal{R}}(\Psi)$ there are infinitely many real algebraic numbers α_1 of degree n satisfying (53), (54) and

$$|x - \alpha_1| \ll H(\alpha_1)^{-w-1}. \tag{56}$$

Let P denote the minimal polynomial of α_1. Then $P(x) = a_n(x - \alpha_1)\cdots(x - \alpha_n)$. By (53), (54), (56) and the fact that $|a_n| \leqslant H(P)$, we get $|P(x)| \ll H \cdot H(P)^{-w-1} H(P)^{-\mu'} = H(P)^{-w-\mu'}$.

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Since $\mu' > \mu$, we have $|P(x)| < H(P)^{-w-\mu}$ for sufficiently large $H(P)$. Moreover,

$$P'(x) = a_n \sum_{i=1}^{n} \frac{(x-\alpha_1)\cdots(x-\alpha_n)}{(x-\alpha_i)}. \tag{57}$$

Again, by (53), (54), (56) and the fact that $|a_n| \leqslant H(P)$, we get that every summand in (57) is $\ll H(P)^{-\mu'}$, which further implies that $|P'(x)| \ll H(P)^{-\mu'}$. Since $\mu' > \mu$, we have $|P'(x)| < H(P)^{-\mu}$ for sufficiently large $H(P)$. The upshot is that the inequalities $|P(x)| < H(P)^{-w-\mu}$ and $|P'(x)| < H(P)^{-\mu}$ hold simultaneously for infinitely many $P \in \mathbb{Z}[x]$ of degree n. Thus (55) is established and the proof is complete.

7. Final remarks

(i) The main body of this paper deals with integral polynomials of degree n. However, it is equally possible to develop a similar theory for linear forms of linearly independent analytic functions. This is due to the fact that Lemma 1, which underlies all the other results, holds for linear forms of analytic functions.

(ii) Clearly, by using the algebraic integers part of Theorem 2, it is possible to establish an analogue of Theorem 3 for monic polynomials. Furthermore, by using the inhomogeneous transference principle of [BV08], it is possible to establish the inhomogeneous version of Theorem 4 and, in particular, the version for monic polynomials.

(iii) Theorem 2 can be used to give quantitative estimates for the number of polynomials with bounded discriminant; see, for example, [BGK08a]. We are going to address this question in more detail in a forthcoming paper.

(iv) Alongside the Hausdorff dimension generalisation of Theorem BKM, it is interesting to develop a Khintchine-type theory; see [BKM01, § 8.3], where the corresponding problem was stated. When $0 < \mu < 1/2$, a result of this kind has been obtained by Kukso [Kuk07] in the so-called case of divergence.

(v) The statement of Theorem 2 can be viewed from a different angle: the algebraic points (α_1, α_2) satisfying (2) lie at a distance $Q^{-\mu}$ from the bisector $y = x$ of the first quadrant. A problem that naturally arises is to investigate the distribution of (α_1, α_2) near other rational lines, such as $y = 2x$ or $y = x/2$. A more general (and challenging) problem is to investigate the distribution of algebraic points (α_1, α_2) with conjugate coordinates of degree n near non-degenerate curves in the plane, such as the parabola $y = x^2$.

(vi) It would be interesting to develop the theory for non-archimedean extensions of $\mathbb{Q}$ and for 'proper' complex algebraic numbers; see [BGK08b] for a related result.

(vii) In previous works (such as [BM04, BM09, Mig83]) on the topic of this paper, examples of algebraic numbers with several very close conjugate algebraic numbers have been given. Lemmas 4 and 5 of this paper can be used to shed further light on this question, which is technically more involved. We will address this in a subsequent paper.

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