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# On Dirichlet non-improvable numbers and shrinking target problems

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*Abstract.* In one-dimensional Diophantine approximation, the Diophantine properties of a real number are characterized by its partial quotients, especially the growth of its large partial quotients. Notably, Kleinbock and Wadleigh [*Proc. Amer. Math. Soc.* **146**(5) (2018), 1833–1844] made a seminal contribution by linking the improvability of Dirichlet's theorem to the growth of the product of consecutive partial quotients. In this paper, we extend the concept of Dirichlet non-improvable sets within the framework of shrinking target problems. Specifically, consider the dynamical system $([0, 1), T)$ of continued fractions. Let $\{z_n\}_{n\geq 1}$ be a sequence of real numbers in $[0, 1]$ and let $B > 1$. We determine the Hausdorff dimension of the following set: $\{x \in [0, 1) : |T^n x - z_n||T^{n+1}x - Tz_n| < B^{-n}$ infinitely often$\}$.

Key words: Dirichlet non-improvable sets, shrinking target problems, Hausdorff dimension
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## 1. *Introduction*

The central question in Diophantine approximation is: how well can a given real number be approximated by rational numbers? In one-dimensional settings, the continued fraction serves as an important tool for this purpose, providing an algorithmic solution for finding the best rational approximation of a given real number. The continued fraction can be computed by the Gauss transformation $T : [0, 1) \to [0, 1)$ defined as

$$T(0) = 0, \quad T(x) = 1/x - \lfloor 1/x \rfloor \text{ if } x \in (0, 1),$$

where $\lfloor 1/x \rfloor$ is the integer part of $1/x$. For $x \in (0, 1)$, put $a_1(x) = \lfloor 1/x \rfloor$ and $a_{n+1}(x) = \lfloor 1/T^n(x) \rfloor = a_1(T^n(x))$ for $n \geq 1$. Then, $x \in (0, 1)$ can be written as the continued fraction expansion

$$x = \cfrac{1}{a_1(x) + \frac{1}{a_2(x) + \cdots}} =: [a_1(x), a_2(x), \ldots], \tag{1.1}$$

where $a_1(x), a_2(x), \ldots$ are positive integers, and called the *partial quotients* of $x$. Let $x = [a_1(x), a_2(x), \ldots]$ be its continued fraction expansion and the truncation $p_n(x)/q_n(x) = [a_1(x), a_2(x), \ldots, a_n(x)]$ be its $n$th *convergent*. The continued fraction expansion of a real number is widely recognized for its significant role in studying one-dimensional homogeneous Diophantine approximation. This can be inferred from the following two fundamental results.

THEOREM 1.1. *We have two results:*

(1) *the optimal rational approximation of the convergent*

$$\min_{1\le q\le q_n(x),p\in\mathbb{Z}} \left|x-\frac{p}{q}\right| = \left|x-\frac{p_n(x)}{q_n(x)}\right|, \qquad \min_{1\le q<q_n(x)} \|qx\| = \|q_{n-1}(x)x\|,$$

*where* $\|\cdot\|$ *denotes the distance to the nearest integer;*

(2) *Legendre's theorem*

$$\left|x-\frac{p}{q}\right| < \frac{1}{2q^2} \Longrightarrow \frac{p}{q} = \frac{p_n(x)}{q_n(x)} \quad \textit{for some } n \ge 1.$$

Building upon these two results, the Diophantine properties of a real number are largely characterized by its partial quotients, especially the growth of its large partial quotients within the consideration of the current paper.

The metrical theory of continued fractions, which concerns the size (in terms of measure or Hausdorff dimension, etc.) of the sets obeying some restrictions on their partial quotients, is an important subject in studying continued fractions. One focus is the study of the following sets:

$$E_m(B) := \{x \in [0,1) : a_n(x)a_{n+1}(x)\cdots a_{n+m-1}(x) \ge B^n \text{ i.o.}\},$$

where $m \in \mathbb{N}$, $B > 1$ and 'i.o.' stands for 'infinitely often'. It is worth noting that the sets $E_1(B)$ and $E_2(B)$ are related to homogeneous Diophantine approximation and Dirichlet non-improvable numbers (see [18, Lemma 2.2]), respectively. The Hausdorff dimension of $E_m(B)$ is completely given in the following result.

THEOREM 1.2. ([26] and [13, Theorem 1.7]) *We have*

$$\dim_{\mathrm{H}} E_m(B) = \inf\{s : P(T, -f_m(s)\log B - s\log|T'|) \le 0\},$$

*where* $\dim_{\mathrm{H}}$ *denotes the Hausdorff dimension,* $P(T,\cdot)$ *is a pressure function defined in* §2.2 *and* $f_m(s)$ *is given by the following iterative formulae:*

$$f_1(s) = s, \qquad f_{k+1}(s) = \frac{sf_k(s)}{1-s+f_k(s)}, \qquad k \ge 1.$$

There are many studies on Hausdorff dimensions of the sets related to $E_m(B)$, for example [2, 3, 8, 13–15, 22, 23, 26].

Since the partial quotients can be obtained through Gauss map, the theory also has close connections with dynamical systems and ergodic theory. Note that

$$a_n(x) = a_1(T^{n-1}x) = \left\lfloor \frac{1}{T^{n-1}x} \right\rfloor \in \left[\frac{1}{2T^{n-1}x}, \frac{1}{T^{n-1}x}\right],$$

and so

$$a_n(x) \geq B^n \Longrightarrow T^{n-1}x \leq B^{-n} \quad \text{and} \quad T^{n-1}x \leq B^{-n} \Longrightarrow a_n(x) \geq B^n/2. \tag{1.2}$$

In other words, the $n$th partial quotient being sufficiently large corresponds to $T^{n-1}x$ being sufficiently close to 0. From this simple observation, Li *et al* [21] came to the following generalization of $E_1(B)$:

$$E_1(\{z_n\}_{n\geq 1}, B) := \{x \in [0,1] : |T^n x - z_n| \leq B^{-n} \text{ i.o.}\},$$

where $\{z_n\}_{n\geq 1}$ is a sequence of real numbers in $[0, 1]$. They further showed that the Hausdorff dimension of this set is the same as that of $E_1(B)$. It is not difficult to deduce from equation (1.2) that $E_1(\{z_n\}_{n\geq 1}, B)$ almost returns to $E_1(B)$ if $z_n \equiv 0$ for all $n \geq 1$. The study of the metrical property of $E_1(\{z_n\}_{n\geq 1}, B)$ is also referred to as a *shrinking target problem*, which is initially introduced by Hill and Velani [11], and has recently gained much attention. See [1, 4, 6, 12, 19, 22] and references therein.

Following this kind of philosophy and starting from yet another observation,

$$a_n(x)a_{n+1}(x) \geq B^n \Longrightarrow T^{n-1}x \cdot T^n x \leq B^{-n}$$

and

$$T^{n-1}x \cdot T^n x \leq B^{-n} \Longrightarrow a_n(x)a_{n+1}(x) \geq B^n/4,$$

we introduce the following generalization of $E_2(B)$:

$$E_2(\{z_n\}_{n\geq 1}, B) := \{x \in [0,1) : |T^n x - z_n||T^{n+1}x - Tz_n| < B^{-n} \text{ i.o.}\}$$

with $\{z_n\}_{n\geq 1}$ and $B > 1$ given above. The Hausdorff dimension of the set $E_2(\{z_n\}_{n\geq 1}, B)$ is completely determined in the current work. For each $n \geq 1$, let us define three quantities as follows:

$$\begin{aligned}
s_{n,1} &= \inf\bigg\{s \in [0,1] : \sum_{a_1,\ldots,a_n\in\mathbb{N}} \frac{1}{q_n(a_1,\ldots,a_n)^{2s} B^{ns^2}} \leq 1\bigg\},\\
s_{n,2} &= \inf\bigg\{s \in [0,1] : \sum_{a_1,\ldots,a_n\in\mathbb{N}} \frac{a_1(z_n)^{1-s}}{q_n(a_1,\ldots,a_n)^{2s} B^{ns}} \leq 1\bigg\},\\
s_{n,3} &= \inf\bigg\{s \in [0,1] : \sum_{a_1,\ldots,a_n\in\mathbb{N}} \frac{1}{q_n(a_1,\ldots,a_n)^{2s} a_1(z_n)^s B^{ns/2}} \leq 1\bigg\}.
\end{aligned} \tag{1.3}$$

We adopt the convention that $a_1(0) = +\infty$, in this case, set $s_{n,2} = 1$ and $s_{n,3} = 0$. The $n$th pre-dimensional number is defined by

$$s_n = \begin{cases} s_{n,1} & \text{if } s_{n,1} \leq s_{n,2}, \\ \max\{s_{n,2}, s_{n,3}\} & \text{if } s_{n,1} > s_{n,2}. \end{cases} \tag{1.4}$$

THEOREM 1.3. *We have*

$$\dim_{\mathrm{H}} E_2(\{z_n\}_{n\geq 1}, B) = \limsup_{n\to\infty} s_n := s^*.$$

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*Remark 1.4.* We consider the approximation of $|T^n x - z_n||T^{n+1}x - Tz_n|$ instead of $|T^n x - z_n||T^{n+1}x - y_n|$ for the following reasons. The latter focuses on the interaction between the approximation of $T^n x$ to $z_n$ and that of $T^{n+1}x$ to $y_n$. Assume for the moment that $T^n x$ and $T^{n+1}x$ are sufficiently close to $z_n$ and $y_n$, respectively. Then, most prefixes of the continued fraction expansions of $T^n x$ and $z_n$ are the same. The same applies to $T^{n+1}x$ and $y_n$. These imply that $y_n$ is sufficiently close to $Tz_n$, and so is $|T^n x - z_n||T^{n+1}x - Tz_n|$ to $|T^n x - z_n||T^{n+1}x - y_n|$. Therefore, our setting is not as restrictive as it may seem.

*Remark 1.5.* The Hausdorff dimension of $E_2(\{z_n\}_{n\geq 1}, B)$ depends on the location of $z_n$, more precisely, on $a_1(z_n)$, which does not happen with $E_1(\{z_n\}_{n\geq 1}, B)$. Let us illustrate it with two simple examples. If $z_n \equiv 0$ for all $n \geq 1$, then $E_2(\{z_n\}_{n\geq 1}, B)$ almost reduces to $E_2(B)$. This motivates the definition of $s_{n,1}$. If $z_n \equiv [1, 1, \ldots]$ for all $n \geq 1$, then by a simple fact from the theory of continued fraction,

$$\begin{aligned}&|T^n x - z_n| \text{ is small enough}\\ &\quad\Longrightarrow |T^n x - z_n| = |T^{n+1}x - Tz_n|, \text{ up to a multiplicative constant.}\end{aligned}$$

Therefore, up to a multiplicative constant,

$$|T^n x - z_n||T^{n+1}x - Tz_n| < B^{-n} \Longrightarrow |T^n x - z_n| < B^{-n/2}.$$

This motivates the definition of $s_{n,3}$. As for $s_{n,2}$, it can be interpreted as follows: as $z_n$ varies from 1 to 0, the optimal cover of $E_2(\{z_n\}_{n\geq 1}, B)$ changes, leading to the definition of $s_{n,2}$.

The structure of this paper is as follows. In §2, we recall several notions and elementary properties of continued fractions. We prove the upper bound and lower bound of Hausdorff dimension of the set $E_2(\{z_n\}_{n\geq 1}, B)$ in §3 and §4, respectively.

## 2. *Preliminaries*

In this section, we recall some basic properties of continued fractions and pressure functions, as well as establishing some basic facts. Throughout, for two variables $f$ and $g$, the notation $f \ll g$ means that $f \leq cg$ for some unspecified constant $c$, and the notation $f \asymp g$ means that $f \ll g$ and $g \ll f$. For a set $A$, $|A|$ stands for the diameter of $A$. We use $\mathcal{L}$ to represent the Lebesgue measure.

### 2.1. *Continued fraction.*

It is well known that if $x \in (0, 1)$ is a rational number, the expansion of $x$ is finite; if $x \in (0, 1)$ is an irrational number, the expansion of $x$ is infinite. A finite truncation on the expansion of $x$ gives rational fraction $p_n(x)/q_n(x) := [a_1(x), a_2(x), \ldots, a_n(x)]$, which is called the $n$th convergents of $x$. With the conventions

$$p_{-1} = 1, \quad q_{-1} = 0, \quad p_0 = 0 \quad \text{and} \quad q_0 = 1,$$

the sequence $p_n = p_n(x)$ and $q_n = q_n(x)$ can be given by the following recursive relations:

$$p_{n+1} = a_{n+1}(x)p_n + p_{n-1}, \quad q_{n+1} = a_{n+1}(x)q_n + q_{n-1}. \tag{2.1}$$

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Clearly, $q_n(x)$ is determined by $a_1(x), \ldots, a_n(x)$. So we may write $q_n(a_1(x), \ldots, a_n(x))$. When no confusion is likely to arise, we write $a_n$ and $q_n$, respectively, in place of $a_n(x)$ and $q_n(x)$ for simplicity.

For an integer vector $(a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$ with $n \geq 1$, denote

$$I_n(a_1, a_2, \ldots, a_n) := \{x \in [0, 1) : a_1(x) = a_1, a_2(x) = a_2, \ldots, a_n(x) = a_n\} \tag{2.2}$$

for the corresponding $n$th level *cylinder*, that is, the set of all real numbers in $[0, 1)$ whose continued fraction expansions begin with $(a_1, \ldots, a_n)$.

We will frequently use the following well-known properties of continued fraction expansion. They are explained in the standard texts [16, 17].

PROPOSITION 2.1. *For any positive integers $a_1, \ldots, a_n$, let $p_n = p_n(a_1, \ldots, a_n)$ and $q_n = q_n(a_1, \ldots, a_n)$ be defined recursively by equation (2.1).*

(1) *We have $q_n \geq 2^{(n-1)/2}$ and*

$$\prod_{i=1}^{n} a_i \leq q_n \leq \prod_{i=1}^{n}(a_i + 1) \leq \prod_{i=1}^{n} 2a_i. \tag{2.3}$$

(2) *It holds that*

$$q_n \leq (a_n + 1)q_{n-1}, \quad 1 \leq \frac{q_{n+k}(a_1, \ldots, a_n, \ldots, a_{n+k})}{q_n(a_1, \ldots, a_n)q_k(a_{n+1}, \ldots, a_{n+k})} \leq 2,$$

$$\frac{a_k + 1}{2} \leq \frac{q_n(a_1, \ldots, a_n)}{q_{n-1}(a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n)} \leq a_k + 1.$$

(3) *We have*

$$I_n(a_1, \ldots, a_n) = \begin{cases} [p_n/q_n, (p_n + p_{n-1})/(q_n + q_{n-1})) & \text{if } n \text{ is even}, \\ ((p_n + p_{n-1})/(q_n + q_{n-1}), p_n/q_n] & \text{if } n \text{ is odd}. \end{cases}$$

*The length of $I_n(a_1, \ldots, a_n)$ is given by*

$$\frac{1}{2q_n^2} \leq |I_n(a_1, \ldots, a_n)| = \frac{1}{q_n(q_n + q_{n-1})} \leq \frac{1}{q_n^2}. \tag{2.4}$$

The next proposition describes the positions of cylinders $I_{n+1}$ of level $n + 1$ inside the $n$th level cylinder $I_n$.

PROPOSITION 2.2. *Let $I_n = I_n(a_1, \ldots, a_n)$ be an nth level cylinder, which is partitioned into sub-cylinders $\{I_{n+1}(a_1, \ldots, a_n, a_{n+1}) : a_{n+1} \in \mathbb{N}\}$. When $n$ is odd, these sub-cylinders are positioned from left to right, as $a_{n+1}$ increases from $1$ to $\infty$; when $n$ is even, they are positioned from right to left.*

2.2. *Pressure function.* Pressure function is an appropriate tool in dealing with dimension problems in infinite conformal iterated function systems. We recall that the pressure function with a continuous potential can be approximated by the pressure functions

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restricted to the sub-systems in continued fractions. For more information on pressure functions, we refer the readers to [9, 24, 25].

Let $\mathbb{A}$ be a finite or infinite subset of $\mathbb{N}$, and define

$$X_{\mathbb{A}} = \{x \in [0, 1) : a_n(x) \in \mathbb{A} \text{ for all } n \geq 1\}.$$

Then, with the Gauss map $T$ restricted to it, $X_{\mathbb{A}}$ forms a dynamical system. The pressure function restricted to this sub-system $(X_{\mathbb{A}}, T)$ with potential $\phi : [0, 1) \to \mathbb{R}$ is defined as

$$P_{\mathbb{A}}(T, \phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{(a_1,\ldots,a_n)\in\mathbb{A}^n} \sup_{x\in X_{\mathbb{A}}} e^{S_n\phi([a_1,\ldots,a_n+x])}, \tag{2.5}$$

where $S_n\phi(x)$ denotes the ergodic sum $\phi(x) + \cdots + \phi(T^{n-1}x)$. When $\mathbb{A} = \mathbb{N}$, we write $P(T, \phi)$ for $P_{\mathbb{N}}(T, \phi)$.

The $n$th variation $\mathrm{Var}_n(\phi)$ of $\phi$ is defined as

$$\mathrm{Var}_n(\phi) := \sup\{|\phi(x) - \phi(y)| : I_n(x) = I_n(y)\}.$$

The existence of the limit in equation (2.5) is due to the following result.

PROPOSITION 2.3. [21, 24] *Let $\phi : [0, 1) \to \mathbb{R}$ be a real function with $\mathrm{Var}_1(\phi) < \infty$ and $\mathrm{Var}_n(\phi) \to 0$ as $n \to \infty$. Then, the limit defining $P_{\mathbb{A}}(T, \phi)$ in equation (2.5) exists and the value of $P_{\mathbb{A}}(T, \phi)$ remains the same even without taking the supremum over $x \in X_{\mathbb{A}}$ in equation (2.5).*

The following proposition states a continuity of the pressure function when the continued fraction system $([0, 1), T)$ is approximated by its sub-systems $(X_{\mathbb{A}}, T)$.

PROPOSITION 2.4. [9, Proposition 2] *Let $\phi : [0, 1) \to \mathbb{R}$ be a real function with $\mathrm{Var}_1(\phi) < \infty$ and $\mathrm{Var}_n(\phi) \to 0$ as $n \to \infty$. We have*

$$P(T, \phi) = P_{\mathbb{N}}(T, \phi) = \sup\{P_{\mathbb{A}}(T, \phi) : \mathbb{A} \text{ is a finite subset of } \mathbb{N}\}. \tag{2.6}$$

The potential functions related to the dimension of $E_2(\{z_n\}_{n\geq1}, B)$ will be taken as the following forms:

(1) $\phi_1(x) = -s \log |T'(x)| - s^2 \log B$ corresponding to the pre-dimensional number $s_{n,1}$ (see equation (4.5));

(2) $\phi_2(x) = -s \log |T'(x)| - s \log B + (1 - s)\alpha$ for some $\alpha > 0$ corresponding to the pre-dimensional number $s_{n,2}$ (see equation (4.22));

(3) $\phi_3(x) = -s \log |T'(x)| - (s/2) \log B - s\beta$ for some $\beta \geq 0$ corresponding to the pre-dimensional number $s_{n,3}$ (see equation (4.31)).

2.3. *Basic facts.* Some lemmas will be established in this subsection for future use. The first one involves a summation that naturally appearing in the proof of the upper bound of $E_2(\{z_n\}_{n\geq1}, B)$.

LEMMA 2.5. *For any $a \in \mathbb{N}^+$ and $t > 1/2$, the following holds:*

$$\sum_{b \in \mathbb{N}^+ \setminus \{a\}} \frac{a^t}{b^t |a-b|^t} \asymp a^{1-t}. \tag{2.7}$$

*Proof.* We are going to decompose $\{b \in \mathbb{N} : b \neq a\}$ appearing in the left summation into four blocks, and in each block using a particular estimate. Write

$$\{b \in \mathbb{N} : b \neq a\} = A + B + C + D,$$

where

$$A = \{b \in \mathbb{N} : 1 \le b \le a/2\}, \quad B = \{b \in \mathbb{N} : a/2 \le b < a\},$$
$$C = \{b \in \mathbb{N} : a < b \le 2a\}, \quad D = \{b \in \mathbb{N} : b > 2a\}.$$

For the block $A$, since $1 \le b \le a/2$ for any $b \in A$, one has $a/2 \le |a-b| < a$. Thus,

$$\sum_{b \in A} \frac{a^t}{b^t |a-b|^t} \asymp \sum_{b \in A} \frac{1}{b^t} \asymp \int_1^{a/2} \frac{1}{x^t}\, dx \asymp a^{1-t}.$$

For the block $B$, using $a/2 \le b < a$ and both $a$ and $b$ are integers, we have

$$\sum_{b \in B} \frac{a^t}{b^t |a-b|^t} \asymp \sum_{b \in B} \frac{1}{|a-b|^t} = \sum_{1 \le c \le a/2} \frac{1}{c^t} \asymp a^{1-t}.$$

The estimation for the block $C$ is similar to $B$, we omit the details.

For the block $D$, since $b > 2a$ for any $b \in D$, it follows that $b/2 \le |a-b| < b$. So, by the condition $t > 1/2$,

$$\sum_{b \in D} \frac{a^t}{b^t |a-b|^t} \asymp a^t \cdot \sum_{b \in D} \frac{1}{b^{2t}} \asymp a^t \cdot \int_{2a}^{\infty} \frac{1}{x^{2t}}\, dx \asymp a^{1-t}.$$

Combine the above four estimations and the proof is completed. □

Next, we explore some properties of the pre-dimension numbers $s_{n,i}$ and their relationship to $a_1(z_n)$. Recall the definitions of $s_{n,i}$ in equation (1.3).

LEMMA 2.6. *Let $n \in \mathbb{N}$ and $f(s) = \sum_{a_1,\ldots,a_n \in \mathbb{N}} q_n(a_1, \ldots, a_n)^{-2s}$. We have*

$$f(s) < \infty \iff s > 1/2.$$

*Moreover, $f(s)$ is continuous on $s > 1/2$ and goes to infinity as $s \downarrow 1/2$. Consequently:*

(1) $s_{n,i} > 1/2$ *for* $i = 1, 2, 3$;

(2) $s_{n,1}$ *satisfies*

$$\sum_{a_1,\ldots,a_n \in \mathbb{N}} \frac{1}{q_n(a_1, \ldots, a_n)^{2s_{n,1}} B^{n s_{n,1}^2}} = 1.$$

*Similar arguments apply to $s_{n,2}$ and $s_{n,3}$, with the summations being replaced in the obvious way according to their definitions.*

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*Proof.* The proofs follow the ideas in [26, Lemma 2.6] and [9, Lemma 3.2].

By Proposition 2.1(1), for any $s > 0$,

$$f(s) = \sum_{a_1,\ldots,a_n\in\mathbb{N}} q_n(a_1,\ldots,a_n)^{-2s} \asymp \sum_{a_1,\ldots,a_n\in\mathbb{N}} \prod_{i=1}^{n} a_i^{-2s} = \bigg(\sum_{a\in\mathbb{N}} a^{-2s}\bigg)^n$$
$$\asymp \bigg(\int_1^\infty x^{-2s}\,dx\bigg)^n = (1-2s)^{-n}, \tag{2.8}$$

which clearly implies the first point of the lemma.

By Hölder inequality, $\log f(s)$ is convex on $s > 1/2$. Hence, $\log f(s)$ is continuous on $s > 1/2$, so is $f(s)$. By equation (2.8), it is easily seen that $f(s)$ goes to infinity as $s \downarrow 1/2$.

Items (1) and (2) follow from the continuity of $f(s)$ and the definitions of $s_{n,i}$ ($i = 1, 2, 3$) directly. □

LEMMA 2.7. *Let $s_{n,i}$ be as in equation (1.3), $i = 1, 2, 3$. The following statements hold:*

(1) *if $s_{n,1} \le s_{n,2}$, then $a_1(z_n) \ge B^{ns_{n,1}}$;*
(2) *if $s_{n,1} > s_{n,2}$, then $a_1(z_n) < B^{ns_{n,2}}$;*
(3) *if $s_{n,2} < s_{n,3}$, then $a_1(z_n) < B^{ns_{n,3}/2}$;*
(4) *if $s_{n,2} \ge s_{n,3}$, then $a_1(z_n) \ge B^{ns_{n,2}/2}$.*

*Proof.* (1) Since $s_{n,1} \le s_{n,2}$, by Lemma 2.6(2) and the definitions of $s_{n,1}$ and $s_{n,2}$,

$$\sum_{a_1,\ldots,a_n\in\mathbb{N}} \frac{1}{q_n(a_1,\ldots,a_n)^{2s_{n,1}} B^{ns_{n,1}^2}} = 1 \le \sum_{a_1,\ldots,a_n\in\mathbb{N}} \frac{a_1(z_n)^{1-s_{n,1}}}{q_n(a_1,\ldots,a_n)^{2s_{n,1}} B^{ns_{n,1}}},$$

which is equivalent to

$$a_1(z_n) \ge B^{ns_{n,1}}.$$

The proofs of the items (2)–(4) follow the same lines as item (1). □

## 3. *Upper bound of* $\dim_{\mathrm{H}} E_2(\{z_n\}_{n\ge1}, B)$

In the remainder of this paper, to simplify the notation, sequences of natural numbers will be denoted by letters in boldface: $\boldsymbol{a}, \boldsymbol{b}, \ldots$.

Let

$$F_n := \bigcup_{\boldsymbol{a}\in\mathbb{N}^n} \{x \in I_n(\boldsymbol{a}) : |T^n x - z_n||T^{n+1}x - Tz_n| < B^{-n}\}.$$

It follows that

$$E_2(\{z_n\}_{n\ge1}, B) = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} F_n.$$

Our next objective is to find a suitable cover of $F_n$ for $n \ge 1$, which depends on the values of $a_1(z_n)$. The following identity will be crucial for this purpose. Note that for any $x = [a_1(x), \ldots, a_n(x), \ldots]$,

$$T^n x = [a_{n+1}(x), \ldots] = \frac{1}{a_{n+1}(x) + T^{n+1}x}.$$

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Then, it follows that for $z_n \neq 0$,

$$|T^n x - z_n| = \left| \frac{1}{a_{n+1}(x) + T^{n+1}x} - \frac{1}{a_1(z_n) + Tz_n} \right| = \left| \frac{(a_1(z_n) - a_{n+1}(x)) + (Tz_n - T^{n+1}x)}{(a_{n+1}(x) + T^{n+1}x)(a_1(z_n) + Tz_n)} \right|. \tag{3.1}$$

The proof of the upper bound relies on investigating the finer structure of $F_n$, which will be divided into two cases presented in the following two subsections, respectively.

Let $s > \limsup_{n\to\infty} s_n$, where $s_n$ is given in equation (1.4). Then, there exists an $\epsilon > 0$ for which

$$s - \epsilon > s_n \quad \text{for all large } n. \tag{3.2}$$

3.1. *Optimal cover of $F_n$ when n is large enough and $s_{n,1} \leq s_{n,2}$.* Suppose that $n$ is large enough so that equation (3.2) is satisfied. Since $s_{n,1} \leq s_{n,2}$, we have

$$s_n = s_{n,1} < s - \epsilon \tag{3.3}$$

with $s$ and $\epsilon$ given in equation (3.2), and by Lemma 2.7

$$a_1(z_n) \geq B^{ns_{n,1}}.$$

Now, consider the intersection of $F_n$ with $(n+1)$th level cylinders. For any $(\boldsymbol{a}, a_{n+1}) \in \mathbb{N}^{n+1}$ with $\boldsymbol{a} \in \mathbb{N}^n$ and $a_{n+1} \leq B^{ns_{n,1}}/2$, let

$$J_{n+1}(\boldsymbol{a}, a_{n+1}) := F_n \cap I_{n+1}(\boldsymbol{a}, a_{n+1}).$$

Let $x \in I_{n+1}(\boldsymbol{a}, a_{n+1})$. Then, we have $a_{n+1}(x) = a_{n+1}$, and so $|a_{n+1}(x) - a_1(z_n)| \geq B^{ns_{n,1}}/2$, which is much larger than $|Tz_n - T^{n+1}x|$. Applying the identity (3.1) and using $0 \leq T^{n+1}x, Tz_n \leq 1$, we get

$$\frac{|a_1(z_n) - a_{n+1}|}{8a_1(z_n)a_{n+1}} \leq |T^n x - z_n| \leq \frac{2|a_1(z_n) - a_{n+1}|}{a_1(z_n)a_{n+1}}. \tag{3.4}$$

If $x$ also belongs to $J_{n+1}(\boldsymbol{a}, a_{n+1})$, then it satisfies the above inequality and $|T^n x - z_n||T^{n+1}x - Tz_n| \leq B^{-n}$. Thus,

$$|T^{n+1}x - Tz_n| \leq \frac{8a_1(z_n)a_{n+1}}{|a_1(z_n) - a_{n+1}|B^n},$$

which implies that

$$J_{n+1}(\boldsymbol{a}, a_{n+1}) \subset \left\{ x \in I_{n+1}(\boldsymbol{a}, a_{n+1}) : |T^{n+1}x - Tz_n| \leq \frac{8a_1(z_n)a_{n+1}}{|a_1(z_n) - a_{n+1}|B^n} \right\} = \left\{ x \in I_{n+1}(\boldsymbol{a}, a_{n+1}) : T^{n+1}x \in B\left( Tz_n, \frac{8a_1(z_n)a_{n+1}}{|a_1(z_n) - a_{n+1}|B^n} \right) \right\}.$$

Since for any $x \in I_{n+1}(\boldsymbol{a}, a_{n+1})$,

$$q_{n+1}(\boldsymbol{a}, a_{n+1})^2 \leq (T^{n+1})'(x) \leq 2q_{n+1}(\boldsymbol{a}, a_{n+1})^2, \tag{3.5}$$

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it follows that

$$|J_{n+1}(\boldsymbol{a}, a_{n+1})| \le \frac{16 a_1(z_n) a_{n+1}}{q_{n+1}(\boldsymbol{a}, a_{n+1})^2 |a_1(z_n) - a_{n+1}| B^n} \le \frac{16 a_1(z_n)}{q_n(\boldsymbol{a})^2 a_{n+1} |a_1(z_n) - a_{n+1}| B^n}. \tag{3.6}$$

We take this opportunity to infer that $J_{n+1}(\boldsymbol{a}, a_{n+1})$ contains an interval of length comparable to equation (3.6), which will be needed in the subsequent proof of the lower bound of $\dim_{\mathrm{H}} E_2(\{z_n\}_{n\ge1}, B)$. Note that equation (3.4) holds for any $x \in I_{n+1}(\boldsymbol{a}, a_{n+1})$. If $x$ further satisfies

$$|T^{n+1}x - Tz_n| \le \frac{a_1(z_n) a_{n+1}}{2|a_1(z_n) - a_{n+1}| B^n},$$

then by the second inequality in equation (3.4), one has $|T^n x - z_n||T^{n+1}x - Tz_n| \le B^{-n}$, which implies

$$J_{n+1}(\boldsymbol{a}, a_{n+1}) \supset \left\{ x \in I_{n+1}(\boldsymbol{a}, a_{n+1}) : T^{n+1}x \in B\left( Tz_n, \frac{a_1(z_n) a_{n+1}}{2|a_1(z_n) - a_{n+1}| B^n} \right) \right\}.$$

By equation (3.5), $J_{n+1}(\boldsymbol{a}, a_{n+1})$ contains an interval with length greater than

$$\frac{a_1(z_n) a_{n+1}}{4 q_{n+1}(\boldsymbol{a}, a_{n+1})^2 |a_1(z_n) - a_{n+1}| B^n} \ge \frac{a_1(z_n)}{16 q_n(\boldsymbol{a})^2 a_{n+1} |a_1(z_n) - a_{n+1}| B^n}. \tag{3.7}$$

Note that $F_n$ can be covered by the union of the interval $\bigcup_{a_{n+1} > B^{ns_{n,1}}/2} I_{n+1}(\boldsymbol{a}, a_{n+1})$ and the sets $J_{n+1}(\boldsymbol{a}, a_{n+1})$ with $a_{n+1} \le B^{ns_{n,1}}/2$. Therefore, by the previous discussion, the $s$-volume of optimal cover of $F_n$ can be estimated as follows:

$$\begin{aligned}
&\sum_{\boldsymbol{a}\in\mathbb{N}^n} \left( \frac{1}{q_n(\boldsymbol{a})^{2s} (B^{ns_{n,1}}/2)^s} + \sum_{a_{n+1} \le B^{ns_{n,1}}/2} \frac{16^s a_1(z_n)^s}{q_n(\boldsymbol{a})^{2s} a_{n+1}^s |a_1(z_n) - a_{n+1}|^s B^{ns}} \right) \\
&= \sum_{\boldsymbol{a}\in\mathbb{N}^n} \left( \frac{1}{q_n(\boldsymbol{a})^{2s} (B^{ns_{n,1}}/2)^s} + \frac{16^s}{q_n(\boldsymbol{a})^{2s} B^{ns}} \cdot \sum_{a_{n+1} \le B^{ns_{n,1}}/2} \frac{a_1(z_n)^s}{a_{n+1}^s |a_1(z_n) - a_{n+1}|^s} \right) \\
&\asymp \sum_{\boldsymbol{a}\in\mathbb{N}^n} \left( \frac{B^{-nss_{n,1}}}{q_n(\boldsymbol{a})^{2s}} + \frac{B^{ns_{n,1}(1-s) - ns}}{q_n(\boldsymbol{a})^{2s}} \right) \asymp \sum_{\boldsymbol{a}\in\mathbb{N}^n} \frac{B^{-nss_{n,1}}}{q_n(\boldsymbol{a})^{2s}},
\end{aligned}$$

where we use the main ideas of the proof of Lemma 2.5 with $t = s > s_{n,1} + \epsilon > 1/2$ (by Lemma 2.6(1)), $a = a_1(z_n)$ and $b = a_{n+1}$ for the inner-most summation in the second formula. By the definition of $s_{n,1}$ and the fact that $s > s_{n,1} + \epsilon$, we see that

$$\sum_{\boldsymbol{a}\in\mathbb{N}^n} \frac{B^{-nss_{n,1}}}{q_n(\boldsymbol{a})^{2s}} \le 2^{-n\delta_1} \tag{3.8}$$

for some $\delta_1 > 0$ depending on $s$ only.

3.2. *Optimal cover of $F_n$ when $n$ is large enough and $s_{n,1} > s_{n,2}$.* Suppose that $n$ is large enough so that equation (3.2) is satisfied. Since $s_{n,1} > s_{n,2}$, we have

$$s_n = \max\{s_{n,2}, s_{n,3}\} < s - \epsilon \tag{3.9}$$

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with $s$ and $\epsilon$ given in equation (3.2), and by Lemma 2.7,

$$a_1(z_n) < B^{ns_{n,2}}.$$

Following the same notation in the last section, still let $J_{n+1}(\boldsymbol{a}, a_{n+1}) := F_n \cap I_{n+1}(\boldsymbol{a}, a_{n+1})$ with $(\boldsymbol{a}, a_{n+1}) \in \mathbb{N}^{n+1}$. Let $x \in I_{n+1}(\boldsymbol{a}, a_{n+1})$, and so $a_{n+1}(x) = a_{n+1}$. The discussion is split into three subcases.

*Subcase (1A)*: $|a_1(z_n) - a_{n+1}| > 1$. Since $|Tz_n - T^{n+1}x| < 1$, applying the identity (3.1), we can get

$$\frac{|a_1(z_n) - a_{n+1}|}{8a_1(z_n)a_{n+1}} \leq |T^n x - z_n| \leq \frac{2|a_1(z_n) - a_{n+1}|}{a_1(z_n)a_{n+1}}.$$

Since $x \in I_{n+1}(\boldsymbol{a}, a_{n+1})$ is arbitrary, by the same reason as equations (3.6) and (3.7), we get that $J_{n+1}(\boldsymbol{a}, a_{n+1})$ is contained in an interval with length at most

$$\frac{16a_1(z_n)}{q_n(\boldsymbol{a})^2 a_{n+1}|a_1(z_n) - a_{n+1}|B^n} \tag{3.10}$$

and contains an interval with length at least

$$\frac{a_1(z_n)}{16q_n(\boldsymbol{a})^2 a_{n+1}|a_1(z_n) - a_{n+1}|B^n}. \tag{3.11}$$

*Subcase (1B)*: $|a_1(z_n) - a_{n+1}| = 1$. Applying the identity (3.1) again, we obtain

$$|T^n x - z_n| = \left| \frac{1 - (Tz_n - T^{n+1}x)}{(a_{n+1} + T^{n+1}x)(a_1(z_n) + Tz_n)} \right| \geq \frac{|1 - (Tz_n - T^{n+1}x)|}{4a_1(z_n)a_{n+1}}.$$

Denote

$$J^{(1)}_{n+1}(\boldsymbol{a}, a_{n+1}) := \{x \in I_{n+1}(\boldsymbol{a}, a_{n+1}) : |T^{n+1}x - Tz_n| \leq 8a_1(z_n)a_{n+1}B^{-n}\}$$

and

$$J^{(2)}_{n+1}(\boldsymbol{a}, a_{n+1}) := \{x \in I_{n+1}(\boldsymbol{a}, a_{n+1}) : |T^{n+1}x - Tz_n| \geq 1 - 8a_1(z_n)a_{n+1}B^{-n}\}.$$

If $8a_1(z_n)a_{n+1}B^{-n} \geq 1/2$, then the union of the above two sets is $I_{n+1}(\boldsymbol{a}, a_{n+1})$, which can obviously cover $J_{n+1}(\boldsymbol{a}, a_{n+1})$.

Now suppose that $8a_1(z_n)a_{n+1}B^{-n} < 1/2$. Let $y \notin J^{(1)}_{n+1}(\boldsymbol{a}, a_{n+1}) \cup J^{(2)}_{n+1}(\boldsymbol{a}, a_{n+1})$. There are two cases, one is $1/2 \leq |T^{n+1}y - Tz_n| < 1 - 8a_1(z_n)a_{n+1}B^{-n}$, and the other is $8a_1(z_n)a_{n+1}B^{-n} < |T^{n+1}y - Tz_n| < 1/2$. For the first case,

$$\begin{aligned} |T^n y - z_n||T^{n+1}y - Tz_n| &\geq \frac{|1 - (Tz_n - T^{n+1}y)|}{4a_1(z_n)a_{n+1}} \cdot |T^{n+1}y - Tz_n| \\ &> \frac{8a_1(z_n)a_{n+1}B^{-n}}{4a_1(z_n)a_{n+1}} \cdot \frac{1}{2} = B^{-n}. \end{aligned}$$

By employing the same strategy on the other case, one has for $y \notin J^{(1)}_{n+1}(\boldsymbol{a}, a_{n+1}) \cup J^{(2)}_{n+1}(\boldsymbol{a}, a_{n+1})$,

$$|T^n y - z_n||T^{n+1}y - Tz_n| > B^{-n},$$

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which implies that $y \notin J_{n+1}(\boldsymbol{a}, a_{n+1})$. Summarizing,

$$J_{n+1}(\boldsymbol{a}, a_{n+1}) \subset J_{n+1}^{(1)}(\boldsymbol{a}, a_{n+1}) \cup J_{n+1}^{(2)}(\boldsymbol{a}, a_{n+1}),$$

and therefore by the same reason as equation (3.6), $J_{n+1}(\boldsymbol{a}, a_{n+1})$ can be covered by two intervals whose length is at most

$$\frac{16 a_1(z_n) B^{-n}}{q_n(\boldsymbol{a})^2 a_{n+1} |a_1(z_n) - a_{n+1}|}. \tag{3.12}$$

*Subcase (1C)*: $a_1(z_n) = a_{n+1}$. Using the identity (3.1), one has

$$\frac{|T^{n+1}x - Tz_n|^2}{4a_1(z_n)^2} \leq |T^n x - z_n||T^{n+1}x - Tz_n| \leq \frac{|T^{n+1}x - Tz_n|^2}{a_1(z_n)^2}.$$

Clearly, the following holds:

$$\begin{aligned}
&\{x \in I_{n+1}(\boldsymbol{a}, a_{n+1}) : |T^{n+1}x - Tz_n| \leq a_1(z_n) B^{-n/2}\} \\
&\quad \subset J_{n+1}(\boldsymbol{a}, a_{n+1}) \\
&\quad \subset \{x \in I_{n+1}(\boldsymbol{a}, a_{n+1}) : |T^{n+1}x - Tz_n| \leq 2a_1(z_n) B^{-n/2}\}.
\end{aligned}$$

This together with equation (3.5) gives

$$|J_{n+1}(\boldsymbol{a}, a_{n+1})| \leq \frac{2a_1(z_n) B^{-n/2}}{q_{n+1}(\boldsymbol{a}, a_1(z_n))^2} \leq \frac{2B^{-n/2}}{q_n(\boldsymbol{a})^2 a_1(z_n)} \tag{3.13}$$

and $J_{n+1}(\boldsymbol{a}, a_{n+1})$ contains an interval with length greater than

$$\frac{a_1(z_n) B^{-n/2}}{q_{n+1}(\boldsymbol{a}, a_1(z_n))^2} \geq \frac{B^{-n/2}}{4q_n(\boldsymbol{a})^2 a_1(z_n)}. \tag{3.14}$$

Combining the estimations (3.10), (3.12) and (3.13), the $s$-volume of optimal cover of $F_n$ can be estimated as follows:

$$\begin{aligned}
&\ll \sum_{\boldsymbol{a} \in \mathbb{N}^n} \left( \sum_{a_{n+1} \neq a_1(z_n)} \frac{a_1(z_n)^s B^{-ns}}{q_n(\boldsymbol{a})^{2s} a_{n+1}^s |a_1(z_n) - a_{n+1}|^s} + \frac{B^{-ns/2}}{q_n(\boldsymbol{a})^{2s} a_1(z_n)^s} \right) \\
&\asymp \sum_{\boldsymbol{a} \in \mathbb{N}^n} \left( \frac{a_1(z_n)^{1-s} B^{-ns}}{q_n(\boldsymbol{a})^{2s}} + \frac{B^{-ns/2}}{q_n(\boldsymbol{a})^{2s} a_1(z_n)^s} \right),
\end{aligned}$$

where we have used Lemma 2.5 with $t = s > \max\{s_{n,2}, s_{n,3}\} + \epsilon > 1/2$ (by Lemma 2.6(1)), $a = a_1(z_n)$ and $b = a_{n+1}$ for the inner-most summation. By the definitions of $s_{n,2}$ and $s_{n,3}$ and $s > \max\{s_{n,2}, s_{n,3}\} + \epsilon$ (see equation (3.9)), we have

$$\sum_{\boldsymbol{a} \in \mathbb{N}^n} \left( \frac{a_1(z_n)^{1-s} B^{-ns}}{q_n(\boldsymbol{a})^{2s}} + \frac{B^{-ns/2}}{q_n(\boldsymbol{a})^{2s} a_1(z_n)^s} \right) \leq 2^{-n\delta_2} \tag{3.15}$$

for some $\delta_2 > 0$ depending on $s$ only.

3.3. *Completing the proof of the upper bound of* $\dim_{\mathrm{H}} E_2(\{z_n\}_{n\geq 1}, B)$. By the previous two subsections, we see that for given $s > \limsup_{n\to\infty} s_n$, by equations (3.8) and (3.15),

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there exists a cover of $F_n$ for which the corresponding $s$-volume

$$\ll 2^{-n \min\{\delta_1,\delta_2\}},$$

where $\delta_1 > 0$ and $\delta_2 > 0$ are respectively given in (3.8) and (3.15) and depend on $s$ only. Note that for any $N \in \mathbb{N}$,

$$E_2(\{z_n\}_{n\geq 1}, B) \subset \bigcup_{n=N} F_n.$$

By the definition of Hausdorff measure, one has

$$\mathcal{H}^s(E_2(\{z_n\}_{n\geq 1}, B)) \ll \lim_{N\to\infty} \sum_{n=N}^{\infty} 2^{-n \min\{\delta_1,\delta_2\}} = 0,$$

which implies that $\dim_{\mathrm{H}} E_2(\{z_n\}_{n\geq 1}, B) \leq s$. By the arbitrariness of $s$ ($> \limsup_{n\to\infty} s_n$), we have

$$\dim_{\mathrm{H}} E_2(\{z_n\}_{n\geq 1}, B) \leq \limsup_{n\to\infty} s_n,$$

which is what we want.

## 4. *Lower bound of* $\dim_{\mathrm{H}} E_2(\{z_n\}_{n\geq 1}, B)$

Before proving the lower bound, we list some definitions and auxiliary results which will be used later.

For any set $E \subset \mathbb{R}$, its $s$-dimensional Hausdorff content is given by

$$\mathcal{H}^s_\infty(E) := \inf \bigg\{ \sum_{i=1}^{\infty} |B_i|^s : E \subset \bigcup_{i=1}^{\infty} B_i \text{ where } B_i \text{ are open balls} \bigg\}.$$

With this definition, the lower bound of the Hausdorff dimension of a limsup set can be estimated by verifying some Hausdorff content bounds. The following lemma is a variant version of Falconer's sets of large intersection condition. For the details, see [10, Corollary 2.6].

LEMMA 4.1. *Let* $\{B_k\}_{k\geq 1}$ *be a sequence of balls in* $[0, 1]$ *that satisfies* $\mathcal{L}(\limsup_{k\to\infty} B_k) = \mathcal{L}([0, 1]) = 1$. *Let* $\{E_n\}_{n\geq 1}$ *be a sequence of open sets in* $[0, 1]$ *and* $E = \limsup_{n\to\infty} E_n$. *Let* $s > 0$. *If for any* $0 < t < s$, *there exists a constant* $c_t$ *such that*

$$\limsup_{n\to\infty} \mathcal{H}^t_\infty(E_n \cap B_k) \geq c_t |B_k| \tag{4.1}$$

*holds for all* $B_k$, *then* $\dim_{\mathrm{H}} E \geq s$.

*Remark 4.2.* In fact, limsup set satisfying equation (4.1) has the so-called large intersection property (see [7] or [10]), which means that the intersection of the sets with countably many similar copies of itself still has Hausdorff dimension at least $s$. In particular, the Hausdorff dimension of such limsup set is at least $s$. This property is beyond the subject of this paper, so we will not go into detail.

The Hausdorff content of a Borel set is typically estimated by putting measures or mass distributions on it, following the mass distribution described below.

PROPOSITION 4.3. (Mass distribution principle [5, Lemma 1.2.8]) *Let $E$ be a Borel subset of $\mathbb{R}$. If $E$ supports a strictly positive Borel measure $\mu$ that satisfies*

$$\mu(B(x,r)) \le cr^s$$

*for some constant $0 < c < \infty$ and for every ball $B(x,r)$, then $\mathcal{H}^s_\infty(E) \ge \mu(E)/c$.*

Since

$$\bigcap_{n=1}^{\infty} \bigcup_{\boldsymbol{u}\in\mathbb{N}^n} I_n(\boldsymbol{u}) = [0,1] \setminus \mathbb{Q},$$

the sequence of balls demanded in Lemma 4.1 can be taken as the set of all cylinders. We will show that for any $0 < t < s^*$ (recall that $s^* = \limsup_{n\to\infty} s_n$), there exists a constant $c_t$ depending on $t$ such that for any $k \ge 1$, and an arbitrary $k$th level cylinder $I_k(\boldsymbol{u})$ with $\boldsymbol{u} \in \mathbb{N}^k$,

$$\limsup_{n\to\infty} \mathcal{H}^t_\infty(F_n \cap I_k(\boldsymbol{u})) \ge c_t |I_k(\boldsymbol{u})|, \tag{4.2}$$

where recall that

$$F_n = \bigcup_{\boldsymbol{a}\in\mathbb{N}^n} \{x \in I_n(\boldsymbol{a}) : |T^n x - z_n||T^{n+1}x - Tz_n| \le B^{-n}\}.$$

In view of mass distribution principle, to establish equation (4.2), we will construct a measure $\mu$ supported on $F_n \cap I_k(\boldsymbol{u})$ and then estimate the $\mu$-measure of arbitrary balls. Here and hereafter, $\boldsymbol{u}$ and $t$ will be fixed. The proof is divided into three cases according to how $s^*$ is attained, presented section by section.

4.1. *Case I: $s^* = \limsup_{n\to\infty} s_n$ is obtained along a subsequence of $\{s_{n,1}\}_{n\ge1}$.* Note that $t < s^*$. There exist infinitely many $n$ such that

$$s_n = s_{n,1} > t. \tag{4.3}$$

For such $n$, by the definition of $s_n$,

$$s_{n,1} \le s_{n,2},$$

which by Lemma 2.7(1) gives

$$a_1(z_n) \ge B^{ns_{n,1}}. \tag{4.4}$$

Following the same argument as [26, Lemma 2.4] with some obvious modification, we have

$$s^* = \lim_{\substack{n \text{ satisfies equation (4.3)}\\ n\to\infty}} s_{n,1} = \inf\{s > 0 : P(T, -s\log|T'| - s^2 \log B) \le 0\}. \tag{4.5}$$

Fix an $\epsilon < s^* - t$. By Proposition 2.4 and Lemma 2.6(2), there exist integers $\ell > \max\{\log_B 4/\epsilon, 2t/\epsilon + 1\}$ and $M > 0$ such that the unique positive number $s = s(\ell, M)$

satisfying the equation

$$\sum_{\boldsymbol{a}\in\{1,\ldots,M\}^\ell} \frac{1}{q_\ell(\boldsymbol{a})^{2s} B^{\ell s^2}} = 1 \tag{4.6}$$

is greater than $t+\epsilon$. It should be emphasized that $\ell > \max\{\log_B 4/\epsilon, 2t/\epsilon + 1\}$ implies that

$$B^{\ell\epsilon} \geq 4 \tag{4.7}$$

and that for any $\boldsymbol{a} \in \mathbb{N}^\ell$,

$$q_\ell(\boldsymbol{a})^{-2s} \leq q_\ell(\boldsymbol{a})^{-2(t+\epsilon)} \leq q_\ell(\boldsymbol{a})^{-2t} 2^{-(\ell-1)\epsilon} \leq q_\ell(\boldsymbol{a})^{-2t} 2^{-2t}. \tag{4.8}$$

Choose $n \in \mathbb{N}$ so that $n-k \geq \ell kt/\epsilon$ and equation (4.3) is satisfied, and write $n-k = m\ell + \ell_0$, where $0 \leq \ell_0 < \ell$. By the choice of $n$, one has $m \geq kt/\epsilon$.

From now on, let $\ell$, $M$ and $n$ be fixed. Let $1^{\ell_0}$ be the word consisting only of 1 and of length $\ell_0$. Let $\tilde{\boldsymbol{u}} = (\boldsymbol{u}, 1^{\ell_0}) \in \mathbb{N}^{k+\ell_0}$ and consider the following auxiliary set defined by $(n+1)$th level cylinders:

$$\begin{aligned} \{I_{n+1}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, b) : \boldsymbol{a}_i \in \{1, \ldots, M\}^\ell, 1 \leq i \leq m,& \\ B^{nt} \leq b \leq 2B^{nt} \text{ and } b \text{ is even}\}.& \end{aligned} \tag{4.9}$$

The additional requirement here, that $b$ be even, is that any two cylinders in the above set are well separated (see Lemma 4.4 below). By equation (3.7), there is an interval, denoted by $\mathcal{I}_1(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, b)$ (given that there are three cases (§4.1–§4.3) to consider, each requiring the construction of subsets and measures, we will use subscripts 1, 2, 3 or superscripts (1), (2), (3) to distinguish and avoid burdening of notation, where there is no risk of ambiguity) such that

$$\mathcal{I}_1(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, b) \subset F_n \cap I_{n+1}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, b). \tag{4.10}$$

It is easy to see that $\mathcal{I}_1(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, b) \subset F_n \cap I_{k+\ell_0}(\tilde{\boldsymbol{u}}) \subset F_n \cap I_k(\boldsymbol{u})$. Moreover, the length of this interval can be estimated as

$$\begin{aligned} |\mathcal{I}_1(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, b)| &\geq \frac{a_1(z_n)}{16 q_n(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m)^2 b |a_1(z_n) - b| B^n} \\ &\geq \frac{1}{16 q_n(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m)^2 b B^n} \\ &\geq \frac{1}{32 q_n(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m)^2 B^{n(1+t)}}, \end{aligned} \tag{4.11}$$

where we use $a_1(z_n) \geq B^{ns_{n,1}} > B^{nt} B^{n\epsilon} \geq 4B^{nt}$ (see equations (4.4) and (4.7)) and $b \leq 2B^{nt}$ in the second and third inequalities. In what follows, we call $\mathcal{I}_1(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, b)$ in $F_n \cap I_{k+\ell_0}(\tilde{\boldsymbol{u}})$ as *fundamental interval*, and the cylinders $I_{n+1}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, b)$ and $I_{k+\ell_0+p\ell}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_p)$ with $0 \leq p \leq m$ as *basic cylinders*.

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Define a probability measure $\mu_1$ supported on $F_n \cap I_{k+\ell_0}(\tilde{\boldsymbol{u}}) \subset F_n \cap I_k(\boldsymbol{u})$ as follows:

$$\mu_1 = \sum_{\boldsymbol{a}_1\in\{1,\ldots,M\}^\ell} \cdots \sum_{\boldsymbol{a}_m\in\{1,\ldots,M\}^\ell} \sum_{\substack{B^{nt}\le b\le 2B^{nt}\\ b \text{ is even}}} \bigg(\prod_{i=1}^{m} \frac{1}{q_\ell(\boldsymbol{a}_i)^{2s} B^{\ell s^2}}\bigg) \cdot \frac{2}{B^{nt}} \cdot \mathcal{L}^{(1)}_{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b}, \tag{4.12}$$

where

$$\mathcal{L}^{(1)}_{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b} := \frac{\mathcal{L}|_{\mathcal{I}_1(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b)}}{\mathcal{L}(\mathcal{I}_1(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b))}$$

denotes the normalized Lebesgue measure on the fundamental interval $\mathcal{I}_1(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b)$. Roughly speaking, we assign each word $(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b)$ a different weight: the weights corresponding to $\boldsymbol{a}_i$ and $b$ are respectively

$$\frac{1}{q_\ell(\boldsymbol{a}_i)^{2s} B^{\ell s^2}} \quad \text{and} \quad \frac{2}{B^{nt}}.$$

The definition of $s$ (see equation (4.6)) ensures that $\mu_1$ is a probability measure.

The next two lemmas describe the gap between fundamental intervals defined in equation (4.10) and their $\mu_1$-measures, respectively.

LEMMA 4.4. *Let $\mathcal{I}_1 = \mathcal{I}_1(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b)$ and $\mathcal{I}_1' = \mathcal{I}_1(\tilde{\boldsymbol{u}},\boldsymbol{a}_1',\ldots,\boldsymbol{a}_m',b')$ be two fundamental intervals defined in equation (4.10). Then, the following statements hold.*

(1) *If $(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{p-1}) = (\boldsymbol{a}_1',\ldots,\boldsymbol{a}_{p-1}')$ but $\boldsymbol{a}_p \ne \boldsymbol{a}_p'$ for some $1\le p\le m$, then*

$$\operatorname{dist}(\mathcal{I}_1,\mathcal{I}_1') \ge \frac{|I_{k+\ell_0+p\ell}(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_p)|}{2(M+2)^4}.$$

(2) *If $(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m) = (\boldsymbol{a}_1',\ldots,\boldsymbol{a}_m')$ but $b\ne b'$, then*

$$\operatorname{dist}(\mathcal{I}_1,\mathcal{I}_1') \ge \frac{|I_{n+1}(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b)|}{32}.$$

*Proof.* (1) Bear in mind that $\mathcal{I}_1$ and $\mathcal{I}_1'$ are two fundamental intervals contained in the $(n+1)$th level cylinders $I_{n+1}(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b)$ and $I_{n+1}(\tilde{\boldsymbol{u}},\boldsymbol{a}_1',\ldots,\boldsymbol{a}_m',b')$, respectively. For further discussion, we write $(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b) = (c_1,c_2,\ldots,c_{m\ell},c_{m\ell+1})$ and $(\boldsymbol{a}_1',\ldots,\boldsymbol{a}_m',b') = (c_1',c_2',\ldots,c_{m\ell}',c_{m\ell+1}')$ for the moment. Assume that $1\le i\le m\ell+1$ is the smallest integer such that $c_i\ne c_i'$. By the assumption in item (1), we have $(p-1)\ell < i \le p\ell \le m\ell$. Therefore, the distance between $\mathcal{I}_1$ and $\mathcal{I}_1'$ is majorized by

$$\operatorname{dist}(I_{k+\ell_0+i+1}(\tilde{\boldsymbol{u}},c_1,\ldots,c_{i-1},c_i,c_{i+1}), I_{k+\ell_0+i+1}(\tilde{\boldsymbol{u}},c_1,\ldots,c_{i-1},c_i',c_{i+1}')). \tag{4.13}$$

Now, we consider two cases.

*Case 1:* $1\le i< m\ell$. Since $i<m\ell$, we have $c_{i+1},c_{i+1}'\le M$. By the distribution of cylinders (see Proposition 2.2), either $I_{k+\ell_0+i+1}(\tilde{\boldsymbol{u}},c_1,\ldots,c_{i-1},c_i,M+1)$ or $I_{k+\ell_0+i+1}(\tilde{\boldsymbol{u}},c_1,\ldots,c_{i-1},c_i',M+1)$ lies between two cylinders listed in equation (4.13). Without loss of generality, assume that this is satisfied by the former one. Then, by equation (4.13) and using Proposition 2.1(2) repeatedly, we have

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$$\begin{aligned}
\operatorname{dist}(\mathcal{I}_1,\mathcal{I}_2) &\geq |I_{k+\ell_0+i+1}(\tilde{\boldsymbol{u}},c_1,\ldots,c_{i-1},c_i,M+1)| \\
&\geq \frac{1}{2q_{k+\ell_0+i+1}(\tilde{\boldsymbol{u}},c_1,\ldots,c_{i-1},c_i,M+1)^2} \\
&\geq \frac{1}{2(M+2)^2(c_i+1)^2q_{k+\ell_0+i-1}(\tilde{\boldsymbol{u}},c_1,\ldots,c_{i-1})^2} \\
&\geq \frac{|I_{k+\ell_0+i-1}(\tilde{\boldsymbol{u}},c_1,\ldots,c_{i-1})|}{2(M+2)^4} \\
&\geq \frac{|I_{k+\ell_0+p\ell}(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_p)|}{2(M+2)^4} \qquad (4.14)
\end{aligned}$$

as desired.

*Case 2: $i=m\ell$.* In this case, $c_{i+1}=b$ and $c'_{i+1}=b'$. By the distribution of cylinders (see Proposition 2.2), we see that either $I_{n+1}(\tilde{\boldsymbol{u}},c_1,\ldots,c_{m\ell-1},c_{m\ell},1)$ or $I_{n+1}(\tilde{\boldsymbol{u}},c_1,\ldots,c_{m\ell-1},c'_{m\ell},1)$ lies between two cylinders listed in equation (4.13). By the same reason as equation (4.14), we can obtain the same conclusion.

(2) Without loss of generality, assume that $b<b'$. Note that by equation (4.9), both $b$ and $b'$ are even. Then, for any $b''$ with $b<b''<b'$, the $(n+1)$th level cylinder $I_{n+1}(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b'')$ lies between $\mathcal{I}_1$ and $\mathcal{I}'_1$. This means the distance between $\mathcal{I}_1$ and $\mathcal{I}'_1$ is at least

$$\begin{aligned}
|I_{n+1}(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b'')| &\geq \frac{1}{2q_{n+1}(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b'')^2} \\
&\geq \frac{1}{2\cdot 4^2q_n(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m)^2B^{2nt}} \\
&\geq \frac{1}{2\cdot 4^2q_n(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m)^2b^2} \\
&\geq \frac{1}{32}|I_{n+1}(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b)|,
\end{aligned}$$

where we use $B^{nt}\leq b''$, $b\leq 2B^{nt}$ in the second and third inequalities. □

LEMMA 4.5. *Let $\mu_1$ be as in equation (4.12). Then, the following statements hold.*

(1) *For any $(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_p)$ with $0\leq p\leq m$,*

$$\mu_1(I_{k+\ell_0+p\ell}(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_p))\leq\frac{q_{k+\ell_0+p\ell}(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_p)^{-2t}}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t}.$$

(2) *For any $(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b)$,*

$$\mu_1(\mathcal{I}_1(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b))\leq\frac{64|\mathcal{I}_1(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b)|^t}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t}.$$

*Proof.* (1) The conclusion is clear if $p=0$. Now suppose that $p\neq 0$. By the definition of $\mu_1$, one has

$$\begin{aligned}
&\mu_1(I_{k+\ell_0+p\ell}(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_p)) \\
&=\prod_{i=1}^{p}\frac{1}{q_\ell(\boldsymbol{a}_i)^{2s}B^{\ell s^2}}\leq\prod_{i=1}^{p}\frac{1}{q_\ell(\boldsymbol{a}_i)^{2s}}\overset{\text{equation (4.8)}}{\leq}\prod_{i=1}^{p}\frac{1}{(2q_\ell(\boldsymbol{a}_i))^{2t}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2^{2t} q_{p\ell}(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_p)^{2t}} = \frac{q_{k+\ell_0}(\tilde{\boldsymbol{u}})^{2t}}{q_{k+\ell_0}(\tilde{\boldsymbol{u}})^{2t}} \cdot \frac{1}{2^{2t} q_{p\ell}(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_p)^{2t}} \\
&\leq \frac{q_{k+\ell_0}(\tilde{\boldsymbol{u}})^{2t}}{q_{k+\ell_0+p\ell}(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_p)^{2t}} \\
&\leq \frac{q_{k+\ell_0+p\ell}(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_p)^{-2t}}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t}. \qquad (4.15)
\end{aligned}
$$

(2) Recall that $n = k + \ell_0 + m\ell$. It should be noticed that equation (4.15) is actually an upper bound of $\prod_{i=1}^{p} q_\ell(\boldsymbol{a}_i)^{-2s}$. In the spirit of equation (4.15), by the definition of $\mu_1$,

$$
\begin{aligned}
\mu_1(\mathcal{I}_1(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b)) &= \bigg(\prod_{i=1}^{m} \frac{1}{q_\ell(\boldsymbol{a}_i)^{2s} B^{\ell s^2}}\bigg) \cdot \frac{2}{B^{nt}} \\
&\leq \frac{q_{k+\ell_0+m\ell}(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m)^{-2t}}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t} \cdot \frac{1}{B^{m\ell s^2}} \cdot \frac{2}{B^{nt}}.
\end{aligned}
$$

Since $m \geq kt/\epsilon$ (which follows from the choice of $n$), one has

$$
\begin{aligned}
m\ell s^2 &\geq m\ell(t+\epsilon)^2 \geq m\ell t^2 + 2m\ell t\epsilon \geq m\ell t^2 + 2\ell k t^2 \\
&\geq m\ell t^2 + (\ell + k)t^2 \geq nt^2. \qquad (4.16)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mu_1(\mathcal{I}_1(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b)) &\leq \frac{q_{k+\ell_0+m\ell}(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m)^{-2t}}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t} \cdot \frac{1}{B^{nt^2}} \cdot \frac{2}{B^{nt}} \\
&= \frac{1}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t} \cdot \frac{2}{q_n(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m)^{2t} B^{nt(1+t)}} \qquad (4.17) \\
&\overset{\text{equation (4.11)}}{\leq} \frac{64|\mathcal{I}_1(\tilde{\mathbf{u}},\mathbf{a}_1,\ldots,\mathbf{a}_m,b)|^t}{|I_{k+\ell_0}(\tilde{\mathbf{u}})|^t}.
\end{aligned}
$$

□

LEMMA 4.6. *Let $\mu_1$ be as in equation (4.12). For any $r > 0$ and $x \in [0, 1]$, we have*

$$
\mu_1(B(x,r)) \leq \frac{16(M+2)^4(M+1)^{2\ell} r^t}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t}.
$$

*Proof.* Without loss of generality, assume that $x \in \mathcal{I}_1(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b)$. Obviously, if $r \geq |I_{k+\ell_0}(\tilde{\boldsymbol{u}})|$, then

$$
\mu_1(B(x,r)) = 1 \leq \frac{r^t}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t}.
$$

Hence, it is sufficient to focus on the case $r < |I_{k+\ell_0}(\tilde{\boldsymbol{u}})|$. Lemma 4.4 suggests that we need to consider three cases.

*Case 1*: $r \leq |I_{n+1}(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b)|/32$. By Lemma 4.4, we see that the distance between $\mathcal{I}_1(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b)$ and other fundamental intervals contained in $F_n \cap I_{k+\ell_0}(\tilde{\boldsymbol{u}})$ is at least $r$. So $B(x,r)$ only intersects the fundamental interval $\mathcal{I}_1(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b)$ to which $x$ belongs. It follows that

$$\begin{aligned}\mu_1(B(x,r)) &= \bigg(\prod_{i=1}^{m} \frac{1}{q_\ell(\boldsymbol{a}_i)^{2s} B^{\ell s^2}}\bigg) \cdot \frac{2}{B^{nt}} \cdot \mathcal{L}^{(1)}_{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b}(B(x,r)) \\ &= \mu_1(\mathcal{I}_1(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b)) \cdot \mathcal{L}^{(1)}_{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b}(B(x,r)).\end{aligned}$$

Since

$$\mathcal{L}^{(1)}_{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b}(B(x,r)) \le \min\bigg\{1, \frac{2r}{|\mathcal{I}_1(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b)|}\bigg\} \le \frac{2r^t}{|\mathcal{I}_1(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b)|^t},$$

where the last inequality follows from $\min\{a,c\} \le a^t c^{1-t}$ for any $t \in [0,1]$. By Lemma 4.5(2), we have

$$\begin{aligned}\mu_1(B(x,r)) &\le \frac{64|\mathcal{I}_1(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b)|^t}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t} \cdot \frac{2r^t}{|\mathcal{I}_1(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b)|^t} \\ &= \frac{128r^t}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t}.\end{aligned}$$

*Case 2*: $|I_{n+1}(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b)|/32 < r \le |I_n(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m)|/(2(M+2)^4)$. In this case, the ball $B(x,r)$ intersects exactly the $n$th level basic cylinder $I_n(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m)$, but may intersect multiple $(n+1)$th level basic cylinders inside it. Therefore, by the definition of $\mu_1$ and equation (4.17),

$$\begin{aligned}\mu_1(B(x,r)) &\le \#\Delta_1(x) \cdot \max_{\substack{B^{nt} \le b \le 2B^{nt} \\ b \text{ is even}}} \mu_1(\mathcal{I}_1(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b)) \\ &\le \#\Delta_1(x) \cdot \frac{1}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t} \cdot \frac{2}{q_n(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m)^{2t} B^{nt(1+t)}},\end{aligned} \tag{4.18}$$

where

$$\Delta_1(x) = \{B^{nt} \le b \le 2B^{nt} :\ b \text{ is even and } I_{n+1}(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b) \cap B(x,r) \neq \emptyset\}.$$

To get the best upper bound for $\mu_1(B(x,r))$, we need to use two methods to bound $\#\Delta_1(x)$ from above. First, there are at most $B^{nt}/2$ choices for $b$ and so

$$\#\Delta_1(x) \le B^{nt}/2. \tag{4.19}$$

On the other hand, each $(n+1)$th level basic cylinder $I_{n+1}(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b)$ is of length at least

$$2^{-1}q_{n+1}(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,b)^{-2} \ge 32^{-1}q_n(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m)^{-2}B^{-2nt},$$

which means that

$$\#\Delta_1(x) \le 2r \cdot 32q_n(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m)^2 B^{2nt}.$$

This together with equation (4.19) gives

$$\begin{aligned}\#\Delta_1(x) &\le \min\{B^{nt}/2, 2r \cdot 32q_n(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m)^2 B^{2nt}\} \\ &\le 64B^{nt(1-t)} \cdot (r \cdot q_n(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m)^2 B^{2nt})^t \\ &= 64B^{nt(1+t)} \cdot r^t \cdot q_n(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m)^{2t}.\end{aligned}$$

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Substituting this upper bound for $\#\Delta_1(x)$ into equation (4.18), we get

$$\begin{aligned}\mu_1(B(x,r)) &\leq 64B^{nt(1+t)} \cdot r^t \cdot q_n(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m)^{2t} \cdot \frac{1}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t} \\ &\quad \cdot \frac{2}{q_n(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m)^{2t} B^{nt(1+t)}} \\ &= 128 \frac{r^t}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t}.\end{aligned}$$

*Case 3*: $|I_{k+\ell_0+(p+1)\ell}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{p+1})|/(2(M+2)^4) \leq r < |I_{k+\ell_0+p\ell}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_p)|/(2(M+2)^4)$ for some $1 \leq p \leq m-1$. In this case, the ball $B(x,r)$ only intersects one $(k+\ell_0+p\ell)$th level basic cylinder, that is, $I_{k+\ell_0+p\ell}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_p)$. Hence, by Lemma 4.5(1), we get

$$\begin{aligned}\mu_1(B(x,r)) &\leq \mu_1(I_{k+\ell_0+p\ell}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_p)) \\ &\leq \frac{q_{k+\ell_0+p\ell}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_p)^{-2t}}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t} \\ &= \frac{q_{k+\ell_0+p\ell}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_p)^{-2t}}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t} \cdot \frac{q_\ell(\boldsymbol{a}_{p+1})^{-2t}}{q_\ell(\boldsymbol{a}_{p+1})^{-2t}} \\ &\leq \frac{4q_{k+\ell_0+(p+1)\ell}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{p+1})^{-2t}}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t} \cdot (M+1)^{2\ell} \\ &\leq \frac{16(M+2)^4(M+1)^{2\ell} r^t}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t}.\end{aligned}$$

Combining the estimation given in Cases 1–3, we arrive at the conclusion. □

*Completing the proof of Theorem 1.3.* Recall that $\tilde{\boldsymbol{u}} = (\boldsymbol{u}, 1^{\ell_0}) \in \mathbb{N}^{k+\ell_0}$. By Proposition 2.1(1), a simple calculation shows that

$$\frac{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t}{|I_k(\boldsymbol{u})|^t} \geq \frac{q_{k+\ell_0}(\tilde{\boldsymbol{u}})^{-2t}}{2^{2t} \cdot q_k(\boldsymbol{u})^{-2t}} \geq \frac{1}{2^{4t} \cdot q_{\ell_0}(1^{\ell_0})^{2t}} \geq \frac{1}{2^{4t} \cdot 2^{t\ell_0}} \geq \frac{1}{2^{\ell+4}}.$$

Therefore, for any $\boldsymbol{u} \in \mathbb{N}^k$, by Lemma 4.6 and mass distribution principle, we have

$$\begin{aligned}\mathcal{H}^t_\infty(F_n \cap I_k(\boldsymbol{u})) &\geq \mathcal{H}^t_\infty(F_n \cap I_{k+\ell_0}(\tilde{\boldsymbol{u}})) \\ &\geq \frac{1}{16(M+2)^4(M+1)^{2\ell}} |I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t \mu_1(F_n \cap I_{k+\ell_0}(\tilde{\boldsymbol{u}})) \\ &\geq \frac{1}{2^{\ell+8}(M+2)^4(M+1)^{2\ell}} |I_k(\boldsymbol{u})|^t \\ &\geq \frac{1}{2^{\ell+8}(M+2)^4(M+1)^{2\ell}} |I_k(\boldsymbol{u})|,\end{aligned}$$

where the last inequality follows from $t < s^* \leq 1$. Since $\ell$ and $M$ depend on $t$ only, and since the above Hausdorff content bound holds for infinitely many $n \in \mathbb{N}$, by Lemma 4.1, we have

$$\dim_{\mathrm{H}} E_2(\{z_n\}_{n\geq 1}, B) \geq s^*.$$

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4.2. *Case II: $s^* = \limsup_{n\to\infty} s_n$ is obtained along a subsequence of $\{s_{n,2}\}_{n\geq 1}$.* In this case, there exist infinitely many $n$ such that

$$s_n = s_{n,2} > t. \tag{4.20}$$

For such $n$, by the definition of $s_n$,

$$s_{n,1} > s_{n,2} \geq s_{n,3},$$

which by Lemma 2.7(2) and (4) gives

$$B^{ns_{n,2}/2} \leq a_1(z_n) < B^{ns_{n,2}}. \tag{4.21}$$

By taking a subsequence, assume that integers satisfying equation (4.20) will ensure the existence of the following limit:

$$(s^* \log B)/2 \leq \lim_{\substack{n \text{ satisfies equation (4.20)} \\ n\to\infty}} \frac{\log a_1(z_n)}{n} =: \alpha \leq s^* \log B.$$

Recall the definition of $s_{n,2}$ and using the continuity of pressure functions, the above discussion gives

$$s^* = \inf\{s \in [0, 1] : P(T, -s \log |T'| - s \log B + (1 - s)\alpha) \leq 0\}. \tag{4.22}$$

Since most of the arguments in this subsection are quite identical to the last subsection, we will follow the same notation when there is no risk of ambiguity. In addition, to keep the paper of a manageable length, some proofs will not be detailed here if they are similar to those in §4.1. Instead, we will present only the main ideas.

Fix an $\epsilon < s^* - t$. By Proposition 2.4, there exist integers $\ell \geq 2t/\epsilon + 1$ and $M$ such that the unique positive number $s = s(\ell, M)$ satisfying the equation

$$\sum_{\boldsymbol{a}\in\{1,\ldots,M\}^\ell} \frac{e^{\alpha\ell(1-s)}}{q_\ell(\boldsymbol{a})^{2s} B^{\ell s}} = 1 \tag{4.23}$$

is greater than $t + \epsilon$.

Choose a large integer $n \in \mathbb{N}$ so that equation (4.20) is satisfied and

$$n - k \geq \ell, \quad e^{n(\alpha-\epsilon)} \leq a_1(z_n) \leq e^{n(\alpha+\epsilon)}. \tag{4.24}$$

Write $n - k = m\ell + \ell_0$, where $0 \leq \ell_0 < \ell$.

Analogously, let $\tilde{\boldsymbol{u}} = (\boldsymbol{u}, 1^{\ell_0}) \in \mathbb{N}^{k+\ell_0}$ and consider the following set defined by $(n + 1)$th level cylinders:

$$\{I_{n+1}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, c) : \boldsymbol{a}_i \in \{1, \ldots, M\}^\ell, 1 \leq i \leq m, \\ 2e^{(\alpha+\epsilon)n} \leq c \leq 3e^{(\alpha+\epsilon)n} \text{ and } c \text{ is even}\}.$$

Similar to §4.1, we restrict $c$ to be even for the only reason that any two cylinders in the above set are well separated (see Lemma 4.7 below). By equation (3.11), there is an interval, denoted by $\mathcal{I}_2(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, c)$ such that

$$\mathcal{I}_2(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, c) \subset F_n \cap I_{n+1}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, c). \tag{4.25}$$

Moreover, the length of this interval can be estimated as

$$\begin{aligned}|\mathcal{I}_2(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, c)| &\geq \frac{a_1(z_n)}{16q_n(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m)^2 c|a_1(z_n) - c|B^n} \\ &\geq \frac{1}{16 \cdot 3e^{2n\epsilon} q_n(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m)^2 cB^n},\end{aligned} \tag{4.26}$$

where the last inequality follows from $c \leq 3e^{(\alpha+\epsilon)n}$ and $e^{n(\alpha-\epsilon)} \leq a_1(z_n) \leq e^{n(\alpha+\epsilon)}$ (see equation (4.24)).

Define a probability measure $\mu_2$ supported on $F_n \cap I_{k+\ell_0}(\tilde{\boldsymbol{u}}) \subset F_n \cap I_k(\boldsymbol{u})$ as follows:

$$\mu_2 = \sum_{\boldsymbol{a}_1 \in \{1,\ldots,M\}^\ell} \cdots \sum_{\boldsymbol{a}_m \in \{1,\ldots,M\}^\ell} \sum_{\substack{2e^{n(\alpha+\epsilon)} \leq c \leq 3e^{n(\alpha+\epsilon)} \\ c \text{ is even}}} \bigg(\prod_{i=1}^m \frac{e^{\ell\alpha(1-s)}}{q_\ell(\boldsymbol{a}_i)^{2s} B^{\ell s}}\bigg) \cdot \frac{2}{e^{n(\alpha+\epsilon)}} \cdot \mathcal{L}^{(2)}_{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,c},$$

where

$$\mathcal{L}^{(2)}_{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,c} := \frac{\mathcal{L}|_{\mathcal{I}_2(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,c)}}{\mathcal{L}(\mathcal{I}_2(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, c))}.$$

The next two lemmas describe the gap between fundamental intervals defined in equation (4.25) and their $\mu_2$-measures, respectively. The first one follows the same lines as the proof of Lemma 4.4.

LEMMA 4.7. *Let $\mathcal{I}_2 = \mathcal{I}_2(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, c)$ and $\mathcal{I}_2' = \mathcal{I}_2(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1', \ldots, \boldsymbol{a}_m', c')$ be two intervals defined in equation (4.25). Then, the following statements hold:*

(1) *if $(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{p-1}) = (\boldsymbol{a}_1', \ldots, \boldsymbol{a}_{p-1}')$ but $\boldsymbol{a}_p \neq \boldsymbol{a}_p'$ for some $1 \leq p \leq m$, then*

$$\operatorname{dist}(\mathcal{I}_2, \mathcal{I}_2') \geq \frac{|I_{k+\ell_0+p\ell}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_p)|}{2(M+2)^4};$$

(2) *if $(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m) = (\boldsymbol{a}_1', \ldots, \boldsymbol{a}_m')$ but $c \neq c'$, then*

$$\operatorname{dist}(\mathcal{I}_2, \mathcal{I}_2') \geq \frac{1}{18}|I_{n+1}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, c)|.$$

Instead of giving a complete proof of the following lemma, we merely point out which changes have to be made.

LEMMA 4.8.

(1) *For any $(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_p)$ with $0 \leq p \leq m$,*

$$\mu_2(I_{k+\ell_0+p\ell}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_p)) \ll \frac{q_{k+\ell_0+p\ell}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_p)^{-2t}}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t}.$$

(2) *For any $(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, c)$,*

$$\mu_2(\mathcal{I}_2(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, c)) \ll \frac{|\mathcal{I}_2(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, c)|^t}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t}.$$

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*Sketch proof.* (1) Note that

$$\mu_2(I_{k+\ell_0+p\ell}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_p)) = \prod_{i=1}^{p} \frac{e^{\ell\alpha(1-s)}}{q_\ell(\boldsymbol{a}_i)^{2s} B^{\ell s}}.$$

By equations (4.21) and (4.24),

$$e^{n\alpha} \le e^{n\epsilon} a_1(z_n) \le e^{n\epsilon} B^{n s_{n,2}} \le e^{n\epsilon} B^{n(s+O(\epsilon))},$$

which is equivalent to

$$e^{\alpha} \le e^{\epsilon} B^{s+O(\epsilon)}. \tag{4.27}$$

By decreasing $\epsilon$ if necessary, we have

$$e^{\alpha(1-s)} \le B^s.$$

Therefore,

$$\mu_2(I_{k+\ell_0+p\ell}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_p)) \le \prod_{i=1}^{p} \frac{1}{q_\ell(\boldsymbol{a}_i)^{2s}} \le \frac{q_{k+\ell_0+p\ell}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_p)^{-2t}}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t}, \tag{4.28}$$

where the last inequality follows the same argument identical to equation (4.15).

(2) By the definition of $\mu_2$,

$$\mu_2(\mathcal{I}_2(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, c)) = \left(\prod_{i=1}^{m} \frac{e^{\ell\alpha(1-s)}}{q_\ell(\boldsymbol{a}_i)^{2s} B^{\ell s}}\right) \cdot \frac{2}{e^{n(\alpha+\epsilon)}}$$
$$= \left(\prod_{i=1}^{m} \frac{1}{q_\ell(\boldsymbol{a}_i)^{2s}}\right) \cdot \frac{e^{m\ell\alpha(1-s)}}{B^{m\ell s}} \cdot \frac{2}{e^{n(\alpha+\epsilon)}}.$$

Although the setting is slightly different, one can follow the proof of equation (4.16) and show that whenever $n$ is large enough,

$$n(1-\epsilon) \le m\ell \le n \tag{4.29}$$

since $n = k + m\ell + \ell_0$ and $\ell$ is fixed. Hence, using $2e^{n(\alpha+\epsilon)} \le c \le 3e^{n(\alpha+\epsilon)}$, we get

$$\frac{e^{m\ell\alpha(1-s)}}{B^{m\ell s}} \cdot \frac{2}{e^{n(\alpha+\epsilon)}} = \frac{e^{n\alpha(1-s)}}{B^{ns}} \cdot \frac{1}{e^{n\alpha}} \cdot e^{O(n\epsilon)} = \frac{1}{e^{n\alpha s} B^{ns}} \cdot e^{O(n\epsilon)}$$
$$= \frac{1}{c^s B^{ns}} \cdot e^{O(n\epsilon)}.$$

Again, by decreasing $\epsilon$ and increasing $s$ if necessary, we have

$$\frac{1}{c^s B^{ns}} \cdot e^{O(n\epsilon)} \le \frac{1}{e^{2nt\epsilon} c^t B^{nt}}.$$

By equations (4.28) and (4.26), it follows that

$$\mu_2(\mathcal{I}_2(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, c)) \le \frac{q_{k+\ell_0+m\ell}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m)^{-2t}}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t} \cdot \frac{1}{e^{2nt\epsilon} c^t B^{nt}}$$
$$\ll \frac{|\mathcal{I}_2(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, c)|^t}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t}.$$

*Remark 4.9.* We present an inequality that will be also used in a later discussion:

$$
\begin{aligned}
&e^{n(\alpha+\epsilon)(1+t)} \cdot q_n(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m)^{2t} \cdot \bigg( \prod_{i=1}^{m} \frac{e^{\ell\alpha(1-s)}}{q_\ell(\boldsymbol{a}_i)^{2s} B^{\ell s}} \bigg) \cdot \frac{2}{e^{n(\alpha+\epsilon)}} \\
&\quad \ll e^{n(\alpha+\epsilon)(1+t)} \cdot \frac{1}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t} \cdot \frac{e^{m\ell\alpha(1-s)}}{B^{m\ell s}} \cdot \frac{2}{e^{n(\alpha+\epsilon)}} \qquad \text{by equation (4.28)} \\
&\quad = \frac{1}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t} \cdot \frac{e^{n\alpha(1+t-s)}}{B^{ns}} \cdot e^{O(n\epsilon)} \qquad \text{by equation (4.29)} \\
&\quad \le \frac{1}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t} \cdot \frac{B^{ns(1+t-s)}}{B^{ns}} \cdot e^{O(n\epsilon)} \qquad \text{by equation (4.27)} \\
&\quad \le \frac{1}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t},
\end{aligned}
$$

where the last inequality follows from the fact that the error term $e^{O(n\epsilon)}$ can be made smaller than $B^{ns(s-t)}$.

Next, the following lemma gives the estimation of the $\mu_2$-measure of any ball $B(x, r)$ with $x \in [0, 1]$ and $r > 0$.

LEMMA 4.10. *For any $r > 0$ and $x \in [0, 1]$, we have*

$$\mu_2(B(x, r)) \ll (M+2)^4 (M+1)^{2\ell} \cdot \frac{r^t}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t},$$

*where the unspecified constant is absolute.*

*Sketch proof.* It suffices to focus on the case $r < |I_{k+\ell_0}(\tilde{\boldsymbol{u}})|$. We need to consider three cases according to Lemma 4.7.

*Case 1*: $r \le |I_{n+1}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, c)|/18$. In view of Lemma 4.7(2), $B(x, r)$ only intersects the fundamental interval $\mathcal{I}_2(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, c)$ to which $x$ belongs. By the same reason as Case 1 in Lemma 4.6, and using Lemma 4.8(2) instead of Lemma 4.5(2), we deduce that

$$\mu_2(B(x, r)) \ll \frac{r^t}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t}.$$

*Case 2*: $|I_{n+1}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, c)|/18 < r \le |I_n(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m)|/(2(M+2)^4)$. In this case, the ball $B(x, r)$ intersects exactly the $n$th level basic cylinder $I_n(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m)$, but may intersect multiple $(n+1)$th level basic cylinders inside this cylinder. Let

$$\Delta_2(x) = \{2e^{n(\alpha+\epsilon)} \le c \le 3e^{n(\alpha+\epsilon)} : \ c \text{ is even and } I_{n+1}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, c) \cap B(x, r) \ne \emptyset\}.$$

In comparison to Case 2 in Lemma 4.6, here the choices for $c$ are at most $e^{n(\alpha+\epsilon)}/2$ and each $(n+1)$th level basic cylinder $I_{n+1}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, c)$ is of length at least

$$\ge 2^{-1} q_{n+1}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, c)^{-2} \ge 2^{-1} 2^{-2} 3^{-2} q_n(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m)^{-2} e^{-2n(\alpha+\epsilon)}.$$

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This gives

$$\begin{aligned}\#\Delta_2(x) &\ll \min\{e^{n(\alpha+\epsilon)}/2, r\cdot q_n(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m)^2 e^{2n(\alpha+\epsilon)}\} \\ &\le e^{n(\alpha+\epsilon)(1-t)}\cdot (r\cdot q_n(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m)^2 e^{2n(\alpha+\epsilon)})^t \\ &= e^{n(\alpha+\epsilon)(1+t)}\cdot r^t\cdot q_n(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m)^{2t}.\end{aligned}$$

By the definition of $\mu_2$ and the inequality presented in Remark 4.9,

$$\begin{aligned}\mu_2(B(x,r)) &\le \#\Delta_2(x)\cdot \max_{\substack{2e^{n(\alpha+\epsilon)}\le c\le 3e^{n(\alpha+\epsilon)} \\ c \text{ is even}}} \mu_2(\mathcal{I}_2(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, c)) \\ &\ll e^{n(\alpha+\epsilon)(1+t)}\cdot r^t\cdot q_n(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m)^{2t}\cdot \bigg(\prod_{i=1}^m \frac{e^{\ell\alpha(1-s)}}{q_\ell(\boldsymbol{a}_i)^{2s}B^{\ell s}}\bigg)\cdot \frac{2}{e^{n(\alpha+\epsilon)}} \\ &\le \frac{r^t}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t}.\end{aligned}$$

*Case 3*: $|I_{k+\ell_0+(p+1)\ell}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{p+1})|/(2(M+2)^4) \le r < |I_{k+\ell_0+p\ell}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_p)|/(2(M+2)^4)$ for some $1\le p\le m-1$. In this case, the ball $B(x,r)$ only intersects one $(k+\ell_0+p\ell)$th level basic cylinder, that is, $I_{k+\ell_0+p\ell}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_p)$. Hence, following the same line as Case 3 in Lemma 4.6, we have

$$\mu_2(B(x,r)) \ll (M+2)^4(M+1)^{2\ell}\cdot \frac{r^t}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t}.$$

Combining the estimation given in Cases 1–3, we arrive at the conclusion.

*Completing the proof of Theorem 1.3.* The proof is the same as that at the end of §4.1, we leave out the details.

4.3. *Case III: $s^* = \limsup_{n\to\infty} s_n$ is obtained along a subsequence of* $\{s_{n,3}\}_{n\ge 1}$. There exist infinitely many $n$ such that

$$s_n = s_{n,3} > t. \tag{4.30}$$

Fix an $\epsilon < s^* - t$. By the definition of $s_n$,

$$s_{n,3} > s_{n,2}.$$

With Lemma 2.7(3), it follows that

$$a_1(z_n) \le B^{ns_{n,3}/2} < B^{n/2}.$$

By taking a subsequence, assume that integers satisfying equation (4.30) will ensure the existence of the following limit:

$$\lim_{\substack{n \text{ satisfies equation (4.20)} \\ n\to\infty}} \frac{\log a_1(z_n)}{n} =: \beta \le \frac{\log B}{2}.$$

Recall the definition of $s_{n,3}$. By the continuity of the pressure functions, we can infer from the above discussion that

$$s^* = \inf\{s\in[0,1] : P(T, -s\log|T'| - (s/2)\log B - s\beta) \le 0\}. \tag{4.31}$$

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By Proposition 2.4, there exist integers $\ell \geq 2t/\epsilon + 1$ and $M$ such that the unique positive number $s = s(\ell, M)$ satisfying the equation

$$\sum_{\boldsymbol{a}\in\{1,\ldots,M\}^{\ell}} \frac{B^{-\ell s/2}}{q_{\ell}(\boldsymbol{a})^{2s} e^{\beta \ell s}} = 1 \tag{4.32}$$

is greater than $t + \epsilon$.

Choose a large integer $n \in \mathbb{N}$ so that equation (4.30) is satisfied and

$$n - k \geq \ell, \quad e^{n(\beta-\epsilon)} \leq a_1(z_n) \leq e^{n(\beta+\epsilon)}. \tag{4.33}$$

Write $n - k = m\ell + \ell_0$, where $0 \leq \ell_0 < \ell$.

Let $\tilde{\boldsymbol{u}} = (\boldsymbol{u}, 1^{\ell_0}) \in \mathbb{N}^{k+\ell_0}$ and consider the following set defined by $(n+1)$th level cylinders:

$$\{I_{n+1}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, a_1(z_n)) : \boldsymbol{a}_i \in \{1, \ldots, M\}^{\ell}, 1 \leq i \leq m\}. \tag{4.34}$$

We stress that for the $(n+1)$th position, there is only one choice, that is, $a_1(z_n)$. This makes the discussion much easier than the previous two. By equation (3.14), there is an interval, denoted by $\mathcal{I}_3(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, a_1(z_n))$ such that

$$\mathcal{I}_3(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, a_1(z_n)) \subset F_n \cap I_{n+1}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, a_1(z_n)). \tag{4.35}$$

Additionally, since $a_1(z_n) \leq B^{n/2}$, by equation (3.14),

$$|\mathcal{I}_3(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, a_1(z_n))| \geq \frac{B^{-n/2}}{4q_n(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m)^2 a_1(z_n)}.$$

Define a probability measure $\mu_3$ supported on $F_n \cap I_{k+\ell_0}(\tilde{\boldsymbol{u}})$ as follows:

$$\mu_3 = \sum_{\boldsymbol{a}_1\in\{1,\ldots,M\}^{\ell}} \cdots \sum_{\boldsymbol{a}_m\in\{1,\ldots,M\}^{\ell}} \left(\prod_{i=1}^{m} \frac{1}{q_{\ell}(\boldsymbol{a}_i)^{2s} e^{\beta\ell s} B^{\ell s/2}}\right) \cdot \mathcal{L}^{(3)}_{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,a_1(z_n)},$$

where

$$\mathcal{L}^{(3)}_{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,a_1(z_n)} := \frac{\mathcal{L}|_{\mathcal{I}_3(\tilde{\boldsymbol{u}},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m,a_1(z_n))}}{\mathcal{L}(\mathcal{I}_3(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, a_1(z_n)))}.$$

The next two lemmas describe the gap between fundamental intervals defined in equation (4.25) and the $\mu_3$-measures of these fundamental intervals, respectively. The first one follows the same lines as the proof of Lemma 4.4.

LEMMA 4.11. *Let $\mathcal{I}_3 = \mathcal{I}_3(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, a_1(z_n))$ and $\mathcal{I}_3' = \mathcal{I}_3(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1', \ldots, \boldsymbol{a}_m', a_1(z_n))$ be two intervals defined in equation (4.25). If $(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{p-1}) = (\boldsymbol{a}_1', \ldots, \boldsymbol{a}_{p-1}')$ but $\boldsymbol{a}_p \neq \boldsymbol{a}_p'$ for some $1 \leq p \leq m$, then*

$$\operatorname{dist}(\mathcal{I}_3, \mathcal{I}_3') \geq \frac{|I_{k+\ell_0+p\ell}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_p)|}{2(M+2)^4}.$$

Instead of giving the complete proof of the following lemma, we merely point out which changes have to be made.

LEMMA 4.12. *The following statements hold.*

(1) *For any* $(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_p)$ *with* $0 \le p \le m$,

$$\mu_3(I_{k+\ell_0+p\ell}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_p)) \ll \frac{q_{k+\ell_0+p\ell}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_p)^{-2t}}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t}.$$

(2) *For any* $(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, a_1(z_n))$,

$$\mu_3(\mathcal{I}_3(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, a_1(z_n))) \ll \frac{|\mathcal{I}_3(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, a_1(z_n))|^t}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t}.$$

*Sketch proof.* (1) Following the same lines as in Lemma 4.5(1), one has

$$\mu_3(I_{k+\ell_0+p\ell}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_p)) = \prod_{i=1}^{p} \frac{1}{q_\ell(\boldsymbol{a}_i)^{2s} e^{\beta\ell s} B^{\ell s/2}} \le \prod_{i=1}^{p} \frac{1}{q_\ell(\boldsymbol{a}_i)^{2s}}$$
$$\ll \frac{q_{k+\ell_0+p\ell}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_p)^{-2t}}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t}.$$

(2) By the definition of $\mu_3$,

$$\mu_3(\mathcal{I}_3(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, a_1(z_n))) = \prod_{i=1}^{m} \frac{1}{q_\ell(\boldsymbol{a}_i)^{2s} e^{\beta\ell s} B^{\ell s/2}}.$$

In view of item (2), we only need to estimate $\prod_{i=1}^{m}(e^{\beta\ell s} B^{\ell s/2})^{-1} = e^{-m\ell\beta s} B^{-m\ell s/2}$. Although the setting is slightly different, one can follow the proof of equation (4.16) and show that whenever $n$ is large enough,

$$n(1-\epsilon) \le m\ell \le n,$$

since $n = k + m\ell + \ell_0$ and $\ell$ is fixed. Hence, by equation (4.33),

$$e^{-m\ell\beta s} B^{-m\ell s/2} = e^{-n\beta s} B^{-ns/2} \cdot e^{O(n\epsilon)} = a_1(z_n)^{-s} B^{-ns/2} \cdot e^{O(\epsilon)}.$$

Since $a_1(z_n) = e^{n(\beta + O(\epsilon))}$, by decreasing $\epsilon$ if necessary, we have

$$a_1(z_n)^{-s} B^{-ns/2} \cdot e^{O(\epsilon)} \le a_1(z_n)^{-t} B^{-nt/2}.$$

This together with item (1) yields the conclusion.

Next, the following lemma gives the estimation of the $\mu_3$-measure of arbitrary ball $B(x, r)$ with $x \in [0, 1]$ and $r > 0$.

LEMMA 4.13. *For any* $r > 0$ *and* $x \in [0, 1]$, *we have*

$$\mu_3(B(x, r)) \ll (M+2)^4 (M+1)^{2\ell} \cdot \frac{r^t}{|I_{k+\ell_0}(\tilde{\boldsymbol{u}})|^t}.$$

*Proof.* Assume that $x \in \mathcal{I}_3(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, a_1(z_n))$ and $r > 0$. To estimate $\mu_3(B(x, r))$, compared with Lemmas 4.6 and 4.10, only two cases need to be considered instead of three, because there is only one choice for the $(n+1)$th position of the basic cylinder (see equation (4.34)), namely $a_1(z_n)$. According to Lemma 4.11, the proof is split into two cases.

*Case 1*: $r \leq (|I_{k+\ell_0+m\ell}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{p+1})|)/2(M+2)^4$.

*Case 2*: $(|I_{k+\ell_0+(p+1)\ell}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{p+1})|)/(2(M+2)^4) \leq r < (|I_{k+\ell_0+p\ell}(\tilde{\boldsymbol{u}}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_p)|)/2(M+2)^4$ for some $1 \leq p \leq m-1$.

The argument is similar to Cases 1 and 3 in Lemma 4.6 (or Cases 1 and 3 in Lemma 4.10), respectively. We omit the details. □

*Completing the proof of Theorem 1.3.* The proof is the same as that at the end of §4.1, we leave out the details.

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