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A GENERALIZED CANTOR THEOREM IN ZF

YINHE PENG AND GUOZHEN SHEN

Abstract. It is proved in ZF (without the axiom of choice) that, for all infinite sets M, there are no surjections from $\omega \times M$ onto $\mathscr{P}(M)$.

§1. Introduction. Throughout this paper, we shall work in ZF (i.e., the Zermelo–Fraenkel set theory without the axiom of choice).

In [1], Cantor proves that, for all sets M, there are no bijections between M and $\mathscr{P}(M)$, and since there is an injection from M into $\mathscr{P}(M)$, it follows that there are no injections from $\mathscr{P}(M)$ into M. In [12], Specker proves a generalization of Cantor's theorem, which states that, for all infinite sets M, there are no injections from $\mathscr{P}(M)$ into M^2. In [2], Forster proves another generalization of Cantor's theorem, which states that, for all infinite sets M, there are no finite-to-one functions from $\mathscr{P}(M)$ to M. In [8–10], several further generalizations of these results are proved, among which are the following:

(i) For all infinite sets M and all $n \in \omega$, there are no finite-to-one functions from $\mathscr{P}(M)$ to M^n or to $[M]^n$.
(ii) For all infinite sets M, there are no finite-to-one functions from $\mathscr{P}(M)$ to $\omega \times M$.
(iii) For all infinite sets M and all sets N, if there is a finite-to-one function from N to M, then there are no surjections from N onto $\mathscr{P}(M)$.

For a set M, let $\mathrm{fin}(M)$ denote the set of all finite subsets of M. Although it can be proved in ZF that, for all infinite sets M, there are no injections from $\mathscr{P}(M)$ into $\mathrm{fin}(M)$ (cf. [5, Theorem 3]), the existence of an infinite set A such that there is a finite-to-one function from $\mathscr{P}(A)$ to $\mathrm{fin}(A)$ and such that there is a surjection from $\mathrm{fin}(A)$ onto $\mathscr{P}(A)$ is consistent with ZF (cf. [8, Remark 3.10] and [5, Theorem 1]). Now it is natural to ask whether the existence of an infinite set A such that there is a surjection from A^2 onto $\mathscr{P}(A)$ or from $[A]^2$ onto $\mathscr{P}(A)$ is consistent with ZF, and these questions are originally asked in [13] and in [4] respectively. In [11, Question 5.6], it is asked whether the existence of an infinite set A such that there is a surjection from $\omega \times A$ onto $\mathscr{P}(A)$ is consistent with ZF, and it is noted there that an affirmative answer to this question would yield affirmative answers to the above two questions. In this paper, we give a negative answer to this question; that is, we prove in ZF that,

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for all infinite sets M, there are no surjections from $\omega \times M$ onto $\mathscr{P}(M)$. We also obtain some related results.

§2. Preliminaries. In this section, we indicate briefly our use of some terminology and notation. For a function f, we use $\mathrm{dom}(f)$ for the domain of f, $\mathrm{ran}(f)$ for the range of f, $f[A]$ for the image of A under f, $f^{-1}[A]$ for the inverse image of A under f, and $f{\upharpoonright}A$ for the restriction of f to A. For functions f and g, we use $g \circ f$ for the composition of g and f. We write $f : A \to B$ to express that f is a function from A to B, and $f : A \twoheadrightarrow B$ to express that f is a function from A *onto* B.

Definition 2.1. Let A, B be arbitrary sets.

(1) $A \preccurlyeq B$ means that there exists an injection from A into B.
(2) $A \preccurlyeq^* B$ means that there exists a surjection from a subset of B onto A.
(3) $\mathrm{fin}(A)$ denotes the set of all finite subsets of A.
(4) $\mathscr{P}_\infty(A)$ denotes the set of all infinite subsets of A.

Clearly, if $A \preccurlyeq B$ then $A \preccurlyeq^* B$, and if $A \preccurlyeq^* B$ then $\mathscr{P}(A) \preccurlyeq \mathscr{P}(B)$ and $\mathscr{P}_\infty(A) \preccurlyeq \mathscr{P}_\infty(B)$.

Fact 2.2. $\omega_1 \preccurlyeq^* \mathscr{P}(\omega)$.

Proof. Cf. [3, Theorem 5.11]. ⊣

In the sequel, we shall frequently use expressions like "one can explicitly define" in our formulations, which is illustrated by the following example.

Theorem 2.3 (Cantor–Bernstein). *From injections $f : A \to B$ and $g : B \to A$, one can explicitly define a bijection $h : A \to B$.*

Proof. Cf. [7, III.2.8]. ⊣

Formally, Theorem 2.3 states that in ZF one can define a class function H without free variables such that, whenever f is an injection from A into B and g is an injection from B into A, $H(f, g)$ is defined and is a bijection between A and B.

We say that a set M is *Dedekind infinite* if there exists a bijection between M and a proper subset of M; otherwise M is *Dedekind finite*. It is well-known that M is Dedekind infinite if and only if there exists an injection from ω into M. We say that a set M is *power Dedekind infinite* if the power set of M is Dedekind infinite; otherwise M is *power Dedekind finite*. Recall Kuratowski's celebrated theorem:

Theorem 2.4 (Kuratowski). *A set M is power Dedekind infinite if and only if there exists a surjection from M onto ω.*

Proof. Cf. [3, Proposition 5.4]. ⊣

§3. The main theorem. In this section, we prove our main theorem, which states that, for all infinite sets M, there are no surjections from $\omega \times M$ onto $\mathscr{P}(M)$. We first recall some well-known results.

Theorem 3.1 (Cantor). *From a function $f : M \to \mathscr{P}(M)$, one can explicitly define an $A \in \mathscr{P}(M) \setminus \mathrm{ran}(f)$.*

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Proof. It suffices to take $A = \{x \in \mathrm{dom}(f) \mid x \notin f(x)\}$. ⊣

Lemma 3.2. *For any infinite ordinal* α, *one can explicitly define an injection* $f : \alpha \times \alpha \to \alpha$.

Proof. Cf. [12, 2.1] or [7, IV.2.24]. ⊣

Lemma 3.3. *For any infinite ordinal* α, *one can explicitly define an injection* $f : \mathrm{fin}(\alpha) \to \alpha$.

Proof. Cf. [3, Theorem 5.19]. ⊣

Lemma 3.4. *For any infinite ordinal* α, *one can explicitly define a bijection* $f : \omega^\alpha \to \alpha$.

Proof. Let α be an infinite ordinal. Let

$$\exp(\omega, \alpha) = \big\{t : \alpha \to \omega \bigm| \{\gamma < \alpha \mid t(\gamma) \neq 0\} \text{ is finite}\big\},$$

and let r be the right lexicographic ordering of $\exp(\omega, \alpha)$. It is easy to verify that r well-orders $\exp(\omega, \alpha)$ and the order type of $\langle \exp(\omega, \alpha), r\rangle$ is ω^α (cf. [7, IV.2.10]). Let g be the unique isomorphism of $\langle \omega^\alpha, \in\rangle$ onto $\langle \exp(\omega, \alpha), r\rangle$. Let h be the function on $\exp(\omega, \alpha)$ defined by

$$h(t) = t{\restriction}\{\gamma < \alpha \mid t(\gamma) \neq 0\}.$$

Then h is an injection from $\exp(\omega, \alpha)$ into $\mathrm{fin}(\alpha \times \omega)$. By Lemmas 3.2 and 3.3, we can explicitly define an injection $p : \mathrm{fin}(\alpha \times \omega) \to \alpha$. Therefore, $p \circ h \circ g$ is an injection from ω^α into α. Now, since the function that maps each $\gamma < \alpha$ to ω^γ is an injection from α into ω^α, it follows from Theorem 2.3 that we can explicitly define a bijection $f : \omega^\alpha \to \alpha$. ⊣

Fact 3.5. *If* $A = B \cup C$ *is a set of ordinals which is of order type* ω^δ, *then* B *or* C *has order type* ω^δ.

Proof. Cf. [7, IV.2.22(vii)]. ⊣

The key step of our proof is the following lemma.

Lemma 3.6. *From a surjection* $f : \omega \times M \twoheadrightarrow \alpha$, *where* α *is an uncountable ordinal, one can explicitly define a surjection from* M *onto* α.

Proof. Let α be an uncountable ordinal and let f be a surjection from $\omega \times M$ onto α. For each $n \in \omega$, let $A_n = f[\{n\} \times M]$, let δ_n be the order type of A_n, and let g_n be the unique isomorphism of δ_n onto A_n. Let $\delta = \bigcup_{n\in\omega} \delta_n$ and let g be the function on $\omega \times \delta$ defined by

$$g(n, \gamma) = \begin{cases} g_n(\gamma), & \text{if } \gamma < \delta_n, \\ 0, & \text{otherwise.} \end{cases}$$

Then g is a surjection from $\omega \times \delta$ onto α, which implies that δ is also an uncountable ordinal. Hence, it follows from Lemma 3.2 that we can explicitly define a surjection from δ onto α. So it suffices to explicitly define a surjection from M onto δ. We consider the following two cases:

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Case 1. There exists an $n \in \omega$ such that $\delta_n = \delta$. Let n_0 be the least natural number such that $\delta_{n_0} = \delta$. Then the function that maps each $x \in M$ to $g_{n_0}^{-1}(f(n_0, x))$ is a surjection of M onto δ.

Case 2. If we are not in Case 1, then, since $\delta = \bigcup_{n\in\omega} \delta_n$, δ is a limit ordinal. Since $\delta > \omega$, without loss of generality, assume that δ_n is infinite for all $n \in \omega$. For each $n \in \omega$, let $\beta_n = \omega^{\delta_n}$. By Lemma 3.4, for each $n \in \omega$, we can explicitly define a bijection $p_n : \beta_n \to \delta_n$. For each $n \in \omega$, let h_n be the function on M defined by $h_n(x) = p_n^{-1}(g_n^{-1}(f(n, x)))$. Then, for any $n \in \omega$, h_n is a surjection from M onto β_n. Let $\beta = \omega^{\delta}$. Clearly, $\beta = \bigcup_{n\in\omega} \beta_n$. By Lemma 3.4, it suffices to explicitly define a surjection $h : M \twoheadrightarrow \beta$.

We first define by recursion two sequences $\langle B_n\rangle_{n\in\omega}$ and $\langle q_n\rangle_{n\in\omega}$ as follows. Let $B_0 = M$. Let $n \in \omega$ and assume that $B_n \subseteq M$ has been defined so that

$$\beta = \bigcup\{\eta \mid \eta = \beta_k \text{ for some } k \in \omega \text{ such that } h_k[B_n] \text{ has order type } \beta_k\}. \tag{1}$$

We define a subset B_{n+1} of B_n and a surjection $q_n : B_n \setminus B_{n+1} \twoheadrightarrow \beta_n$ as follows. Since $\beta_n < \beta$, by (1), there is a least $k \in \omega$ such that $\beta_n < \beta_k$ and $h_k[B_n]$ has order type β_k. Let t be the unique isomorphism of $h_k[B_n]$ onto β_k, and let

$$D = \{x \in B_n \mid t(h_k(x)) < \beta_n\}.$$

Since $\beta_k = \omega^{\delta_k}$ is closed under ordinal addition, it follows that $\beta_n \cdot 2 < \beta_k$. Now, if (1) holds with B_n replaced by D, we define $B_{n+1} = D$ and let q_n be the function on $B_n \setminus D$ defined by

$$q_n(x) = \begin{cases} \text{the unique } \gamma < \beta_n \text{ such that } t(h_k(x)) = \beta_n + \gamma, & \text{if } t(h_k(x)) < \beta_n \cdot 2,\\ 0, & \text{otherwise.} \end{cases}$$

Otherwise, it follows from (1) and Fact 3.5 that (1) holds with B_n replaced by $B_n \setminus D$, and then we define $B_{n+1} = B_n \setminus D$ and let q_n be the function on D defined by $q_n(x) = t(h_k(x))$. Clearly, in either case, $B_{n+1} \subseteq B_n$, (1) holds with B_n replaced by B_{n+1}, and q_n is a surjection from $B_n \setminus B_{n+1}$ onto β_n. Now, it suffices to define $h = \bigcup_{n\in\omega} q_n \cup (\bigcap_{n\in\omega} B_n \times \{0\})$. ⊣

Lemma 3.7. *For all infinite sets M and all sets N, if there is a finite-to-one function from N to M, then there are no surjections from N onto $\mathscr{P}(M)$.*

Proof. Cf. [8, Theorem 5.3]. ⊣

Now we are ready to prove our main theorem.

Theorem 3.8. *For all infinite sets M, there are no surjections from $\omega \times M$ onto $\mathscr{P}(M)$.*

Proof. Assume towards a contradiction that there exist an infinite set M and a surjection $\Phi : \omega \times M \twoheadrightarrow \mathscr{P}(M)$. We first prove that M is power Dedekind infinite. Let Ψ be the restriction of Φ to the set $\{(n, x) \in \omega \times M \mid \Phi(n, x) = \Phi(k, x) \text{ for no } k < n\}$. Clearly, Ψ is a surjection from a subset of $\omega \times M$ onto $\mathscr{P}(M)$ such that, for all $x \in M$, $\Psi{\restriction}(\omega \times \{x\})$ is injective. If M is power Dedekind finite, then $\mathrm{dom}(\Psi) \cap (\omega \times \{x\})$ is finite for all $x \in M$, and thus there exists a finite-to-one function from $\mathrm{dom}(\Psi)$ to M, contradicting Lemma 3.7. Hence, M is power Dedekind infinite.

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Now, it follows from Theorem 2.4 that $\omega \preccurlyeq^* M$, and thus, by Fact 2.2, $\omega_1 \preccurlyeq^* \mathscr{P}(\omega) \preccurlyeq \mathscr{P}(M) \preccurlyeq^* \omega \times M$, which implies that $\omega_1 \preccurlyeq^* M$ by Lemma 3.6 and hence $\omega_1 \preccurlyeq \mathscr{P}(M)$. Let h be an injection from ω_1 into $\mathscr{P}(M)$. In what follows, we get a contradiction by constructing by recursion an injection H from Ord (the class of all ordinals) into $\mathscr{P}(M)$.

For $\gamma < \omega_1$, we take $H(\gamma) = h(\gamma)$. Now, we assume that α is an uncountable ordinal and that $H{\restriction}\alpha$ is an injection from α into $\mathscr{P}(M)$. Then $(H{\restriction}\alpha)^{-1} \circ \Phi$ is a surjection from a subset of $\omega \times M$ onto α and hence can be extended by zero to a surjection $f : \omega \times M \twoheadrightarrow \alpha$. By Lemma 3.6, f explicitly provides a surjection $g : M \twoheadrightarrow \alpha$. Since $(H{\restriction}\alpha) \circ g$ is a surjection from M onto $H[\alpha]$, it follows from Theorem 3.1 that we can explicitly define an $H(\alpha) \in \mathscr{P}(M) \setminus H[\alpha]$ from $H{\restriction}\alpha$ (and Φ). ⊣

§4. A further generalization. In [6], Kirmayer proves that, for all infinite sets M, there are no surjections from M onto $\mathscr{P}_\infty(M)$. In this section, we generalize this result by showing that, for all infinite sets M, there are no surjections from $\omega \times M$ onto $\mathscr{P}_\infty(M)$, which is also a generalization of Theorem 3.8. The proof is similar to that of Theorem 3.8, but first we have to prove that Lemma 3.7 holds with $\mathscr{P}(M)$ replaced by $\mathscr{P}_\infty(M)$.

LEMMA 4.1. *For any infinite ordinal α, one can explicitly define an injection $f : \mathscr{P}(\alpha) \to \mathscr{P}_\infty(\alpha)$.*

PROOF. By Lemma 3.2, we can explicitly define an injection $p : \alpha \times \alpha \to \alpha$. Let f be the function on $\mathscr{P}(\alpha)$ defined by

$$f(A) = \begin{cases} p[A \times \{0\}], & \text{if } A \text{ is infinite,} \\ p[(\alpha \setminus A) \times \{1\}], & \text{otherwise.} \end{cases}$$

Then it is easy to see that f is an injection from $\mathscr{P}(\alpha)$ into $\mathscr{P}_\infty(\alpha)$. ⊣

LEMMA 4.2. *From a set M, a finite-to-one function $f : N \to M$, and a surjection $g : N \twoheadrightarrow \alpha$, where α is an infinite ordinal, one can explicitly define a surjection $h : M \twoheadrightarrow \alpha$.*

PROOF. Cf. [8, Lemma 5.2]. ⊣

LEMMA 4.3. *For all infinite sets M and all sets N, if there is a finite-to-one function from N to M, then there are no surjections from N onto $\mathscr{P}_\infty(M)$.*

PROOF. Assume towards a contradiction that there exist an infinite set M and a set N such that there exist a finite-to-one function $f : N \to M$ and a surjection $\Phi : N \twoheadrightarrow \mathscr{P}_\infty(M)$. Clearly, the function that maps each cofinite subset A of M to the cardinality of $M \setminus A$ is a surjection from a subset of $\mathscr{P}_\infty(M)$ onto ω, and hence $\omega \preccurlyeq^* \mathscr{P}_\infty(M) \preccurlyeq^* N$, which implies that $\omega \preccurlyeq^* M$ by Lemma 4.2. Thus $\omega \preccurlyeq \mathscr{P}_\infty(\omega) \preccurlyeq \mathscr{P}_\infty(M)$. Let h be an injection from ω into $\mathscr{P}_\infty(M)$. In what follows, we get a contradiction by constructing by recursion an injection H from Ord into $\mathscr{P}_\infty(M)$.

For $n \in \omega$, we take $H(n) = h(n)$. Now, we assume that α is an infinite ordinal and that $H{\restriction}\alpha$ is an injection from α into $\mathscr{P}_\infty(M)$. Then $(H{\restriction}\alpha)^{-1} \circ \Phi$ is a surjection from

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a subset of N onto α and hence can be extended by zero to a surjection $g : N \twoheadrightarrow \alpha$. By Lemma 4.2, from M, f, and g, we can explicitly define a surjection $p : M \twoheadrightarrow \alpha$. Then the function q on $\mathscr{P}_\infty(\alpha)$ defined by $q(A) = p^{-1}[A]$ is an injection from $\mathscr{P}_\infty(\alpha)$ into $\mathscr{P}_\infty(M)$, and thus it follows from Lemma 4.1 that we can explicitly define an injection $t : \mathscr{P}(\alpha) \to \mathscr{P}_\infty(M)$. Then $t^{-1} \circ (H{\restriction}\alpha)$ is a bijection between a subset of α and $t^{-1}[H[\alpha]]$, and thus can be extended by zero to a function $u : \alpha \to \mathscr{P}(\alpha)$. By Theorem 3.1, we can explicitly define a $B \in \mathscr{P}(\alpha) \setminus \operatorname{ran}(u)$. Since $t^{-1}[H[\alpha]] \subseteq \operatorname{ran}(u)$, it follows that $B \notin t^{-1}[H[\alpha]]$, which implies that $t(B) \notin H[\alpha]$. Now, it suffices to define $H(\alpha) = t(B)$. ⊣

We are now in a position to prove the result mentioned at the beginning of this section.

THEOREM 4.4. *For all infinite sets M, there are no surjections from $\omega \times M$ onto $\mathscr{P}_\infty(M)$.*

PROOF. We proceed along the lines of the proof of Theorem 3.8. Assume towards a contradiction that there exist an infinite set M and a surjection $\Phi : \omega \times M \twoheadrightarrow \mathscr{P}_\infty(M)$. We first prove that M is power Dedekind infinite. Let Ψ be the restriction of Φ to the set $\{(n, x) \in \omega \times M \mid \Phi(n, x) = \Phi(k, x) \text{ for no } k < n\}$. Clearly, Ψ is a surjection from a subset of $\omega \times M$ onto $\mathscr{P}_\infty(M)$ such that, for all $x \in M$, $\Psi{\restriction}(\omega \times \{x\})$ is injective. If M is power Dedekind finite, then $\operatorname{dom}(\Psi) \cap (\omega \times \{x\})$ is finite for all $x \in M$, and thus there exists a finite-to-one function from $\operatorname{dom}(\Psi)$ to M, contradicting Lemma 4.3. Hence, M is power Dedekind infinite.

Now, it follows from Theorem 2.4 that $\omega \preccurlyeq^* M$, and thus, by Fact 2.2 and Lemma 4.1, $\omega_1 \preccurlyeq^* \mathscr{P}(\omega) \preccurlyeq \mathscr{P}_\infty(\omega) \preccurlyeq \mathscr{P}_\infty(M) \preccurlyeq^* \omega \times M$, which implies that $\omega_1 \preccurlyeq^* M$ by Lemma 3.6 and hence $\omega_1 \preccurlyeq \mathscr{P}_\infty(\omega_1) \preccurlyeq \mathscr{P}_\infty(M)$. Let h be an injection from ω_1 into $\mathscr{P}_\infty(M)$. In what follows, we get a contradiction by constructing by recursion an injection H from Ord into $\mathscr{P}_\infty(M)$.

For $\gamma < \omega_1$, we take $H(\gamma) = h(\gamma)$. Now, we assume that α is an uncountable ordinal and that $H{\restriction}\alpha$ is an injection from α into $\mathscr{P}_\infty(M)$. Then $(H{\restriction}\alpha)^{-1} \circ \Phi$ is a surjection from a subset of $\omega \times M$ onto α and hence can be extended by zero to a surjection $f : \omega \times M \twoheadrightarrow \alpha$. By Lemma 3.6, f explicitly provides a surjection $g : M \twoheadrightarrow \alpha$. Then the function q on $\mathscr{P}_\infty(\alpha)$ defined by $q(A) = g^{-1}[A]$ is an injection from $\mathscr{P}_\infty(\alpha)$ into $\mathscr{P}_\infty(M)$, and thus it follows from Lemma 4.1 that we can explicitly define an injection $t : \mathscr{P}(\alpha) \to \mathscr{P}_\infty(M)$. Then $t^{-1} \circ (H{\restriction}\alpha)$ is a bijection between a subset of α and $t^{-1}[H[\alpha]]$, and thus can be extended by zero to a function $u : \alpha \to \mathscr{P}(\alpha)$. By Theorem 3.1, we can explicitly define a $B \in \mathscr{P}(\alpha) \setminus \operatorname{ran}(u)$. Since $t^{-1}[H[\alpha]] \subseteq \operatorname{ran}(u)$, it follows that $B \notin t^{-1}[H[\alpha]]$, which implies that $t(B) \notin H[\alpha]$. Now, it suffices to define $H(\alpha) = t(B)$. ⊣

Using the method presented here, we can also show that the statements (i)–(iii) in Section 1 hold with $\mathscr{P}(M)$ replaced by $\mathscr{P}_\infty(M)$ (Lemma 4.3 is just the statement (iii) for $\mathscr{P}_\infty(M)$). We shall omit the details.

The questions whether the existence of an infinite set A such that there is a surjection from A^2 onto $\mathscr{P}(A)$ or from $[A]^2$ onto $\mathscr{P}(A)$ is consistent with ZF are still open.

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ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE
CHINESE ACADEMY OF SCIENCES
EAST ZHONG GUAN CUN ROAD NO. 55
BEIJING 100190, CHINA
E-mail: pengyinhe@amss.ac.cn

SCHOOL OF PHILOSOPHY
WUHAN UNIVERSITY
NO. 299, BAYI ROAD
WUHAN 430072, HUBEI PROVINCE, CHINA
E-mail: shen_guozhen@outlook.com

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