Certain classes of univalent functions with negative coefficients

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The subclasses $S^*(\alpha, \beta)$ and $C^*(\alpha, \beta)$ of T, the class of analytic and univalent functions of the form

$f(z) = z - \sum\limits_{n=2}^{\infty} |a_n| z^n$ have been considered. Sharp results

concerning coefficients, distortion of functions belonging to $S^*(\alpha, \beta)$ and $C^*(\alpha, \beta)$ are determined along with a representation formula for the functions in $S^*(\alpha, \beta)$. Furthermore, it is shown that the classes $S^*(\alpha, \beta)$ and $C^*(\alpha, \beta)$ are closed under arithmetic mean and convex linear combinations.

1. Introduction

Let S denote the class of functions of the form equal to

$z + \sum\limits_{n=2}^{\infty} |a_n| z^n$ that are analytic and univalent in the unit disk $|z| < 1$.

We denote by $S(\alpha)$ and $C(\alpha)$ the subclasses of S consisting of functions which are, respectively, starlike of order α and convex of order α. A function $f \in S$ is in $S(\alpha, \beta)$, the class of starlike functions of order α $(0 \le \alpha < 1)$ and type β $(0 < \beta \le 1)$ if and only if

$$\left| \left[\frac{zf'(z)}{f(z)} - 1 \right] / \left[\frac{zf'(z)}{f(z)} + (1-2\alpha) \right] \right| < \beta, \quad (|z| < 1).$$

Further, $f \in S$ is in $C(\alpha, \beta)$, the class of convex functions of order α and type β if and only if $zf'(z) \in S(\alpha, \beta)$.

Let T denote the subclass of S consisting of functions whose non-zero coefficients, from the second on, are negative; that is, an analytic and univalent function f is in T if and only if it can be expressed as $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$. We denote by $S*(\alpha)$, $C*(\alpha)$, $S*(\alpha, \beta)$, $C*(\alpha, \beta)$ the classes obtained by taking intersections, respectively, of the classes $S(\alpha)$, $C(\alpha)$, $S(\alpha, \beta)$, and $C(\alpha, \beta)$ with T . In [1], Schild considered a subclass of T consisting of polynomials having $|z| = 1$ as radius of univalence. For this class, he obtained a necessary and sufficient condition in terms of the coefficients, and with the aid of this he derived better results for certain quantities connected with the conformal mapping of univalent functions. Silverman [2] determined coefficient inequalities, distortion, and covering theorems for the classes $S*(\alpha)$ and $C*(\alpha)$.

In this paper, sharp results concerning coefficients and distortion theorems for the classes $S*(\alpha, \beta)$ and $C*(\alpha, \beta)$ are determined. We also show that these classes are closed under 'arithmetic mean' and 'convex linear combinations'.

2. Coefficients theorems

THEOREM 1. *A function* $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ *is in* $S*(\alpha, \beta)$ *if and only if*

$$\sum_{n=2}^{\infty} \{(n-1)+\beta(n+1-2\alpha)\} |a_n| \le 2\beta(1-\alpha) .$$

This result is sharp.

Proof. Let $|z| = 1$. Then

$$\left| zf'(z) - f(z) \right| - \beta \left| zf'(z) + (1-2\alpha)f(z) \right|$$

$$= \left| \sum_{n=2}^{\infty} (1-n) |a_n| z^n \right| - \beta \left| 2(1-\alpha)z - \sum_{n=2}^{\infty} (n+1-2\alpha) |a_n| z^n \right|$$

$$\leq \sum_{n=2}^{\infty} \{(n-1) + \beta(n+1-2\alpha)\} |a_n| - 2\beta(1-\alpha) \leq 0 .$$

Hence, by the maximum modulus theorem, $f \in S^*(\alpha, \beta)$.

For the converse, assume that

$$\left| \left[\frac{zf'(z)}{f(z)} - 1 \right] \Big/ \left[\frac{zf'(z)}{f(z)} + (1-2\alpha) \right] \right| = \frac{\displaystyle\sum_{n=2}^{\infty} (n-1) |a_n| z^n}{2(1-\alpha)z - \displaystyle\sum_{n=2}^{\infty} (n+1-2\alpha) |a_n| z^n}$$

$$< \beta , \quad (|z| < 1) .$$

Since $|\mathrm{Re}(z)| \leq |z|$ for all z, we have

$$(2.1) \qquad \mathrm{Re}\left\{ \frac{\displaystyle\sum_{n=2}^{\infty} (n-1) |a_n| z^n}{2(1-\alpha)z - \displaystyle\sum_{n=2}^{\infty} (n+1-2\alpha) |a_n| z^n} \right\} < \beta .$$

Choose values of z on the real axis so that $\left(zf'(z)/f(z) \right)$ is real. Upon clearing the denominator in (2.1) and letting $z \to 1$ through real values, we obtain

$$\sum_{n=2}^{\infty} (n-1) |a_n| \leq \beta \left\{ 2(1-\alpha) - \sum_{n=2}^{\infty} (n+1-2\alpha) |a_n| \right\} .$$

This gives the required condition.

Finally, the function

$$f(z) = z - \sum_{n=2}^{\infty} \frac{2\beta(1-\alpha)}{(n-1) + \beta(n+1-2\alpha)} z^n ,$$

is an extremal function for the theorem.

COROLLARY. $T = S^*(0, 1)$.

THEOREM 2. *A function* $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ *is in* $C^*(\alpha, \beta)$ *if and only if*

$$\sum_{n=2}^{\infty} n\{(n-1)+\beta(n+1-2\alpha)\}|a_n| \leq 2\beta(1-\alpha) .$$

   Proof. It immediately follows by appealing to Theorem 1.

3. A representation formula

   We now proceed to prove a theorem which gives a representation for functions in $S^*(\alpha, \beta)$.

   THEOREM 3. *A function* $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ *is in* $S^*(\alpha, \beta)$ *if and only if*

(3.1) $$f(z) = z \exp\left\{2(1-\alpha) \int_0^z \varphi(t)/\bigl(1-t\varphi(t)\bigr)dt\right\} ,$$

where $\varphi(z)$ *is analytic and satisfies* $|\varphi(z)| \leq \beta$, *for* $|z| < 1$.

   Proof. Let $f \in S^*(\alpha, \beta)$. Then

$$\left|\left[\frac{zf'(z)}{f(z)} - 1\right]/\left[\frac{zf'(z)}{f(z)} + (1-2\alpha)\right]\right| < \beta , \quad (|z| < 1) .$$

Since the absolute value vanishes for $z = 0$, we have

(3.2) $$\left(\frac{zf'(z)}{f(z)} - 1 / \frac{zf'(z)}{f(z)} + (1-2\alpha)\right) = V(z) ,$$

where $V(z)$ is analytic and $|V(z)| \leq \beta$, for $|z| < 1$. The 'only if' part is easily obtained by integrating (3.2) with $V(z) = z\varphi(z)$, and the other part by differentiating (3.1).

4. Distortion theorems

   THEOREM 4. *If* $f \in S^*(\alpha, \beta)$, *then*

(4.1) $$r - \frac{2\beta(1-\alpha)}{1+\beta(3-2\alpha)} r^2 \leq |f(z)| \leq r + \frac{2\beta(1-\alpha)}{1+\beta(3-2\alpha)} r^2 , \quad (|z| = r)$$

and

(4.2) $$1 - \frac{4\beta(1-\alpha)}{1+\beta(3-2\alpha)} r \leq |f'(z)| \leq 1 + \frac{4\beta(1-\alpha)}{1+\beta(3-2\alpha)} r, \quad (|z| = r) .$$

   Proof. In view of Theorem 1, we have

$$1 + \beta(3-2\alpha) \sum_{n=2}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} \{(n-1)+\beta(n+1-2\alpha)\}|a_n|$$
$$\leq 2\beta(1-\alpha) \ ,$$

which gives

$$\sum_{n=2}^{\infty} |a_n| \leq \frac{2\beta(1-\alpha)}{1+\beta(3-2\alpha)} \ .$$

Therefore

$$|f(z)| \leq r + \sum_{n=2}^{\infty} |a_n| r^n \leq r + \frac{2\beta(1-\alpha)}{1+\beta(3-2\alpha)} \ r^2$$

and

$$|f(z)| \geq r - \sum_{n=2}^{\infty} |a_n| r^n \geq r - \frac{2\beta(1-\alpha)}{1+\beta(3-2\alpha)} \ r^2 \ .$$

Hence (4.1) follows.

Further

$$|f'(z)| \leq 1 + \sum_{n=2}^{\infty} n|a_n||z|^{n-1} \leq 1 + r \sum_{n=2}^{\infty} n|a_n|$$

and

$$|f'(z)| \geq 1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1} \geq 1 - r \sum_{n=2}^{\infty} n|a_n| \ .$$

Also, from Theorem 1, we note that

$$\sum_{n=2}^{\infty} n|a_n| \leq \frac{4\beta(1-\alpha)}{1+\beta(3-2\alpha)} \ .$$

Hence (4.2) follows also.

REMARK. The bounds in (4.1) and (4.2) are sharp, since the equalities are attained for the functions:

$$f(z) = z - \frac{2\beta(1-\alpha)}{1+\beta(3-2\alpha)} \ z^2 \ , \quad (z = \pm r) \ .$$

Using the same technique as in the proof of Theorem 4 we obtain the following:

THEOREM 5. *If* $f \in C^*(\alpha, \beta)$, *then*

(4.3) $r - \dfrac{\beta(1-\alpha)}{1+\beta(3-2\alpha)} r^2 \leq |f(z)| \leq r + \dfrac{\beta(1-\alpha)}{1+\beta(3-2\alpha)} r^2$, $(|z| = r)$

and

(4.4) $1 - \dfrac{2\beta(1-\alpha)}{1+\beta(3-2\alpha)} r \leq |f'(z)| \leq 1 + \dfrac{2\beta(1-\alpha)}{1+\beta(3-2\alpha)} r$, $(|z| = r)$

with equalities for $f(z) = z - \beta(1-\alpha)[1+\beta(3-2\alpha)]^{-1}z^2$, $(z = \pm r)$.

THEOREM 6. *Let* $f \in S^*(\alpha, \beta)$. *Then the disk* $|z| < 1$ *is mapped onto a domain that contains the disk* $|w| < (1+\beta)/1+\beta(3-2\alpha)$. *The result is sharp with extremal function* $z - 2\beta(1-\alpha)[1+\beta(3-2\alpha)]^{-1}z^2$.

THEOREM 7. *Let* $f \in C^*(\alpha, \beta)$. *Then the disk* $|z| < 1$ *is mapped onto a domain that contains the disk* $|w| < \left(1+\beta(2-\alpha)\right)/1+\beta(3-2\alpha)$. *The result is sharp with extremal function* $z - \beta(1-\alpha)[1+\beta(3-2\alpha)]^{-1}z^2$.

Proofs of Theorems 6 and 7. Proofs follow by letting $r \rightarrow 1$ in (4.1) and (4.3).

5.   Closure theorems

In this section, we shall prove that the classes $S^*(\alpha, \beta)$ and $C^*(\alpha, \beta)$ are closed under 'arithmetic mean' and 'convex linear combinations'.

THEOREM 8. *If* $f(z) = z - \displaystyle\sum_{n=2}^{\infty} |a_n|z^n$ *and* $g(z) = z - \displaystyle\sum_{n=2}^{\infty} |b_n|z^n$

are in $S^*(\alpha, \beta)$, *then* $h(z) = z - \dfrac{1}{2} \displaystyle\sum_{n=2}^{\infty} |a_n + b_n|z^n$ *is also in* $S^*(\alpha, \beta)$.

Proof. The proof follows directly by appealing to Theorem 1. In fact, f and g being in $S^*(\alpha, \beta)$, we have

(5.1) $\displaystyle\sum_{n=0}^{\infty} \{(n-1)+\beta(n+1-2\alpha)\}|a_n| \leq 2\beta(1-\alpha)$

and

(5.2) $\displaystyle\sum_{n=2}^{\infty} \{(n-1)+\beta(n+1-2\alpha)\}|b_n| \leq 2\beta(1-\alpha)$.

It is sufficient, for h to be a member of $S^*(\alpha, \beta)$, to show

$$\frac{1}{2} \sum_{n=2}^{\infty} \{(n-1)+\beta(n+1-2\alpha)\} |a_n+b_n| \leq 2\beta(1-\alpha) \ ,$$

which will follow immediately by the use of (5.1) and (5.2).

THEOREM 9. *Let*

$$f_1(z) = z \ , \quad f_n(z) = \frac{2\beta(1-\alpha)}{(n-1)+\beta(n+1-2\alpha)} z^n \ , \quad (n = 2, 3, \ldots) \ .$$

Then $f \in S^*(\alpha, \beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) \ , \ where \quad \lambda_n \geq 0 \quad and \quad \sum_{n=1}^{\infty} \lambda_n = 1 \ .$$

Proof. Suppose

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$$

$$= z - \sum_{n=2}^{\infty} \frac{2\beta(1-\alpha)}{(n-1)+\beta(n+1-2\alpha)} \lambda_n z^n \ .$$

Then

$$\sum_{n=2}^{\infty} \left\{ \frac{(n-1)+\beta(n+1-2\alpha)}{2\beta(1-\alpha)} \lambda_n \frac{2\beta(1-\alpha)}{(n-1)+\beta(n+1-2\alpha)} \right\} \leq 1 \ .$$

Thus, by Theorem 1, $f \in S^*(\alpha, \beta)$.

Conversely, suppose $f \in S^*(\alpha, \beta)$. Again, by Theorem 1, we have

$$|a_n| \leq \frac{2\beta(1-\alpha)}{(n-1)+\beta(n+1-2\alpha)} \ , \quad (n = 2, 3, \ldots) \ .$$

Setting

$$\lambda_n = \frac{(n-1)+\beta(n+1-2\alpha)}{2\beta(1-\alpha)} |a_n| \ , \quad (n = 2, 3, \ldots)$$

and

$$\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n \ ,$$

we have

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) \ .$$

This completes the proof.

The analogues of Theorems 8 and 9 for the class $C^*(\alpha, \beta)$ are:

THEOREM 10. *If* $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ *and* $g(z) = z - \sum_{n=2}^{\infty} |b_n| z^n$

are in $C^*(\alpha, \beta)$ *, then* $h(z) = z - \frac{1}{2} \sum_{n=2}^{\infty} |a_n + b_n| z^n$ *is also in* $C^*(\alpha, \beta)$ *.*

THEOREM 11. *Let*

$$f_1(z) = z \ , \quad f_n(z) = z - \frac{2\beta(1-\alpha)}{n\{(n-1)+\beta(n+1-2\alpha)\}} z^n \ , \quad (n = 2, 3, \ldots) \ .$$

Then $f \in C^*(\alpha, \beta)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) \ \textit{where} \ \lambda_n \geq 0 \ \textit{and} \ \sum_{n=1}^{\infty} \lambda_n = 1 \ .$$

References

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