NOTE ON NEWTONIAN FORCE-FIELDS

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1. Introduction.[1] Although the behaviour of Newtonian potentials *inside* n-dimensional distributions of mass or charge has been discussed in the sense of Lebesgue-Stieltjes integrals by various authors,[2] the discussion of various important theorems seems to have been made only in the sense of Riemann integration, and assuming the Hölder conditions[3] (or at least piecewise continuity) for the volume density ρ. We shall generalize these theorems below.

The discussion may be divided into three parts. First, we discuss Gauss' Integral Theorem. Then, we prove that the force components exist and are the derivatives of a potential, under very general conditions. In both cases, the Fubini Theorem is the principal tool.[4] Finally, we discuss Poisson's equation.

$$\nabla^2 U + (n-2)\omega_n \rho = 0, \quad [n \geqq 3], \tag{1}$$

$$\nabla^2 U + 2\pi\rho = 0, \quad [n = 2]. \tag{1'}$$

Here ω_n is the surface area of the unit sphere in n-space. We also extend the results of H. Petrini in various ways.[5]

2. Gauss' Integral Theorem. Consider a charge distribution of variable density $\rho(x) = \rho(x_1, \ldots, x_n)$, zero outside a bounded region R of Euclidean n-space.[6] We define the *potential* at $a = (a_1, \ldots, a_n)$ as usual by

$$U(a) = \int_R \rho(x) dV / r^{n-2} \quad \text{if } n \geqq 3, \tag{2}$$

and the attractive *force*-components by

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[1]The results of **2-3** were announced at the Princeton Bicentennial Conference in 1946; see Abstract 52-5-137 of *Bull. Amer. Math. Soc.*

[2]Notably Lichtenstein [2]; G. C. Evans, Rice Institute Pamphlet 7 (1920), 252-329; F. Riesz, *Acta Math.*, vol. 48 (1926), 329-343 and vol. 54 (1930), 321-360; T. Rado, *Subharmonic Functions* (Berlin, 1937). We are indebted to Professors Evans, Rado, and Zygmund for various comments with bibliographical suggestions.

[3]O. Hölder, *Beiträge zur Potentialtheorie* (Stuttgart, 1882), 10; Kellogg [1], 150-156; Courant-Hilbert, *Methoden der math. physik*, vol. 2, 228; J. Kravtchenko, *J. Math. Pures Appl.*, vol. 23 (1944), 97-210.

[4]It has been used before in potential theory. See G. C. Evans, *Trans. Amer. Math. Soc.*, vol. 37 (1935), 256; J. M. Thompson, *Bull. Amer. Math. Soc.*, vol. 41 (1935), 744-752; M. O. Reade, *ibid.*, vol. 53 (1947), 321-331.

[5]H. Petrini [3] proved that $\nabla^2 U$ was equal to $-4\pi\rho$ wherever ρ was continuous and $\nabla^2 U$ existed. He also showed that a "generalized Laplacian" existed wherever ρ was continuous. Lichtenstein [2] showed that if ρ was of class L_2, then Poisson's equation was valid except on a set of measure zero.

[6]We could easily represent ρ as the sum of a positive and negative distribution, and so restrict ourselves to potentials of positive mass.

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$$F_i(a) = (n-2)\int_R (x_i - a_i)\rho dV/r^n = \int_R [\partial/\partial a_i(1/r^{n-2})]\rho dV. \tag{3}$$

If $n = 2$, we define correspondingly

$$U(a) = -\int_R \rho(x)\ln r dV, \tag{2'}$$

$$F_i(a) = \int_R (x_i - a_i)\rho dV/r^2 = -\int_R [\partial/\partial a_i(\ln r)]\rho dV. \tag{3'}$$

Here $dV = dx_1 \ldots dx_n$ and $r = |x - a| = [\sum_i (x_i - a_i)^2]^{\frac{1}{2}}$. The integrals may be interpreted either in the sense of Lebesgue or of (improper) Riemann integration. For the Lebesgue integral of a function $f(x)$ to exist, $f(x)$ must be *measurable*; for the Riemann integral to exist, it must be[7] *continuous almost everywhere*—i.e., except on a set of Lebesgue measure zero. Since all factors except ρ in definitions (2), (3), (2′), (3′) are continuous except at $x = a$, these conditions are satisfied by the integrands if they are satisfied by ρ. — For the integrals to exist it is necessary and sufficient that in addition the upper integral of $|f|$ be finite. Hence it is necessary that the total charge $C = \int_R |\rho| dV$ be finite, interpreted in the sense of Lebesgue integration. We recall also that if the Riemann integral exists, then so does the Lebesgue integral, and the two are equal.

Now let N denote the force-component in the direction of the outward normal to a (hyper) surface and let ω_n denote the hyper-area of the unit sphere in n-space. Further, let the concept of "regular" surface defined in [1] be extended in the natural way to n dimensions; let $\partial/\partial n$ denote the outward normal directional derivative. For any regular region R (i.e., region bounded by a regular surface S),

$$\int_S |\partial(1/r^{n-2})/\partial n|\, ds = K \text{ is finite.} \tag{4}$$

Thus it is $(n - 2)\omega_n$ for a convex region.

THEOREM 1. *If $\rho(x)$ is Lebesgue integrable over R then for any "regular" hypersurface bounding a region T,*

$$\int_S N dS = (2 - n)\omega_n \int_T \rho dV. \tag{5}$$

Proof. Form the $(2n - 1)$-dimensional product space $S \times R$. Clearly $\rho(x)\partial(1/r^{n-2})/\partial n$ is measurable over $S \times R$, since the factors are measurable over R and S respectively. Further, by (4) and our earlier remarks,

$$\int_{S\times R} |\rho(x)\partial(1/r^{n-2})/\partial n| dS dV \leqq K\int_R |\rho| dV \leqq KC$$

is finite; hence the Fubini Theorem ([5], p. 87) applies. Taking this as an

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[7]We could not locate an explicit statement of these results, for $n > 1$, in the literature.

iterated integral in the two possible orders, we get (5) precisely.—Note that we can disregard points on $S \times S$, which has $(2n-1)$-dimensional measure zero.

By the Fubini Theorem again, if the $\partial F_i/\partial x_i$ are all Lebesgue integrable over a coordinate cuboid (i.e., rectangle in n dimensions), C, the Divergence Theorem holds[8] for C. Hence, by (5), Div F and $(2-n)\omega_n\rho$ have the same integral over any coordinate cuboid C. From this it follows immediately (by the Lebesgue density theorem or otherwise) that the integrands are equal almost everywhere. This proves the following result.

COROLLARY 1. *If $\rho(x)$ and the $\partial F_i/\partial x_i$ exist almost everywhere and are Lebesgue integrable on a region R, then*

(1) $$\text{Div } F = (2-n)\omega_n\rho, \text{ almost everywhere in } R.$$

A much more elementary argument of the same type shows that if ρ is continuous, and if $\nabla^2 U$ is also continuous, then (1) holds everywhere.

The argument of Theorem 1 gives another interesting result in case S is a sphere of radius c and centre O. It is well-known, and easy to show,[9] that the *average* $\overline{U}(S)$ on S of the potential due to a charge e at x is equal to the potential e/r^{n-2} at the centre, if x is outside S, and is e/c^{n-2} if x is inside S. Using the Fubini Theorem as before, we conclude

COROLLARY 2. *If $\rho(x)$ is Lebesgue integrable, then $U(a)$ exists almost everywhere on S, and*

(6) $$\overline{U}(S) = \int_{r=c}^{\infty} \rho(x)dV/r^{n-2} + (1/c^{n-2})\int_{r=0}^{c} \rho dV.$$

Moreover if (2) exists as a Lebesgue integral when $a = O$

(7) $$\overline{U}(S) - U(O) = \int_{r=0}^{c} (1/c^{n-2} - 1/r^{n-2})\rho dV.$$

A similar argument may be applied to solid spherical averages, which we denote $U^*(S)$. It gives, in three dimensions

(8) $$U^*(S) - U(O) = \int_{r=0}^{c} \frac{1}{2}\left(\frac{3}{c} - \frac{r^2}{c^3} - \frac{2}{r}\right)\rho dV = \frac{-1}{2c^3}\int_{r=0}^{c} \frac{1}{r}(c-r)^2(2c+r)\zeta dV.$$

These differences can be used to define generalized Laplacians.[10]

3. Force and derivatives of potential. The assumptions involved in Corollary 1 are inconvenient, because they involve both ρ and the F_i. We shall now give some results involving only assumptions about ρ.

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[8]See H. Federer, "The Gauss-Green Theorem," *Trans. Amer. Math. Soc.*, vol. 58 (1945), 44-76, esp. pp. 48-50, 60-62; G. G. Lorentz, *Bericht über die Mathematiker-Tagung in Tübingen*, 23-7 Sept., 1946, pp. 94-6.

[9]The average potential on S due to a charge at x is equal by the symmetry of $1/r$ to the potential at x due to the same *total* charge uniformly distributed on S.

[10]Introduced by Evans and F. Riesz, *op. cit., supra*. See Courant-Hilbert, *op. cit., supra*, p. 258; and especially S. Saks, *Mat. Sbornik*, vol. 51 (1941), 451-456, and I. Privaloff, *ibid.*, 457-460.

THEOREM 2. *The $F_i(a)$ defined by* (3) *all exist if and only if ρ is measurable* (*continuous a.e.*) *and*

(9) $$\int_R |\rho| dV/r^{n-1} < +\infty.$$

Explanation. In the case of Lebesgue integrals, ρ must be measurable; with improper Riemann integrals, it must be continuous a.e. In the latter case, (9) must be interpreted as meaning that the proper Riemann integrals of $|\rho| dV/r^{n-1}$ over $R - S(\delta)$, where $S(\delta)$ denotes a sphere of radius δ with centre a, have a finite upper bound. In any case, we note that (9) implies

(9′) $$g(\delta) = \int_{S(\delta)} |\rho| dV/r^{n-1} \to 0 \text{ as } \delta \to 0.$$

Proof. As above, a measurable (continuous a.e.) function in $n > 1$ dimensions is Lebesgue integrable (respectively Riemann integrable) if and only if the upper integral of its absolute value is finite. But

$$\left|\rho \frac{x_i - a_i}{r^n}\right| = \frac{|\rho|}{r^{n-1}} \cdot \frac{|x_i - a_i|}{r} \leqq \frac{|\rho|}{r^{n-1}} \leqq \sum_{i=1}^{n} \left|\rho \frac{x_i - a_i}{r^n}\right|.$$

Hence the integral (9) is finite if and only if all the force integrals in (3) are finite—i.e., exist.

COROLLARY 1. *The theorem holds if* (9) *is replaced by*

$$\rho = 0\left(\frac{1}{r^{1-\gamma}}\right) \text{ or even } \rho = 0\left(\frac{1}{r\,|\log r|^{1+\gamma}}\right)$$

in the neighbourhood of the origin. Here $\gamma > 0$.

However, the existence of the force integrals (3) does not imply that the potential integral (2) is differentiable. We now investigate sufficient conditions for this.

THEOREM 3. *In order that the two terms of*

(10) $$F_i(a) = \partial U/\partial a_i(a)$$

should exist and be equal and continuous, it is sufficient that for some $f(\delta)$ tending to zero with δ,

(11) $$\int_{S(\delta)} |\rho| dV/r^{n-1} < f(\delta), \text{ for all } \delta,$$

where $r = |x - c|$, for all c in some sphere about a. Here $S(\delta)$ denotes the sphere with radius δ and centre c.

Explanation. In view of Theorem 2 equation (11) is simply a technical way of saying that the force integrals (3) converge *uniformly* in an open set containing a. It follows that $F_i(b)$ are continuous near a.

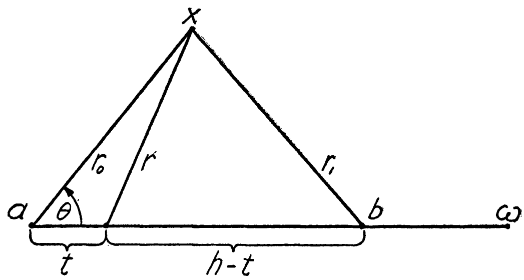
Proof. Without loss of generality, we can take $i = 1$ and suppose a is the origin. Then, writing $b = (h, 0, \ldots, 0)$, we define

$$Q(h) = [U(b) - U(a)]/h - F_1(a).$$

The existence of all terms of $Q(h)$ for b near a follows directly from (11), which implies (9) and the finiteness of (2) and hence the existence of $U(a)$, $U(b)$, $F_1(a)$. Our problem is to show that $Q(h) \to 0$ as $h \to 0$. But if $n > 3$

$$U(b) - U(a) = \int_R [r_1^{2-n} - r_0^{2-n}]\,\rho dV$$

where $r_0 = |x - a|$ and $r_1 = |x - b|$ as in the figure.



Substituting also for $F_1(a)$ from (3), we get

$$Q(h) = \int_R I(x, h)\rho dV, \text{ where } I(x, h) = \frac{1}{h}\left[\frac{1}{r_1^{n-2}} - \frac{1}{r_0^{n-2}}\right] - \frac{(n-2)x_1}{r_0^n}.$$

We now estimate the two preceding terms separately. We have

$$-\frac{1}{h}\left[\frac{1}{r_0^{n-2}} - \frac{1}{r_1^{n-2}}\right] = \frac{(r_0 - r_1)\sum_{k=0}^{n-3} r_0^k r_1^{n-3-k}}{h(r_0 r_1)(r_0 r_1)^{n-3}}.$$

But $\qquad |r_0 - r_1| \leq |h|$, and $r_0^k r_1^{n-3-k} \leqq (r_0^{n-3} + r_1^{n-3})$.

Substituting and adding up the $n - 2$ terms, we get

$$\left|\frac{1}{h}\left[\frac{1}{r_0^{n-2}} - \frac{1}{r_1^{n-2}}\right]\right| \leqq (n-2)\left[\frac{1}{r_0^{n-2}r_1} + \frac{1}{r_0 r_1^{n-2}}\right].$$

Finally, since $1/r_0^{n-2}r_1 + 1/r_0 r_1^{n-2} \leqq r_0^{1-n} + r_1^{1-n}$ and $|x_1/r_0| \leqq 1$, we get the inequality

$$(*) \qquad |I(x, h)| \leqq (n-2)\left|\frac{2}{r_0^{n-1}} + \frac{1}{r_1^{n-1}}\right|.$$

Hence, using Lebesgue integrals, if $|h| < \delta$, and $S(\delta)$ and $S(b, h)$ denote spheres with centres a, b and radii δ, h respectively, we have

$$\int_{S(\delta)} |I(x, h)\rho|\, dV = (n-2)\left[2\int_{S(\delta)} \frac{|\rho| dV}{r_0^{n-1}} + \int_{S(b,h)} \frac{|\rho| dV}{r_1^{n-1}}\right] \leqq 3(n-2)f(2\delta),$$

since $S(2\delta)$ contains $S(b, h)$ if $|h| < \delta$. For this we require only that (11) hold along the x_1-axis to make the contribution to $Q(h)$ arbitrarily small provided δ is small enough and $|h| < \delta$.

But if h is small enough, the contribution from outside $S(\delta)$ is arbitrarily small. For we have, by Taylor's Theorem with remainder for the function r^{2-n} considered as a function of h at $h = 0$,

$$|I(x, h)| \leqq \tfrac{1}{2} h \sup_{0<t<h} |\partial^2(r^{2-n})/\partial t^2| \leqq \tfrac{1}{2} hK(\delta),$$

since, if $h < \delta/2$, the second derivative is bounded outside $S(\delta)$. Hence the total contribution to $Q(h)$ from outside $S(\delta)$ is bounded by $\frac{1}{2}hK(\delta)C$, where $C = \int |\rho| dV$ as before, and can be made arbitrarily small. Adding together the contributions from inside and outside $S(\delta)$, we see that $Q(h) \to 0$ as $h \to 0$.

In case $n = 2$, $U(a)$ and $U(b)$ exist by (11) because

$$U = \int_R - \rho \log r dV \text{ and } |\rho \log r| \leq \frac{|\rho|}{r}.$$

Also, we have $-I(x, h) = \dfrac{1}{h}[\log r_1 - \log r_0] + \dfrac{x_1}{r_0^2}$. Now $|\log r_1 - \log r_0|$ $\leq |r_1 - r_0| \left[\dfrac{1}{r_1} + \dfrac{1}{r_0}\right]$ from the theorem of the mean; hence

$$|I(x, h)| \leq \frac{1}{r_1} + \frac{2}{r_0}, \text{ and } \int_{S(\delta)} |I(x, h, \rho)| dV = 0(\delta) \text{ if } |h| < \text{ as before.}$$

COROLLARY 1. *We have* (10) *provided uniformly*

$$(11') \qquad |\rho(x)| \leqq g(r), \text{ where} \int_0^1 g(r) dr < \infty.$$

COROLLARY 2. *We have* (10) *provided* $\rho(x) = 0(r_0^{\gamma-1})$ for $\gamma < 0$ near a.

Proof. The inequality (11) follows from the hypothesis, because for c sufficiently near a

$$\int_{S(\delta)} \frac{|\rho|\, dV}{r^{n-1}} \leq \int_{S(\delta)} \frac{AdV}{r_0^{1-\gamma} r^{n-1}} \leq \int_{S(\delta)} \frac{AdV}{r_0^{n-\gamma}} + \int_{S(\delta)} \frac{AdV}{r^{n-\gamma}} \leq \int_{S(\delta)} \frac{2AdV}{r^{n-\gamma}}$$

by [1], p. 148, Lemma III (a) generalized to n dimensions, and the last integral approaches 0 as $\delta \to 0$ uniformly in c.

Further inspection shows that for (10) to hold at a, it is sufficient for (11) to hold along the coordinate axes.

Clearly $F_1(x) = \lim\limits_{h \to 0} \left[\dfrac{U(x+h) - U(x)}{h}\right]$ exists as a uniform limit for x in the sphere about a, and hence $F_1(x)$ is continuous, and similarly all $F_i(x)$ are continuous in the hyper-sphere about a. Hence N is continuous in Theorem 1, and we have

COROLLARY 3. *If* $\rho(x)$ *is continuous a.e., and* (11) *or* (11′) *is valid for all* a *in* S, *then Gauss' Integral Theorem* (5) *is valid in the sense of improper Riemann integration. Thus it is sufficient that* $\rho(x)$ *be continuous.*

We shall now prove a somewhat weaker result under more general hypotheses.

THEOREM 4. *For almost every direction $w = (w_1, \ldots, w_n)$, the directional derivative $\partial U/\partial w(a)$ exists and satisfies*

$$\partial U/\partial w(a) = w_1 F_1(a) + \ldots + w_n F_n(a), \tag{12}$$

provided that (9) *holds at a; i.e., provided the $F_i(a)$ are defined.*[11]

Explanation. We mean that the set of unit vectors w for which (12) fails, has spherical measure zero on the unit sphere Ω in n-space.

Proof. By Theorem 2, the right-hand side of (12) and $U(O)$ are defined. Moreover without losing generality, we can set $a = O$, and restrict our attention to an arbitrarily small sphere $S(\delta)$ with centre a, as in Theorem 3. Also as in Theorem 3, we can define (cf. the figure)

$$Q(h, w) = \frac{1}{h}[U(hw) - U(0)] - \sum_{i=1}^{n} w_i F_i(a)$$

$$= \int_{S(\delta)} I(x, h, w)\rho dV, \text{ where, if } n \geqq 3,$$

$$I(x, h, w) = \frac{1}{h}\left[\frac{1}{r_1^{n-2}} - \frac{1}{r_0^{n-2}}\right] - (n-2)\sum_{i=1}^{n} w_i x_i / r_0^{n}.$$

Moreover $U(hw)$ will exist if $\int_{S(\delta)} I(x, h, w)\rho dV$ exists. Further, we define the new function

$$J(x, h, w) = \sup_{0 < t \leqq h} |I(x, t, w)|.$$

Clearly $I(x, t, w)$ is continuous, except when x is in the direction w. Hence ([4], p. 42) for fixed h, $J(x, h, w)$ is *lower semicontinuous* in the product-space $\Omega \times S(\delta)$ of couples (w, x), except on the set of $(2n-1)$-dimensional measure zero where x is in the direction w. Hence ([4], p. 43) $J(x, h, w)$ is *measurable* on $\Omega \times S(\delta)$, and we can define the multiple integral (J is non-negative)

$$Q_1 = \frac{1}{\omega_n}\int_{\Omega}\int_{S(\delta)} J(x, h, w)\, |\rho|\, dV d\Omega \leqq +\infty. \tag{13}$$

But by the inequality (*) of the proof of Theorem 3,

$$J(x, h, w) \leqq (n-2)\left[\frac{2}{r_0^{n-1}} + \left(\frac{1}{r^{n-1}}\right)\max\right].$$

While by trigonometry, $r \geqq r_0 \sin\theta = \theta r_0/2\pi$, if θ is acute, while $r \geqq r_0 \geqq \theta r_0/2\pi$ if θ is obtuse (cf. fig.). Substituting we get

$$Q_1 = \frac{n-2}{\omega_n}\int_{\Omega} d\Omega \int_{S(\delta)}\left[\frac{2}{r_0^{n-1}} + \frac{B}{r_0^{n-1}\theta^{n-1}}\right]|\rho|\, dV,$$

where the use of the iterated integral is permissible by the Fubini Theorem

[11]This result is closely related to the discussion of G. C. Evans, *Trans. Amer. Math. Soc.*, vol. 37 (1935), 234.

([4], p. 82) provided the right-hand integral is finite. But $d\Omega \leqq \omega_{n-1}\theta^{n-1}d\theta$ by spherical geometry; hence by (9′)

$$Q_1 \leqq 2(n-2)g(\delta) + \frac{(n-2)\omega_{n-1}}{\omega_n}\int_{S(\delta)} |\rho|\, dV \int_{\Omega} \frac{B}{r_0^{n-1}\theta^{n-1}}\,\theta^{n-1}d\theta$$
$$= (n-2)\,[-2 + \pi B\omega_{n-1}/\omega_n]\, g(\delta).$$

But by (9′), for δ sufficiently small and all $h < \delta$, this can be made arbitrarily small; hence it is finite.

By the Fubini Theorem and (13),

$$\int_{\Omega} d\Omega \int_{S(\delta)} J(x, h, w)\, |\rho|\, dV = Q_1\omega_n$$

is arbitrarily small if δ is sufficiently small. Hence the integrand $\int_{S(\delta)} J(x, h, w) \cdot |\rho|\, dV$ can be made less than ϵ except on a set of directions of Ω-measure at most ϵ. Hence

$$\sup_{0<t<h} |Q(t, w)| \leqq \int_{S(\delta)} J(x, h, w,)\, |\rho|\, dV < \epsilon$$

except on the same set of directions. This immediately implies (12), including the existence of $\partial U/\partial w(a)$, for almost every direction. If $n = 2$, $|I(x, h)| \leq \frac{1}{r} + \frac{2}{r_0}$, and the proof follows as above.

COROLLARY 1. *Theorem* 4 *holds if* (9) *is replaced by* $\rho = 0\left(\frac{1}{r_0^{1-\gamma}}\right)$ *or even* $\rho = 0\left(\frac{1}{r_0|\log r_0|^{1+\gamma}}\right)$ *for* r_0 *sufficiently small.*

It should be noted that in Theorems 3 and 4 dealing with the first partial derivatives of U no mention is made of the Hölder condition on ρ, namely that there exist A and $\alpha > 0$ such that $P(x) \leqq Ar^{\alpha}$ if $|x| = r \leqq r_0$. (See footnote 3.)

4. Poisson equation. We now extend Petrini's result on the existence of the "generalized Laplacian"

$$\nabla_P^2 U(a) = \lim_{h\to 0} \frac{1}{h^2}\sum_{i=1}^{n} [U(a + he_i) + U(a - he_i) - 2U(a)], \tag{14}$$

defined by second differences, instead of second derivatives. Define $\rho(x)$ to be *mean continuous* at a, if and only if

$$\epsilon(c) = \int_{S(c)} |\rho - \rho_0|\, dV/c^n \to 0 \text{ as } c \to 0. \tag{15}$$

If ρ is bounded, this means that ρ differs from ρ_0 by less than $\epsilon(c)$ except on a set of arbitrarily small relative measure, in sufficiently small spheres $S(c)$ with radius c and centre a.

THEOREM 5. *Let ρ be measurable and bounded near a, and "mean continuous" at a. Then the generalized Laplacian* (14) *exists and satisfies the Poisson equation* (1) *with* $\rho = \rho_0$.

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Remark. If ρ is continuous, then it is bounded and measurable, and (15) holds. Hence Theorem 5 contains the result of Petrini; moreover measurability, and continuity at a are sufficient to imply our result.

Proof. For any $h > 0$, the sum in square brackets in (14) as defined by (2) exists, since ρ is bounded and measurable, whence ρr^{2-n} is integrable. Again, the case of constant density is covered by elementary formulas; hence we can suppose $\rho_0 = 0$ without loss of generality; this we now do.

If we represent each term of (14) by (2), and sum under the integral sign, the right-hand quantity in (14) becomes

$$(16) \qquad \frac{1}{h^2}\int J(x;\, h)\rho(x)dV, \text{ where}$$

$$J(x, h) = \sum_{i=1}^{n}\left[\frac{1}{|a + he_i - x|^{n-2}} + \frac{1}{|a - he_i - x|^{n-2}} - \frac{2}{|a - x|^{n-2}}\right].$$

We now divide the definite integral of (16) into the contribution $I_1(h)$ from inside a sphere of centre a and radius $8h$, and the contributions $I_k(h)$ from the spherical shells $S(h, k)$: $2^k h \leqq |x - a| \leqq 2^{k+1}h$ $[k = 3, 4, 5, \dots]$. We write $|x - a| = r$.

In $S(h, k)$, expanding $|x - y|^{2-n}$ by Taylor's Theorem with remainder out to third partial derivatives, we get $|J(x, h)| \leqq Kh^3/r^{n+1} \leqq K'h^3/2^{(n+1)k}k^{n+1}$. Also, $\int_{S(h,k)} |\rho|\, dV$ is at most $(2^{k+1}h)^n\epsilon(2^{k+1}h)$. Hence $|I_k(h)| = h^2K^*\epsilon(2^{k+1}h)2^{-k}$, where K^* is an absolute constant. Hence by (15),

$$\lim_{h\to 0}\frac{1}{h^2}\sum_{k=3}^{\infty} I_k(h) \leqq K^* \lim_{h\to 0}\sum_{k=3}^{\infty} 2^{-k}\epsilon(2^{k+1}h) = 0.$$

It remains to show that $\lim_{h\to 0} I_1(h)/h^2 = 0$. Breaking up $J(x, h)$ into its $3n$ summands, it is clearly sufficient to show that whether b is a, $a + he_i$, or $a - he_i$, we have

$$(17) \qquad \lim_{h\to 0}\frac{1}{h^2}\int_{S(9h)} r'^{2-n}|\rho(x)|\, dV = 0 \qquad [r' = |x - b|].$$

Here $S(9h)$ denotes the sphere with centre b and radius $9h$.

But since ρ is bounded, $|\rho_i| \leqq M$, and so for any $\delta > 0$,

$$\frac{1}{h^2}\int_{S(\delta)} \frac{|\rho(x)|dV}{r'^{n-2}} \leqq \frac{1}{h^2}\int_{S(\delta)} Mr'^{2-n}r'^{n-1}dr \leqq 41\, M\delta^2/h^2.$$

Also,
$$\frac{1}{h^2}\int_{r'=\delta}^{9h} r'^{2-n}|\rho|dV \leqq \frac{1}{h^2}\delta^{2-n}\int_{r'=\delta}^{9h} |\rho|dV$$

$$\leqq (10h)^n\epsilon(10h)/h^2\delta^{n-2} \leqq K^*(h/\delta)^{n-2}\epsilon(10h).$$

By choosing $\delta/h = \sqrt[n]{\epsilon(10h)}$, we infer (17). — Note that we used $\epsilon(10h)$ and not $\epsilon(9h)$, because (15) refers to spheres with centre a and not spheres with centre b.

COROLLARY 1. *Under the preceding hypothesis, using the notation of* (7), $\lim_{c \to 0} [\overline{U}(S) - U(O)]/c^2$ *exists and is* $\omega_n \nabla_p^2 U(O)$.

Proof. We average with respect to all choices of axes, and observe that the inequalities above hold for *all* choices of axes.

COROLLARY 2. *If* $\nabla^2 U$ *exists and* $\rho(x)$ *satisfies the preceding hypotheses, then* (1) *holds.*

Proof. Expanding in Taylor's series with a remainder $o(h^2)$, the existence of $\nabla^2 U$ implies that $\nabla_p{}^2 U$ exists and has the same value. It is of course known that even the continuity everywhere of ρ need not imply the existence of $\nabla^2 U$.

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