19

The holographic renormalisation group

We saw in the previous chapter that the 'holographic'[286, 287] duality[315] between AdS_5 physics and the physics of the conformally invariant four dimensional Yang–Mills theory can be extended to the properties of solutions which are only asymptotically AdS_5, in keeping with the basic dictionary of the correspondence. We studied the properties of Schwarzschild and Reissner–Nordstrom black holes in AdS, arising naturally as limits of non-extremal and spinning D3-branes, and found that their properties make considerable physical sense in the holographically dual field theory.

It is very clear that this duality between gravitational physics and that of gauge theory is potentially a powerful tool for studying gauge theory. The prototype example is, of course, a highly specialised sort of gauge theory, since it has sixteen supercharges, and is conformally invariant. Of great interest is the study of gauge theories which might be closer to the theories we use to model interactions in particle physics, such as QCD. Perhaps there are gravitational duals of such theories. More generally, of course, we would like to also find and study full string theory duals, if we want to study more than just very large N. At the time of writing, this is subject of considerable research effort.

In this chapter we shall have a brief look at extending the intuition we have developed about the AdS/CFT correspondence a bit further, and address the issue of studying less symmetric gauge theories by deforming the AdS/CFT example.

19.1 Renormalisation group flows from gravity

Recall that, in section 18.1.2, we took a five dimensional perspective, and recognised AdS_5 with gauge symmetry $SO(6)$ as a special fixed point

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solution of the gauged supergravity which preserves the full gauge symmetry. It should be clear from that discussion that other fixed points of the potential will have an intuitive explanation as other conformally invariant theories with fewer supersymmetries. We will again have $\partial\phi_i/\partial x^\mu = 0$, and some set of the scalars approaching some non-zero constants. Since the scalars are charged under the $SO(6)$, such non-zero expectation values will mean that some amount of the $SO(6)$ will be broken, leaving a subgroup G. The scalar potential will take some value $-C/\ell^2$. The solution will be AdS_5 with a new value of the cosmological constant and hence the AdS radius for this solution, $\hat{\ell}$, will be given by: $\Lambda = -C/\ell^2 = -6/\hat{\ell}^2$. The expectation is that this defines a dual conformal field theory, with fewer supersymmetries and global symmetry G.

We can imagine a solution that is asymptotically AdS_5, with all of the scalars being asymptotically zero, but at smaller radius, approaches this new solution. Since the radial parameter has been identified with an energy scale in the theory, we have the intuitive picture that this solution represents a collection of snapshots (one for each radial slice) of the evolution of the gauge theory as a function of energy scale. It begins in the UV with the symmetric theory and then at lower energies approaches a new theory, which is less symmetric. This picture is just what we would call *renormalisation group flow* (RG flow) [319, 320] in the context of the field theory. Our example is one of flowing from a UV fixed point, using a 'relevant operator' (see insert 3.1, p. 84), to an IR fixed point. The five dimensional gravitational dual picture of this (and its ten dimensional extension) therefore deserves to be called *holographic renormalisation group flow*, and we shall do so.

In fact, we can be even bolder than this. There may be other solutions which are viable vacua which are not AdS_5 in the interior. If they are connected at large radius to the familiar $SO(6)$ AdS_5, we can also think of them as the result of deforming the UV fixed point by relevant operators and undergoing RG flow to some new non-conformal field theory. Evidently, the utility of such a tool is worth exploring. Ultimately, we can see that this leads us to even consider the existence of well-defined solutions that are not AdS_5 in either radial limit, which are nevertheless holographic duals of gauge theories. In fact, gauge theories of considerable phenomenological interest – perhaps the entire Standard Model – may perhaps be represented in this way. It is prudent to develop the tools to find and study these holographic duals.

The flow between fixed points has a precise example which we shall study in section 19.3. It breaks the supersymmetry from $\mathcal{N} = 4$ to $\mathcal{N} = 1$. First, we shall study a simpler RG flow, which is just the switching on of an operator which preserves supersymmetry and merely takes the theory

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out onto its Coulomb branch. Last, we shall exhibit a flow to a theory which is non-conformal and $\mathcal{N} = 2$ supersymmetric.

First, we will uncover a little of the basic technology that we will need, and emphasise a bit further aspects of the physics of the gravitational side. Before proceeding, we should note that many of the powerful techniques which underlie the construction of the solutions we present here are well beyond the scope of this book, and we must refer the reader to the literature for the details. We shall merely exhibit solutions and hope that our discussion will at least make their properties seem natural and reasonable in the present context. Also, we will not have space to introduce in a self-contained manner some of the more advanced dual field theory properties that we examine. The reader should not regard this as discouragement, but instead as an opportunity to see some of these advanced field theory concepts and properties emerging in an interesting setting, which may, in some cases, serve as a useful introduction.

19.1.1 A BPS domain wall and supersymmetry

Since every radial slice should be dual to the gauge theory at some energy scale set by the radius, we expect that the metric should be of the form:

$$ds^2_{1,4} = e^{2A(r)}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + dr^2, \qquad u = \frac{\ell}{\alpha'}e^{r/\ell}, \quad (19.1)$$

where we have preserved the Poincaré invariant form of the metric. The function $A(r)$ is chosen such that as $r \to \infty$, $A(r) \to r/\ell$, and so we recover the metric of AdS_5, where we show how to return to the more familiar local AdS_5 coordinates in terms of the variable u.

Let us consider the possibility that one of the 42 scalars, φ, has been switched on, and has a non-trivial profile as we go in to smaller r. The function $A(r)$ will deviate from the AdS behaviour of r/ℓ to some non-trivial behaviour. Generically, it is useful to think of $A(r)$ as parametrising some interpolating region, with AdS_5 located at $r \to +\infty$ being one region. On the other side, there are a number of possibilities for what $A(r)$ might do, and we shall see three types by example as we proceed. One possibility is that we get $A(r) \propto r$ again, (with the scalar running to a constant) giving an AdS region in the interior. As discussed before, this is another fixed point, and is expected to be dual to a conformal field theory again. We shall see this later. Away from the asymptotic behaviour, we should still think of A as giving us an interpolating solution, forming a 'domain wall' separating two types of asymptotic behaviour. See figure 19.1, and recall the kink example of insert 1.4 (p. 18).

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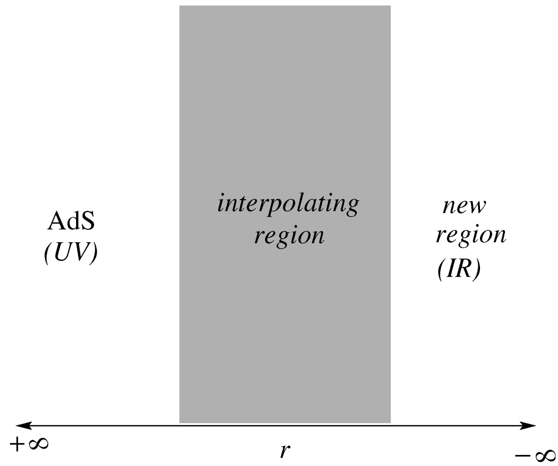

Fig. 19.1. The function $A(r)$ in the metric parametrises a departure from the UV's AdS behaviour, and may be thought of in terms of a 'domain wall' separating it from a new region in the IR.

Let us study some of the physics of this wall[316]. The supergravity action is:

$$S = \frac{1}{16\pi G_5}\int d^5x\sqrt{-G}\,[R - 2\partial_\mu\varphi\,\partial^\mu\varphi - V(\varphi)]. \tag{19.2}$$

If we insert the form of the metric given in equation (19.1), we get the following equations of motion:

$$\begin{aligned} 12\dot{A}^2 - 2\dot{\varphi}^2 + V &= 0 \\ 6\ddot{A} + 12\dot{A}^2 + 2\dot{\varphi}^2 + V &= 0 \\ \ddot{\varphi} + 4\dot{A}\dot{\varphi} - \frac{1}{4}\frac{\partial V}{\partial\varphi} &= 0, \end{aligned} \tag{19.3}$$

where a dot denotes a derivative with respect to r. It is interesting to note that differentiating the first equation and then using the third equation gives the second, and in fact

$$\ddot{A} = -\frac{2}{3}\dot{\varphi}^2. \tag{19.4}$$

It is most interesting to substitute the equation (19.1) into the action itself. Since the only non-trivial behaviour of the metric is as a function of r, the problem reduces to a one dimensional one, since the integral over the directions (t, x_1, x_2, x_3) is trivial. Throwing away the (infinite)

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constant from performing that integral, the action reduces to an energy functional:

$$\mathcal{E} = \frac{1}{16\pi G_5}\int_{-\infty}^{+\infty} dr\, e^{4A}\left[2\dot{\varphi}^2 - 12\dot{A}^2 + V(\varphi)\right]. \tag{19.5}$$

Let us consider the possibility that V depends upon an auxiliary function W, in the following manner*:

$$V(\varphi) = \frac{4}{\ell^2}\left[\frac{1}{2}\left(\frac{\partial W}{\partial\varphi}\right)^2 - \frac{4}{3}W^2\right]. \tag{19.6}$$

Let us substitute this into the energy functional, to get

$$\begin{aligned}\mathcal{E} &= \frac{1}{16\pi G_5}\int_{-\infty}^{+\infty} dr\, e^{4A}\left\{2\dot{\varphi}^2 - 12\dot{A}^2 + \frac{4}{\ell^2}\left(\frac{\partial W}{\partial\varphi}\right)^2 - \frac{16}{3\ell^2}W^2\right\}\\ &= \frac{1}{16\pi G_5}\int_{-\infty}^{+\infty} dr\, e^{4A}\left\{2\left(\dot{\varphi} \pm \frac{1}{\ell}\frac{\partial W}{\partial\varphi}\right)^2 - 12\left(\dot{A}^2 \mp \frac{2}{3\ell}W\right)^2\right.\\ &\qquad\qquad\left. \mp\frac{4}{\ell}\dot{\varphi}\frac{\partial W}{\partial\varphi} \mp \frac{16}{\ell}\dot{A}W\right\}.\end{aligned} \tag{19.7}$$

We have obligingly completed the square, as suggested by four of the terms, and collected the remainder at the end. Since $\dot{\varphi}(\partial W/\partial\varphi) = \dot{W}$, we can write the last two terms under the integral as $\mp d(12e^{4A}W)/dr$, and therefore it may be integrated and replaced by a boundary term.

So the functional is extremised if the following *first order* equations are satisfied:

$$\frac{\partial A}{\partial r} = -\frac{2}{3\ell}W, \qquad \frac{\partial\varphi}{\partial r} = \frac{1}{\ell}\frac{\partial W}{\partial\varphi}. \tag{19.8}$$

In fact, (by analogy with many other cases in earlier chapters) the reader should expect that finding a solution to these equations means that we have found a BPS solution of the system, preserving some of the supersymmetries of the original $\mathcal{N} = 8$ supergravity. The precise number of unbroken supersymmetries depends upon the details of W and the dependence on the scalars.

* The function W is called the 'superpotential' in the context of supersymmetric domain wall technology. It should not be confused with the W we shall later use for field theory superpotentials.

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19.2 Flowing on the Coulomb branch

Recall that the $\mathcal{N}=4$ supersymmetric Yang–Mills theory's gauge multiplet has bosonic fields (A_μ, ϕ_i), $i=1,\ldots,6$, where the scalars ϕ_i transform as a vector of the $SO(6)$ R-symmetry, and fermions λ_i, $i=1,\ldots,4$ which transform as the **4** of the $SU(4)$ covering group of $SO(6)$.

As we know from other examples in chapters 13 and 15, it is interesting to give vacuum expectation values of the scalars in the gauge multiplet. Generically, switching on vevs in the Cartan subalgebra of the $SU(N)$ gauge group will break the theory to $U(1)^{N-1}$, while keeping the scalar potential $\sum_{i,j}\mathrm{Tr}[\phi_i,\phi_j]$ vanishing and hence preserving supersymmetry. This is the Coulomb branch of vacua of the theory.

In the AdS/CFT context, the 42 $\mathcal{N}=8$ gauged supergravity scalars decompose as $\mathbf{1}+\mathbf{1}+\mathbf{10}+\overline{\mathbf{10}}+\mathbf{20}$ of the $SO(6)\simeq SU(4)$ gauge group, coupling to operators which have those R-charges in the gauge theory. Their translation is given in the dictionary extracts in table 18.1. Let us consider a family of vacua which are dual to the case of having switched on some components of this operator. If the AdS/CFT dictionary is to be believed, we should expect to find a non-trivial five dimensional supergravity solution which is asymptotically AdS_5 (since in the UV any relevant operator's vev should be negligible), and in the bulk there should be a non-trivial profile for supergravity scalars in the **20**. In ten dimensional type IIB supergravity terms, since we are exciting an $SO(6)$ spherical harmonic, we expect that the supergravity solution is asymptotically $\mathrm{AdS}_5\times S^5$, but in the interior, it deviates from it. In particular, the S^5 should be deformed in such a way which represents the turning on of the **20**.

19.2.1 A five dimensional solution

The scalar which will have a non-trivial profile will be called α. It should be zero as $r\to\infty$, and according to the dictionary entry (18.11), it should go as

$$\alpha \to \frac{a_1}{\sqrt{6}}e^{-2r/\ell}+\cdots, \tag{19.9}$$

since the **20** is an operator of dimension two.

In fact, there are complete solutions which can be written down[329, 330]. One of them is as follows. The scalar α will correspond to a particular part of the **20**:

$$\alpha: \qquad \sum_{i=1}^{4}\mathrm{Tr}(\phi_i\phi_i)-2\sum_{i=5}^{6}\mathrm{Tr}(\phi_i\phi_i). \tag{19.10}$$

This operator, which we can write as $\mathrm{diag}(1,1,1,1,-2,-2)$, in an $SO(6)$

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basis, splits the $\mathbb{R}^6$ transverse to the brane into an $\mathbb{R}^4$ and an $\mathbb{R}^2$, and so we expect that the supergravity solution will preserve an $SO(4)\times SO(2)$ of the $SO(6)$. The dependence of α and A can be written as first order differential equations representing a flow from their initial values at $r\to\infty$ to the interior, all the way to $r\to-\infty$. Defining $\rho=e^{\alpha}$, we have:

$$\frac{\partial\rho}{\partial r}=\frac{1}{6\ell}\rho^2\frac{\partial W}{\partial\rho}=\frac{1}{3\ell}\left(\frac{1}{\rho}-\rho^5\right),\qquad \frac{\partial A}{\partial r}=-\frac{2}{3\ell}W=\frac{2}{3\ell}\left(\frac{1}{\rho^2}+\frac{\rho^4}{2}\right),\tag{19.11}$$

where the auxiliary function

$$W=-\left(\frac{1}{\rho^2}+\frac{\rho^4}{2}\right)$$

can be used to write the scalar potential:

$$V=\frac{4}{\ell^2}\left[\frac{1}{2}\left(\frac{\partial W}{\partial\varphi}\right)^2-\frac{4}{3}W^2\right]=\frac{1}{3\ell^2}\left(\frac{\partial W}{\partial\alpha}\right)^2-\frac{16}{3\ell^2}W^2.\tag{19.12}$$

The functions W and V are plotted in figure 19.2.

In fact, the flow equations can be solved explicitly. Since we can write a differential equation for ρ in terms of A:

$$\frac{\partial\rho}{\partial A}=\left(\frac{\rho-\rho^7}{2+\rho^6}\right),$$

we can write

$$e^{2A}=\frac{l^2}{\ell^2}\frac{\rho^4}{\rho^6-1},\tag{19.13}$$

where l is a conveniently chosen integration constant. This initial value flow problem completely specifies the five dimensional supergravity solution.

Recall that we have two pictures, a five dimensional one in which we just have the gauged supergravity, and a ten dimensional one in which we have some type IIB solution. The first can be obtained from the latter, of course, although as we get more complicated gauged supergravity solutions, it will be harder to find the 'lift' to the full ten dimensional geometry. We shall, in a number of examples, wish to probe the geometry with D3-branes in order to determine more information about the physics. This is appropriate since the solutions are, after all, supposed to be made of D3-branes, in the sense that we discussed as early as in chapter 10. The D3-brane itself is best understood in a ten dimensional setting, and so the full ten dimensional picture is very useful to have, in

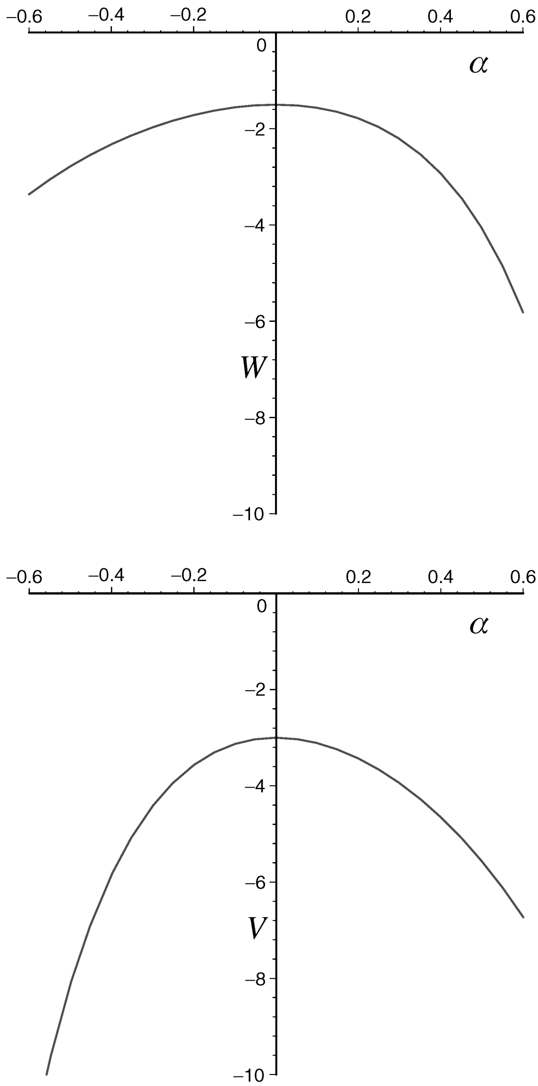

Fig. 19.2. The superpotential and potential, W and V, as a function of the scalar α.

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order to do the probe computation. Notice also that ultimately we would like to study the full string theory beyond the tree level gravitational limit, which is again naturally done in a ten dimensional setting. Let us therefore write the ten dimensional lift of that which we have uncovered here.

19.2.2 A ten dimensional solution

We expect that the ten dimensional solution will be of the form,

$$ds_{10}^2 = \Omega^2 ds_{1,4}^2 + ds_5^2, \tag{19.14}$$

where Ω is a 'warp' factor, which can depend upon the angles on the S^5 and r, and ds_5^2 is a deformed metric on the transverse space. Since we expect an $SO(4) \times SO(2)$ invariance, it is sensible to break things up into the metric $d\Omega_3^2$ on a round S^3, and two other angles θ and ϕ which control the rest of the S^5, which is now deformed. The solution is[329, 330]:

$$\begin{aligned}
\Omega^2 &= \frac{\bar{X}_1^{1/2}}{\rho} \\
\bar{X}_1 &= \cos^2\theta + \rho^6 \sin^2\theta,
\end{aligned} \tag{19.15}$$

with

$$ds_5^2 = \Omega^2 \frac{\ell^2}{\rho^2}\left[d\theta^2 + \frac{\sin^2\theta}{\bar{X}_1}d\phi^2 + \frac{\rho^6\cos^2\theta}{\bar{X}_1}d\Omega_3^2\right]. \tag{19.16}$$

The other supergravity fields all vanish except:

$$e^\Phi = g_\mathrm{s}, \qquad C_{(4)} = \frac{e^{4A}\bar{X}_1}{g_\mathrm{s}\rho^2} dt \wedge dx^1 \wedge dx^2 \wedge dx^3. \tag{19.17}$$

19.2.3 Probing the geometry

The geometry above is very interesting, but there is more physics to be uncovered. It is meant to govern the physics of the Coulomb branch of the moduli space of the $\mathcal{N} = 4$ gauge theory. Going onto the Coulomb branch, recall, is merely the process of pulling the N branes apart, away from the origin at $u = 0$. Recall also our result from chapter 10 that because the branes are all BPS, there is no potential for an individual brane's motion transverse to all the other branes and, furthermore, because we have sixteen supercharges, the actual metric on this moduli space should be flat. This should be true here. It is a simple exercise (see e.g. section 10.3) to probe the supergravity geometry presented in the previous subsection

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with a D3-brane[318]. In Einstein frame, some of the terms in the D3-brane world-volume action are:

$$\begin{aligned} S = &-\tau_3 \int_{\mathcal{M}_4} d^4\xi \; \det{}^{1/2}[G_{ab} + e^{-\Phi/2}\mathcal{F}_{ab}] \\ &+ \mu_3 \int_{\mathcal{M}_4} \left(C_{(4)} + C_{(2)} \wedge \mathcal{F} + \frac{1}{2} C_{(0)}\, \mathcal{F} \wedge \mathcal{F} \right), \end{aligned} \tag{19.18}$$

where $\mathcal{F}_{ab} = B_{ab} + 2\pi\alpha' F_{ab}$, and $\mathcal{M}_4$ is the world-volume of the D3-brane, with coordinates $\xi^0, \ldots, \xi^3$. As usual, the parameters μ_3 and τ_3 are the basic R–R charge and tension of the D3-brane:

$$\mu_3 = \tau_3 g_{\mathrm{s}} = (2\pi)^{-3}(\alpha')^{-2}. \tag{19.19}$$

Also, G_{ab} and B_{ab} are the pulls-back of the ten dimensional metric (in Einstein frame) and the NS–NS two-form potential, respectively.

A quick computation shows that the potential vanishes, and the result for the metric on its moduli space is

$$ds^2_{\mathcal{M}} = \frac{\tau_3}{2} \frac{\bar{X}_1 e^{2A}}{\rho^2} \left[dr^2 + \frac{\ell^2}{\rho^2} \left(d\theta^2 + \frac{\sin^2\theta}{\bar{X}_1} d\phi^2 + \frac{\rho^6 \cos^2\theta}{\bar{X}_1} d\Omega_3^2 \right) \right], \tag{19.20}$$

which looks very far from being flat. The way around this problem must simply be an issue of coordinates. There must be new coordinates more clearly adapted to the dual gauge theory physics in which this geometry is manifestly flat space. We expect to be able to find a new radial coordinate v and a new angle ψ which replace r and θ so that the metric is simply[318]:

$$\begin{aligned} ds^2_{\mathcal{M}} &= \frac{\tau_3}{2} \left[dv^2 + v^2 \left(d\psi^2 + \sin^2\psi d\phi^2 + \cos^2\psi d\Omega_3^2 \right) \right] \\ &= \frac{\tau_3}{2} \left[dv^2 + v^2 d\Omega_5^2 \right]. \end{aligned} \tag{19.21}$$

Equating coefficients requires us to show that the following equations can be solved:

$$\begin{aligned} \frac{\bar{X}_1 e^{2A}}{\rho^2} dr^2 &= dv^2, \\ \frac{\bar{X}_1 e^{2A} \ell^2}{\rho^4} d\theta^2 &= d\psi^2, \\ \frac{e^{2A}\ell^2}{\rho^4} \sin^2\theta &= v^2 \sin^2\psi, \\ e^{2A}\ell^2\rho^2 \cos^2\theta &= v^2 \cos^2\psi. \end{aligned} \tag{19.22}$$

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In fact, we can now perform this change of variables on the supergravity solution itself. After some algebra, and after using the flow equations themselves, the result is:

$$ds_{10}^2 = \left(\frac{\rho^2}{\bar{X}_1 e^{4A}}\right)^{-1/2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + \left(\frac{\rho^2}{\bar{X}_1 e^{4A}}\right)^{1/2}\left(dv^2 + v^2 d\Omega_5^2\right). \tag{19.23}$$

Looking at the form of the other supergravity fields in equation (19.17), we see that we have returned to the standard form for the brane solution, where we now have[318]

$$H_3 = \frac{\rho^2}{\bar{X}_1 e^{4A}} = \frac{\ell^4}{l^2 v^2}\left[\frac{\rho^6 - 1}{(v^2 + l^2)\rho^6 + 2v^2\cos^2\psi(\rho^6 - 1)}\right], \tag{19.24}$$

which we have partially translated into the new coordinates using the change of variables (19.22). A useful equation from there is a quadratic in ρ^6 obtained by eliminating θ from the last two lines in equations (19.22), to give:

$$\sin^2\psi\rho^{12} + \left[\cos^2\psi - \sin^2\psi - \frac{l^2}{v^2}\right]\rho^6 - \cos^2\psi = 0.$$

In the new variables, $H_3(\vec{v})$ is in fact harmonic. One way to see its explicit form is to expand the above equation for ρ^6 in large v. To do this, observe first that to a first approximation, the third term in the square braces vanishes, and so we have the solution $\rho^6 = 1$. Substitute $\rho = 1 + (l^2/v^2)g(l, \psi, v)$, and solve at the next order. The result is $g = 1 + \mathcal{O}(l^2/v^2)$. Recursive substitution like this will give[318]:

$$\rho^6 = 1 + \frac{l^2}{v^2} + (1 - \sin^2\psi)\frac{l^4}{v^4} + (1 - 3\sin^2\psi + 2\sin^4\psi)\frac{l^6}{v^6} + \cdots.$$

Using this, H_3 may be expanded to give:

$$H_3(v) = \frac{\ell^4}{v^4}\left(1 + \frac{l^2}{v^2}(3\sin^2\psi - 1) + \frac{l^4}{v^4}(1 - 8\sin^2\psi + 10\sin^4\psi) + \cdots\right), \tag{19.25}$$

which suggests the form:

$$H_3(v) = \frac{\ell^4}{v^4}\left(1 + \sum_{n=0}^{\infty}\frac{c_{2n}}{|\vec{v}|^{2n}}Y_{2n}(\cos^2\psi)\right), \qquad c_{2n} = (-1)^n l^{2n}, \tag{19.26}$$

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where the $Y_k(\cos^2\psi)$ (with $Y_k(1) = 1$) are the scalar spherical harmonics on S^5. In the above, we see explicitly the **20** ($n = 1$), and the **50** ($n = 2$).

This is remarkable, since we are seeing explicitly that non-zero l turns on precisely the operator which we want, with subleading mixing with higher order harmonics[318].

19.2.4 Brane distributions

The analysis we carried out above, where we found variables which took us from a complicated solution to one of the simple D3-brane standard form (but with a complicated harmonic function H_3), should remind the reader of the discussion presented in insert 18.2. Let us take the case where we only have one of the ℓ_i, say $\ell_1 = l$ non-zero. This corresponds (before the limit of insert 18.2) to a rotation in only one plane, and hence, after the limit, we expect an $SO(4) \times SO(2)$ invariant configuration. Let us study this.

The metric before the change of variables is:

$$\begin{aligned} ds^2_{10} &= H_3^{-1/2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) \\ &\quad + H_3^{1/2}\left[\frac{r^2 + l^2\cos^2\theta}{r^2 + l^2}dr^2 + (r^2 + l^2\cos^2\theta)d\theta^2\right. \\ &\quad \left. + (r^2 + l^2)\sin^2\theta d\phi_1^2 + r^2\cos^2\theta d\Omega_3^2\right], \\ H_3 &= 1 + \frac{\ell^4}{r^2(r^2 + l^2\cos^2\theta)}, \\ C_{(4)} &= g_s^{-1}(H_3^{-1} - 1)\, dt \wedge dx_1 \wedge dx_2 \wedge dx_3, \quad e^\Phi = g_s. \end{aligned} \tag{19.27}$$

The change of variables[314]:

$$\begin{aligned} y_1 &= \sqrt{(r^2 + l^2)}\,\mu_1\cos\phi_1 = \sqrt{(r^2 + l^2)}\,\sin\theta\cos\phi_1 \\ y_2 &= \sqrt{(r^2 + l^2)}\,\mu_1\sin\phi_1 = \sqrt{(r^2 + l^2)}\,\sin\theta\sin\phi_1 \\ y_3 &= r\,\mu_2\cos\phi_2 = r\cos\theta\sin\psi\cos\phi_2 \\ y_4 &= r\,\mu_2\sin\phi_2 = r\cos\theta\sin\psi\sin\phi_2 \\ y_5 &= r\,\mu_3\cos\phi_3 = r\cos\theta\cos\psi\cos\phi_3 \\ y_6 &= r\,\mu_3\sin\phi_3 = r\cos\theta\cos\psi\sin\phi_3, \end{aligned} \tag{19.28}$$

places the solution back into the familiar form:

$$ds^2 = H_3^{-1/2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + H_3^{1/2}\,(d\vec{y}\cdot d\vec{y}).$$

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Let us examine the harmonic (in the y_i) function:

$$H_3 = 1 + \frac{\ell^4}{r^2(r^2 + l^2\cos^2\theta)}. \tag{19.29}$$

Notice that when $r = 0$ there is a quadratic singularity for all θ. From the coordinate transformation (19.28), it is clear that this singularity occurs on the flat plane $\mathbb{R}^4$ given by $y_3 = y_4 = y_5 = y_6 = 0$, and the locus of points $y_1^2 + y_2^2 \leq l^2$. This is a disk.

The singularity in the harmonic function should signal the presence of the source – the D3-branes themselves – and it is tempting to conclude that they are distributed on that disk, and we can write[313] the appropriate uniform density function to go into the integral form (18.35):

$$\rho_3(\vec{v}) = \frac{1}{\pi l^2}\Theta\left(l - \sqrt{y_1^2 + y_2^2}\right)\delta^{(4)}(\vec{y}_\perp). \tag{19.30}$$

In fact, since a pointlike source in six dimensions produces a quartic singularity, a smeared two dimensional source should indeed produce a quadratic singularity so we are clearly on the right track. See figure 19.3.

We can check that our density function is correct by working perpendicular to the (y_1, y_2) plane of the disc, $\theta = 0$, to show that we recover

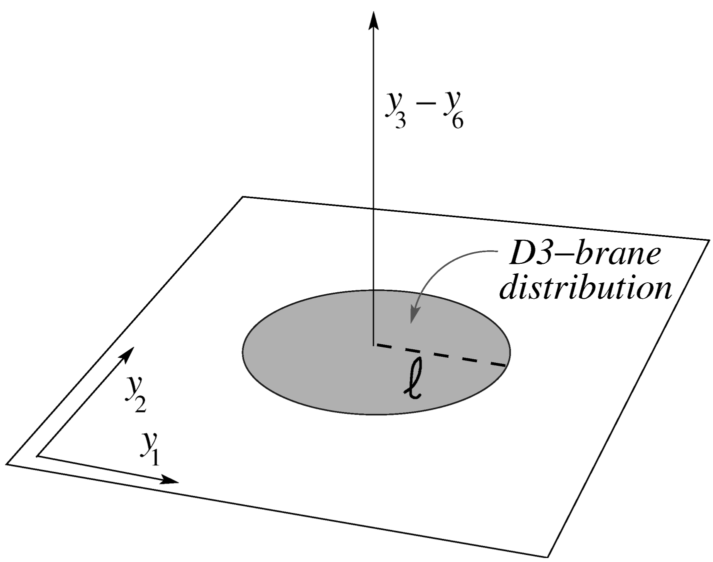

Fig. 19.3. The uniform disc distribution in $\mathbb{R}^6$ of D3-branes produced by switching on an operator in the **20**. This is a part of the Coulomb branch of the dual gauge theory.

expression (19.29) by explicitly integrating the the integral form (18.35). The θ dependence is forced to come out right by standard harmonic analysis: separation of variables, and the uniqueness of the expansion in terms of spherical harmonics.

There remains to establish a direct connection to the geometry of the previous subsection. So far, they have some of the same symmetries, but we have not shown that they are directly related. In fact, despite the differing form of the harmonic functions, they contain precisely the same physics. This can be shown by explicit computation. Notice that from the change of variables (19.28), we can write that

$$r^2 = y^2 - l^2 \sin^2\theta.$$

We can easily expand the harmonic function in $1/r^2$, and then use

$$\frac{1}{r^2} = \frac{1}{y^2}\left(1 - \frac{l^2 \sin^2\theta}{y^2}\right)^{-1} = \frac{1}{y^2}\sum_{m=0}^{\infty}\left(\frac{l^2}{y^2}\sin^2\theta\right)^m .$$

After some algebra, we find precisely the expression (19.26) we wrote earlier in terms of spherical harmonics, with (ψ, v) replaced by (θ, y).

19.3 An $\mathcal{N} = 1$ gauge dual RG flow

To recapitulate, the $\mathcal{N} = 4$ supersymmetric Yang–Mills theory's gauge multiplet has bosonic fields (A_μ, ϕ_i), $i = 1, \ldots, 6$, where the scalars ϕ_i transform as a vector of the $SO(6)$ R-symmetry, and fermions λ_i, $i = 1, \ldots, 4$ which transform as the $\mathbf{4}$ of the $SU(4)$ covering group of $SO(6)$. In $\mathcal{N} = 1$ language, there is a vector supermultiplet (A_μ, λ_4), and three chiral multiplets made of a fermion and a complex scalar ($k = 1, 2, 3$):

$$\Phi_k \equiv (\lambda_k, \varphi_k = \phi_{2k-1} + i\phi_{2k}), \tag{19.31}$$

and they have a superpotential

$$W = h\mathrm{Tr}(\Phi_3[\Phi_1, \Phi_2]) + \mathrm{h.c.}, \tag{19.32}$$

('h.c.' means Hermitian conjugate) where h is related to g_{YM} in a specific way consistent with superconformal symmetry.

Let us study the case of giving a mass to Φ_3,

$$L_{\mathrm{ft}} \to L_{\mathrm{ft}} + \int d^2\theta\, \frac{1}{2} m \Phi_3^2 + \mathrm{h.c}, \tag{19.33}$$

where 'h.c.' is the hermitian conjugate. The resulting spectrum (both massive and massless) can now have at most an $\mathcal{N} = 1$ multiplet structure.

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The resulting $SU(N)$ theory has matter multiplets in two flavours, Φ_1 and Φ_2, transforming in the adjoint of $SU(N)$. The $SU(4) \simeq SO(6)$ R-symmetry of the $\mathcal{N} = 4$ gauge theory is broken to $SU(2)_{\mathrm{F}} \times U(1)_{\mathrm{R}}$, the latter being the R-symmetry of the $\mathcal{N} = 1$ theory, and the former a flavour symmetry under which the matter multiplet forms a doublet.

This mass perturbation is a relevant one and so upon flowing to the IR it becomes more significant. Eventually we fall to scales where the mass is effectively infinite, and we are close to the pure $\mathcal{N} = 1$ theory we discussed in the previous paragraph.

In a supergravity dual, via the dictionary this maps to turning on certain scalar fields in the supergravity, their values being close to zero in the UV ($r \to +\infty$), they develop non-trivial profiles as a function of r, becoming more significantly different from zero as one goes deeper into the IR, $r \to -\infty$. The supergravity equations of motion require that there be a non-trivial back-reaction on the geometry, which deforms the spacetime metric in a way given by $A(r)$, in equation (19.1)

There is a supergravity dual which achieves this[322]. It turns on two scalars, which turn on a combination of the operator which we want, and a vev of the operator we discussed in the previous section:

$$\begin{aligned} \alpha: &\quad \sum_{i=1}^{4} \mathrm{Tr}(\phi_i\phi_i) - 2\sum_{i=5}^{6} \mathrm{Tr}(\phi_i\phi_i) \\ \chi: &\quad \mathrm{Tr}(\lambda_3\lambda_3 + \varphi_1[\varphi_2, \varphi_3]) + \mathrm{h.c.}, \end{aligned} \tag{19.34}$$

At a low enough scale, we can legitimately integrate out the massive scalar Φ_3 , and this results in the quartic superpotential[325, 326]

$$W = \frac{h^2}{4m}\mathrm{Tr}([\Phi_1, \Phi_2]^2), \tag{19.35}$$

which is in fact a marginal operator of the theory[325]. So the theory we get in the IR is also a conformal field theory, as is confirmed by the following considerations. If the operator, represented by the sum of the terms in equations (19.32) and (19.33), is marginal in the IR, then as it is a superpotential, it must have dimension three. This can be achieved if the fields developed anomalous dimensions γ_i (the fields's dimension is $1 + \gamma_i$ in this notation) once they left the UV and went to the IR. Since Φ_3 was treated differently from Φ_1 and Φ_2, it can have a different value for its anomalous dimension. An appropriate assignment is[325, 326] :

$$\gamma_1 = \gamma_2 = -\frac{1}{4}, \gamma_3 = \frac{1}{2}. \tag{19.36}$$

We should also check that the β-function vanishes. In fact, it is

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proportional [331] to $3-\sum_i(1-2\gamma_i)$, and so we see that it vanishes, showing that our operator is in fact exactly marginal[325].

From what we have already learned about AdS/CFT, it is natural to expect that the gravity dual to this conformal field theory is again AdS$_5$. It cannot be the same AdS$_5$ as before, and so it must have a different value for the cosmological constant and for the gauge symmetry associated to the supergravity. In the language of the discussion presented at the beginning of this chapter, it must simply be another fixed point of the $\mathcal{N}$=8 gauged supergravity, which has $\mathcal{N}=2$ supersymmetry and $SU(2)\times U(1)$ gauge symmetry. In the ten dimensional language, it must be that the transverse geometry is no longer S^5, but some deformation of the sphere which preserves $SU(2)\times U(1)$ isometry.

19.3.1 The five dimensional solution

Just as in the previous sections, the radial dependences of scalars and the function $A(r)$ are given in terms of first order 'flow' equations (recall that $\rho\equiv e^{\alpha}$):

$$\begin{aligned}
\frac{d\rho}{dr} &= \frac{1}{6\ell}\rho^2\frac{\partial W}{\partial\rho} = \frac{1}{6\ell}\left(\frac{\rho^6(\cosh(2\chi)-3)+2\cosh^2\chi}{\rho}\right)\\
\frac{d\chi}{dr} &= \frac{1}{\ell}\frac{\partial W}{\partial\chi} = \frac{1}{2\ell}\left(\frac{(\rho^6-2)\sinh(2\chi)}{\rho^2}\right)\\
\frac{dA}{dr} &= -\frac{2}{3\ell}W = -\frac{1}{6\ell\rho^2}\Big(\cosh(2\chi)(\rho^6-2)-(3\rho^6+2)\Big), \qquad (19.37)
\end{aligned}$$

where the function

$$W=\frac{1}{4\rho^2}\Big(\cosh(2\chi)(\rho^6-2)-(3\rho^6+2)\Big)$$

can be used to construct the potential via:

$$V=\frac{4}{\ell^2}\left[\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial W}{\partial\varphi_i}\right)^2-\frac{4}{3}W^2\right]=\frac{1}{3\ell^2}\left(\frac{\partial W}{\partial\alpha}\right)^2+\frac{1}{2}\left(\frac{\partial W}{\partial\chi}\right)^2-\frac{16}{3\ell^2}W^2. \qquad (19.38)$$

The functions W and V are plotted as contour maps in figure 19.4, and as three dimensional figures in figure 19.5.†

It is clear that the values $\chi=0$, $\alpha=0$ ($\rho=1$) define a stationary point for the scalars. After a bit of thought, one can find another fixed point

† The reader should not take the small scale variations of the contours near the fixed points seriously. They are due to loss of numerical accuracy.

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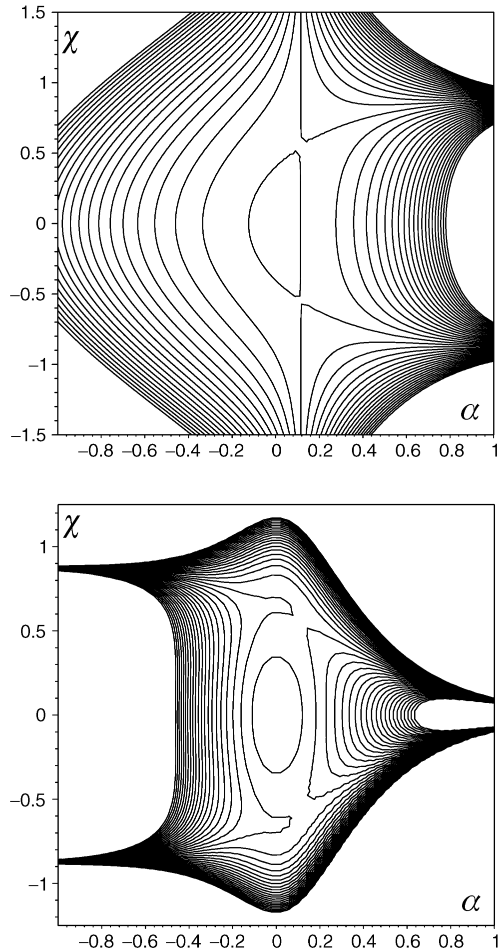

Fig. 19.4. Contour plots of the superpotential and potential, W and V, as functions of the scalars α, χ. This is dual to the RG flow from the $\mathcal{N} = 4$ conformally invariant gauge dual (the fixed point at $\chi = 0, \alpha = 0$) to an $\mathcal{N} = 1$ conformally invariant gauge dual (either of the fixed points at $\alpha = \log 2^{1/6}, \chi = \pm \log 3^{1/2}$).

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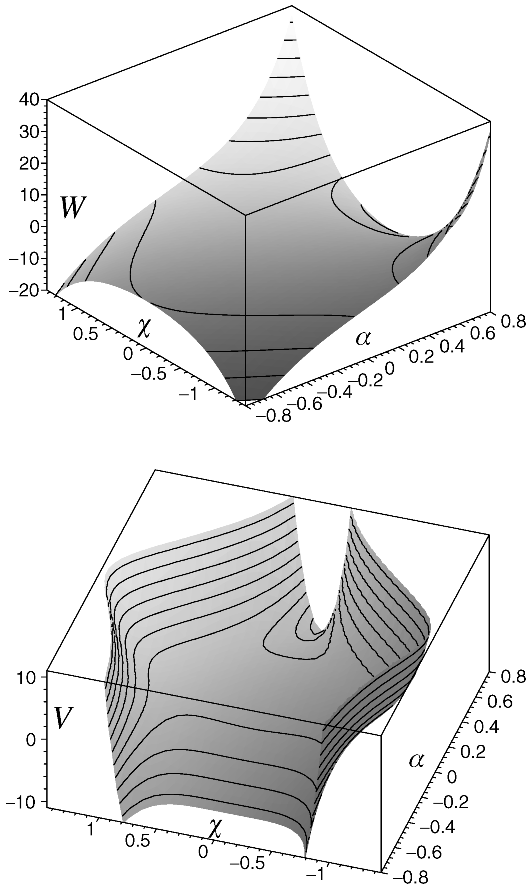

Fig. 19.5. Three dimensional figures of the superpotential and potential, W and V, as functions of the scalar α, χ, for the RG flow from the $\mathcal{N} = 4$ gauge dual to an $\mathcal{N} = 1$ gauge dual.

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solution: $\chi = \pm\log 3^{1/2}$, $\alpha = \log 2^{1/6}$. In the first case, those values give

$$\frac{\partial A}{\partial r} = \frac{1}{\ell} \quad \longrightarrow \quad A(r) = e^{r/\ell},$$

after throwing away an integration constant, which gives AdS_5 with cosmological constant $\Lambda_{\rm UV} = -6/\ell^2$. In the second case, the fixed point values give:

$$\frac{\partial A}{\partial r} = \frac{2^{5/3}}{3\ell} \quad \longrightarrow \quad A(r) = e^{r/\tilde{\ell}} \quad \text{where} \quad \tilde{\ell} = \frac{3\ell}{2^{5/3}},$$

after throwing away in integration constant, which gives AdS_5 with cosmological constant $\Lambda_{\rm IR} = -6/\tilde{\ell}^2$. So the ratio between the two cosmological constants is in fact

$$\frac{\Lambda_{\rm UV}}{\Lambda_{\rm IR}} = \frac{9}{32^{2/3}}. \tag{19.39}$$

This flow can be recognised as a generalisation of the pure ρ Coulomb branch case from before, by setting $\chi = 0$. Unlike that case, there is no known exact solution for these particular equations, but much can be deduced about the structure of the solution by resorting to numerical methods which we shall not explore much here. It is possible to extract that the asymptotic UV ($r \to +\infty$) behaviour of the fields $\chi(r)$ and $\alpha(r) = \log(\rho(r))$ is given by:

$$\chi(r) \to a_0 e^{-r/\ell} + \cdots; \qquad \alpha(r) \to \frac{2}{3}a_0^2\frac{r}{\ell}e^{-2r/\ell} + \frac{a_1}{\sqrt{6}}e^{-2r/\ell} + \cdots. \tag{19.40}$$

This behaviour of χ is, according to the dictionary 18.11, characteristic of an operator of dimension three representing a mass term (controlled by a_0), while that of α represents a *mixture* of both a dimension two mass operator (again through a_0) and a vacuum expectation value (vev) of an operator of mass two (through a_1).

Actually, the values of the constant

$$\hat{a} = \frac{a_1}{a_0^2} + \sqrt{\frac{8}{3}}\log a_0 \tag{19.41}$$

characterise a family of different solutions for $(\rho(r), \chi(r), A(r))$ representing different flows to the gauge theory in the IR. Meanwhile, in the IR ($r \to -\infty$) the asymptotic behaviour is:

$$\begin{aligned} &\chi(r) \to \frac{1}{2}\log 3 - b_0 e^{\lambda r/\ell} + \cdots, \\ &\alpha(r) \to \frac{1}{6}\log 2 - \frac{\sqrt{7}-1}{6}b_0 e^{\lambda r/\ell} + \cdots, \\ &\text{where} \quad \lambda = \frac{2^{5/3}}{3}(\sqrt{7}-1). \end{aligned} \tag{19.42}$$

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At this end of the flow, there is also a combination which is characteristic of the flow, and this is $b_0 a_0^\lambda$. This may be thought of as characterising the width of the region interpolating between the two AdS asymptotes[322].

The critical value $\hat{a}_c \simeq -1.4694$ represents the particular flow which starts out at the $\mathcal{N}=4$ critical point and ends precisely on the $\mathcal{N}=1$ critical point. It has been proposed[317] that the solutions with $\hat{a} > \hat{a}_c$ describe the gauge theory at different points on the Coulomb branch of moduli space. The combination $\hat{a}_c$ then, is pure mass and no vev, while other values are a mixture of both. The vev is that of a combination of massless fields which take us out onto the Coulomb branch.

For the flows with $\hat{a} < \hat{a}_c$, the five dimensional supergravity potential is no longer bounded above by the asymptotic UV value. They are believed to correspond to attempting to give a positive vev to the massive field.

19.3.2 The ten dimensional solution

The ten dimensional solution can be parameterised in the same way as before, in equation (19.14). This time we have[323, 324]:

$$ds_5^2 = \ell^2 \frac{\Omega^2}{\rho^2 \cosh^2\chi}\left[d\theta^2 + \rho^6\cos^2\theta\left(\frac{\cosh\chi}{\bar{X}_2}\sigma_3^2 + \frac{\sigma_1^2+\sigma_2^2}{\bar{X}_1}\right)\right.$$
$$\left. + \frac{\bar{X}_2\cosh\chi\sin^2\theta}{\bar{X}_1^2}\left(d\phi + \frac{\rho^6\sinh\chi\tanh\chi\cos^2\theta}{\bar{X}_2}\sigma_3\right)^2\right], \quad (19.43)$$

with

$$\begin{aligned}\Omega^2 &= \frac{\bar{X}_1^{1/2}\cosh\chi}{\rho}\\ \bar{X}_1 &= \cos^2\theta + \rho^6\sin^2\theta\\ \bar{X}_2 &= \mathrm{sech}\chi\cos^2\theta + \rho^6\cosh\chi\sin^2\theta.\end{aligned} \quad (19.44)$$

The σ_i are the standard $SU(2)$ left-invariant forms (see insert 7.4, p. 180), the sum of the squares of which give the standard metric on a round three-sphere. They are normalised such that $d\sigma_i = \epsilon_{ijk}\sigma_j \wedge \sigma_k$. For future use, we shall denote the coordinates on the S^3 as (ψ_1, ψ_2, ψ_3).

It is easily seen that the non-trivial radial dependences of $\rho(r)$ and $\chi(r)$ deform the metric of the supergravity solution from $\mathrm{AdS}_5 \times S^5$ at $r = +\infty$ where there is an obvious $SO(6)$ symmetry (the round S^5 is restored), to a spacetime which only has an $SU(2)\times U(1)$ symmetry, which is manifest in the metric of equation (19.43). The $SU(2)$ is the left-invariance of the σ_i and the $U(1)$ rotates σ_1 into σ_2. The obvious extra $U(1)$ symmetry, as

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$\partial/\partial\phi$ is also a Killing vector, but this is not a symmetry of the other fields in the full solution.

The fields Φ and $C_{(0)}$, the ten dimensional dilaton and R–R scalar, are gathered into a complex scalar field $\lambda = C_{(0)} + ie^{-\Phi}$, which is constant all along the flow. There are non-zero parts of the two–form potential, $C_{(2)}$, and the NS–NS two-form potential $B_{(2)}$ also, but for our study we won't need them.

Part of $C_{(4)}$ may be written as:

$$C_{(4)} = -\frac{4}{g_{\rm s}} w(r,\theta)\, dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3, \tag{19.45}$$

$$\text{where} \quad w(r,\theta) = \frac{e^{4A}}{8\rho^2}[\rho^6 \sin^2\theta(\cosh(2\chi) - 3) - \cos^2\theta(1 + \cosh(2\chi))].$$

We have only displayed the part of it which will be pertinent to the physics of a D3-brane probe. The part that is missing does not give a non-zero contribution to the probe Lagrangian.

19.3.3 Probing with a D3-brane

In order to understand this geometry a bit better, we shall do what we did in the previous example, and probe the geometry with a D3-brane. Again, this has a natural interpretation[321]. The Coulomb branch moduli space of the $\mathcal{N} = 1$ $SU(N)$ gauge theory is parameterised by the vevs of the complex adjoint scalars $\phi_{1,2}$ which set the potential $\mathrm{Tr}([\phi_1, \phi_2]^2)$ to zero. This generically breaks the theory to a product of $U(1)$s. Probing with a D3-brane will single out a four dimensional subspace of the full moduli space here since our moduli space is the space of allowed zero-cost transverse movements of our single D3-brane probe. These directions are parameterised by the scalars $(\phi_1, \phi_2, \phi_3, \phi_4)$, which make up the complex doublet (φ_1, φ_2). That hyperplane corresponds to the choice $\theta = 0$.

Using the very familiar probe methods from before (see e.g. section 10.3), we get the following result for the effective Lagrangian for the probe moving *slowly* in the transverse directions $y^m = (r, \phi, \theta, \psi_1, \psi_2, \psi_3)$ (we restrict ourselves to considering $F_{ab} = 0$ here):

$$\mathcal{L} \equiv T - V = \frac{\tau_3}{2}\Omega^2 e^{2A} G_{mn}\dot{y}^m\dot{y}^n - \tau_3 \sin^2\theta e^{4A}\rho^4(\cosh(2\chi) - 1). \tag{19.46}$$

The G_{mn} refer to the Einstein frame metric components.

It is clear that the case $\theta = 0$ indeed makes the potential vanish, picking out the four dimensional moduli space of the probe. The case $\rho = 0$, which is $\alpha = -\infty$, lies outside the physically allowed values of the flow.

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19.3.4 The Coulomb branch

It is worthwhile considering the case of large vevs. This should correspond to large r, and we should get a familiar result, flatness in all four (moduli space) transverse directions to the brane. The metric on this moduli space is simply the flat metric on $\mathbb{R}^4$:

$$ds^2_{\mathcal{M}_{\rm UV}} = \frac{1}{8\pi^2 g^2_{\rm YM}}[dv^2 + v^2 d\Omega_3^2], \qquad \text{with} \quad v = \frac{\ell}{\alpha'}e^{r/\ell}, \tag{19.47}$$

where we have defined the energy scale v.

A general point on the flow has $\theta = 0$ as the family of flat directions. This moduli space is the Coulomb branch of the gauge theory anywhere along the flow. We see that we have movement on a (stretched) S^3, with coordinates (ψ_1, ψ_2, ψ_3), and the radial direction r. These give an $\mathbb{R}^4$, topologically, exploring the vevs of the complex scalar fields in the adjoint, ϕ_1 and ϕ_2. The metric on this moduli space for arbitrary $(r, \psi_1, \psi_2, \psi_3)$ is:

$$ds^2 = \frac{\tau_3}{2}\frac{\cosh^2\chi}{\rho^2}e^{2A}dr^2 + \frac{\tau_3}{2}\ell^2 e^{2A}\rho^2\left(\cosh^2\chi\,\sigma_3^2 + \sigma_1^2 + \sigma_2^2\right). \tag{19.48}$$

We can study this metric in the limit of small vevs: $r \to -\infty$. Inserting the IR values of the functions and defining:

$$u = \frac{\rho_0 \ell}{\alpha'}e^{r/\tilde{\ell}}, \qquad \tilde{\ell} = \frac{3}{2^{5/3}}\ell, \qquad \rho_0 \equiv \rho_{\rm IR} = 2^{1/6} \tag{19.49}$$

we get

$$ds^2_{\mathcal{M}_{\rm IR}} = \frac{1}{8\pi^2 g^2_{\rm YM}}\left[\frac{3}{4}du^2 + u^2\left(\frac{4}{3}\sigma_3^2 + \sigma_1^2 + \sigma_2^2\right)\right]. \tag{19.50}$$

This is an interesting result[321] which encodes information about the filed theory in a way which it would be nice to understand better. In order to do this, we ought to find better coordinates in which various field theory quantities are more manifest.[321] In a low-energy sigma model, the metric on the moduli space is the quantity which controls the kinetic terms for the scalar fields. In superspace, the kinetic terms are written in terms of a single function, the Kähler potential K:

$$\mathcal{L} = \int d^4\theta\, K(\Phi^i, \Phi^{j\dagger}) - \left\{\int d^2\theta\, W(\Phi^i) + \text{h.c.}\right\}, \tag{19.51}$$

where Φ^i are chiral superfields whose lowest components are the scalars whose vevs we are exploring and $W(\Phi)$ is the superpotential. Our next task is to prove the existence of a Kähler potential for the probe metric. It is not at all manifest that this is the case, so we should spend some time on this next.

19.3.5 Kähler structure of the Coulomb branch

Let us start again with some new assignments of coordinates. The moduli space is parametrised by the vevs of the complex massless scalars, which we shall write as z_1 and z_2. The z_i transform as an $SU(2)$ doublet (i.e. in the fundamental), while their complex conjugates transform in the antifundamental. The $SU(2)$ flavour symmetry implies that the Kähler potential is a function of u^2 only where we define,

$$u^2 = z_1\bar{z}_1 + z_2\bar{z}_2. \tag{19.52}$$

This is *not* necessarily the coordinate u we used as the AdS coordinate, or in the small vev presentation of the moduli space in the previous subsection. We shall see how they are related in various limits later.

We can divide the coordinates (and indices) into holomorphic and antiholomorphic (those without and those with a bar). If the Kähler structure exists then the metric is given by

$$ds^2 = g_{\mu\bar{\nu}}dz^\mu dz^{\bar{\nu}} = g_{1\bar{1}}dz_1 d\bar{z}_1 + g_{1\bar{2}}dz_1 d\bar{z}_2 + g_{2\bar{1}}dz_2 d\bar{z}_1 + g_{2\bar{2}}dz_2 d\bar{z}_2,$$

where

$$g_{\mu\bar{\nu}} = \partial_\mu\partial_{\bar{\nu}}K(u^2) = \partial_\mu(\partial_{\bar{\nu}}(u^2)K') = \partial_\mu(\partial_{\bar{\nu}}(u^2))K' + \partial_\mu(u^2)\partial_{\bar{\nu}}(u^2)K',$$

where the primes denote differentiation with respect to u^2, and we have inserted our assumption about the u dependence of K. Notice that since

$$\partial_i(u^2) = \bar{z}_i \quad \text{and} \quad \bar{\partial}_i(u^2) = z_i, \tag{19.53}$$

we have

$$\begin{aligned} g_{1\bar{1}} &= \partial_1\bar{\partial}_1 K = K' + z_1\bar{z}_1 K'', \\ g_{1\bar{2}} &= \bar{z}_1 z_2 K', \end{aligned} \tag{19.54}$$

and so on. Some algebra shows that the metric can be written as

$$ds^2 = (dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2)K' + (\bar{z}_1 dz_1 + \bar{z}_2 dz_2)(z_1 d\bar{z}_1 + z_2 d\bar{z}_2)K'.$$

Now notice that[82]

$$\begin{aligned} du &= \frac{1}{2u}(\bar{z}_1 dz_1 + \bar{z}_2 dz_2 + z_1 d\bar{z}_1 + z_2 d\bar{z}_2) \qquad \text{and} \\ u\sigma_3 &= \frac{1}{2u}(-i\bar{z}_1 dz_1 - i\bar{z}_2 dz_2 + iz_1 d\bar{z}_1 + iz_2 d\bar{z}_2). \end{aligned} \tag{19.55}$$

This is convenient, since we can write

$$du + iu\sigma_3 = \frac{1}{u}(\bar{z}_1 dz_1 + \bar{z}_2 dz_2) \quad \text{and} \quad du - iu\sigma_3 = \frac{1}{u}(z_1 d\bar{z}_1 + z_2 d\bar{z}_2).$$

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Some more algebra puts the metric in the following form:

$$ds^2 = (K' + u^2 K')du^2 + u^2(K'(\sigma_1^2 + \sigma_2^2) + (K' + u^2 K')\sigma_3^2). \qquad (19.56)$$

Looking at the form of the probe result in equation (19.48), we see that in order to put the metric into Kähler form we need a change of radial coordinate relating r and u. Equating coefficients, we obtain three equations:

$$(K' + u^2 K')du^2 = \frac{\tau_3}{2}\frac{\cosh^2\chi}{\rho^2}e^{2A}dr^2, \qquad (19.57)$$

$$u^2(K' + u^2 K') = \frac{\tau_3}{2}\ell^2\rho^2 e^{2A}\cosh^2\chi, \qquad (19.58)$$

$$u^2 K' = \frac{\tau_3}{2}\ell^2\rho^2 e^{2A}. \qquad (19.59)$$

Using the first two equations we find

$$dr^2 = \frac{\ell^2\rho^4}{u^2}du^2, \qquad (19.60)$$

with solution:

$$u = \frac{\ell}{\alpha'}e^{f(r)/\ell}, \quad \text{with} \quad \frac{df}{dr} = \frac{1}{\rho^2}. \qquad (19.61)$$

Since the latter is always positive it defines a sensible radial coordinate u. We can now define K by the differential equation (19.59):

$$K' = \frac{dK}{d(u^2)} = \frac{\tau_3}{2}\frac{\ell^2\rho^2 e^{2A}}{u^2}, \qquad (19.62)$$

and we have to check that such a K obeys equation (19.58), which can be written as

$$u^2\frac{d}{d(u^2)}(u^2 K') = \frac{\tau_3}{2}\ell^2\rho^2 e^{2A}\cosh^2\chi. \qquad (19.63)$$

From the definition of u in equation (19.61), we have that

$$\frac{d}{d(u^2)} = \frac{\ell\rho^2}{2u^2}\frac{d}{dr}, \qquad (19.64)$$

and so we need to show that

$$\frac{\ell\rho^2}{2}\frac{d}{dr}(u^2 K') = \frac{\tau_3}{2}\ell^2\rho^2 e^{2A}\cosh^2\chi. \qquad (19.65)$$

From our definition of K in equation (19.62) this amounts to requiring us to show that:

$$\frac{d}{dr}(\rho^2 e^{2A}) = \frac{2}{\ell}e^{2A}\cosh^2\chi. \qquad (19.66)$$

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We can achieve this by performing the derivative on the left hand side and substituting the flow equations for $\rho(r)$ and $A(r)$ listed in equations (1.13) gives precisely the result on the right[321].

We have demonstrated the existence of the Kähler potential. In fact, using the equation (19.64) we can write an alternative form for the definition of K, to accompany (19.62), which is:

$$\frac{dK}{dr} = \tau_3 \ell e^{2A(r)}. \tag{19.67}$$

N.B. This remarkably simple equation has been shown[321] to be satisfied by the Kähler potentials of all of the holographic RG flow examples that are (currently) known in ten dimensions. It would be interesting to learn what lies beneath this apparent universality, and the direct meaning of this equation in field theory.

In fact, one can readily write down an exact solution to this equation everywhere along the flow. Up to additive constants, it is:

$$K = \frac{\tau_3 \ell^2 e^{2A}}{4}\left(\rho^2 + \frac{1}{\rho^4}\right). \tag{19.68}$$

Let us unpack some of the content of this solution[321]. For large u (i.e. in the limit of large vevs), $\rho \sim 1$ so that, from equation (19.61), we have $u \sim \frac{\ell}{\alpha'}\exp(r/\ell)$, and to leading order:

$$K \sim \frac{\tau_3}{2}\ell^2 e^{2r/\ell} = \frac{1}{8\pi^2 g_{YM}^2}u^2, \tag{19.69}$$

which implies the expected flat four dimensional metric (19.47) that we obtained before. We can also look at next-to-leading order corrections to the Kähler potential. Recalling the asymptotic solutions for α and χ in equations (19.40) and also the flow equations (19.37) gives:

$$A(r) \simeq \frac{r}{\ell} - \frac{a_0^2}{6}e^{-2r/\ell} + O(e^{-4r/\ell}), \tag{19.70}$$

so that

$$K \simeq \tau_3 \ell^2 \left(\frac{1}{2}e^{2r/\ell} - \frac{a_0^2}{3}\frac{r}{\ell}\right). \tag{19.71}$$

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We have discarded terms of order $\exp(-2r/\ell)$ as well as constant terms. Similarly, the corresponding expression for u^2 is from (19.61):

$$u^2 \simeq \frac{\ell^2}{\alpha'^2}\left(e^{2r/\ell} + \frac{4a_0^2}{3}\frac{r}{\ell}\right). \tag{19.72}$$

Returning to the Kähler potential, we find that:

$$K \simeq \frac{1}{8\pi^2 g_{YM}^2}\left[u^2 - \frac{a_0^2\ell^2}{\alpha'^2}\ln\left(\frac{\alpha'^2 u^2}{\ell^2}\right)\right], \tag{19.73}$$

an expression which looks like a one-loop field theory result. Further comparison requires some knowledge of how a_0^2 corresponds to the mass for Φ_3. To deduce this we can look at the probe result at large u more closely. The result of the probe calculation was given in equation (19.46). To leading order, we have

$$\begin{aligned} |z_1|^2 + |z_2|^2 &= \frac{\ell^2}{\alpha'^2} e^{2r/\ell}\cos^2\theta, \quad \text{and} \\ |z_3|^2 &= \frac{\ell^2}{\alpha'^2} e^{2r/\ell}\sin^2\theta, \end{aligned} \tag{19.74}$$

and so

$$\mathcal{L} = \frac{1}{8\pi^2 g_{YM}^2}\left((|\dot{z}_1|^2 + |\dot{z}_2|^2 + |\dot{z}_3|^2) - \frac{4a_0^2}{\ell^2}|z_3|^2\right), \tag{19.75}$$

where we used the asymptotic solution (19.40) for α and for χ. The mass of Φ_3 is therefore

$$m_3 = \frac{2a_0}{\ell}. \tag{19.76}$$

Inserting this into the Kähler potential, we obtain

$$K \simeq \frac{1}{8\pi^2 g_{YM}^2}u^2 - \frac{N m_3^2}{16\pi^2}\ln\left(\frac{\alpha'^2 u^2}{\ell^2}\right), \tag{19.77}$$

which is of the form expected for the tree level plus one loop correction, since (it turns out that) the $\mathcal{N} = 4$ field content ensures that $u^2 \ln u^2$ terms cancel exactly. For small u, $\rho \to 2^{1/6}$ and we have

$$u \sim \frac{\ell}{\alpha'}\exp\left(\frac{r}{2^{1/3}\ell}\right). \tag{19.78}$$

This gives us:

$$K \sim \frac{\tau_3}{2}\ell^2 \frac{3}{2^{5/3}}\left(\frac{u^2\alpha'^2}{\ell^2}\right)^{4/3} = \frac{1}{8\pi^2 g_{YM}^2}\frac{3}{2^{5/3}}\left(\frac{\alpha'^2}{\ell^2}\right)^{1/3}(u^2)^{4/3}, \tag{19.79}$$

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and so the metric in this limit is:

$$ds^2 \sim \frac{1}{8\pi^2 g_{YM}^2} 2^{1/3} \left(\frac{u^2 \alpha'^2}{\ell^2}\right)^{1/3} \left(\frac{4}{3} du^2 + u^2 \left(\sigma_1^2 + \sigma_2^2 + \frac{4}{3}\sigma_3^2\right)\right), \tag{19.80}$$

which can be converted to the original form (19.50) after the redefinition $u \to u^{3/4}$ and an overall rescaling.

So now we understand that the curious form of this metric is simply a consequence of the power, 4/3, of u^2 which appears in the Kähler potential. This power in turn follows from simple supergravity scaling, which translates nicely into the field theory data we already discussed.

At the UV end of the flow we have the standard $\mathrm{AdS}_5 \times S^5$ geometry. The AdS_5 part of the metric given in equation (19.1) with $A = \frac{r}{\ell}$ which has a scaling symmetry under

$$x \to \frac{1}{\alpha} x \qquad e^A \to \alpha e^A \qquad u \to \alpha u, \tag{19.81}$$

where we have used that $u \sim e^A$ for large r. In other words the fields on moduli space have scaling dimension one, and so match with the dual field theory values for the scalar components of these chiral superfields in the $\mathcal{N} = 4$ theory. Next we consider the IR end of the flow. Here the solution again has the scaling symmetry (19.81) except that $A = 2^{5/3} r/3\ell$ in this case. The coordinate u goes like $u \sim \exp\left(\frac{r}{2^{1/3}\ell}\right) \sim (e^A)^{3/4}$ and thus the scaling symmetry becomes

$$x \to \frac{1}{\alpha} x \qquad u \to \alpha^{3/4} u. \tag{19.82}$$

Therefore, we see that the massless fields have scaling dimension 3/4 here. Again this agrees with the field theory, as it includes the anomalous dimensions discussed earlier in equation 19.36.

Let's put it another way. Consider the Kähler potential at either end (UV or IR) of the flow. From the $SU(2)$ flavour symmetry we know that K is a function of u^2 only. We also know the scaling dimension of u^2 at each end of the flow. The action's kinetic term is:

$$S = \int d^4x \, \partial_\varphi \, \partial_{\bar{\varphi}} K \, \partial_\mu \varphi \, \partial^\mu \bar{\varphi}, \tag{19.83}$$

where φ are the massless scalars with some scaling dimension. For the action to be invariant under scaling, $K(u^2)$ must have scaling dimension 2. At the UV end of the flow u has scaling dimension 1, so $K \sim u^2$, as expected. At the IR end of the flow, u has scaling dimension 3/4 and so $K \sim (u^2)^{4/3}$, matching our earlier results.

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19.4 An $\mathcal{N}=2$ gauge dual RG flow and the enhançon

It is worthwhile studying just one more flow example. This time it will not flow to a fixed point, and will preserve twice the supersymmetry as the previous example. This is achieved by turning on operators which correspond to giving equal masses to the $\mathcal{N}=1$ multiplets Φ_1, Φ_2. Together, these form an $\mathcal{N}=2$ hypermultiplet. This leaves one adjoint chiral multiplet Φ_3, together with the vector $\mathcal{N}=1$ supermultiplet (A_μ, λ_4), forming the $\mathcal{N}=2$ vector supermultiplet. So the deformation preserves an $\mathcal{N}=2$ structure.

As before, this should correspond to an appropriate combination of scalars being switched on in supergravity, and the solution is known[322]. Again there are two scalars, and they correspond to the following operators:

$$\begin{aligned} \alpha: &\qquad \sum_{i=1}^{4}\mathrm{Tr}(\phi_i\phi_i) - 2\sum_{i=5}^{6}\mathrm{Tr}(\phi_i\phi_i) \\ \chi: &\qquad \mathrm{Tr}(\lambda_1\lambda_2+\lambda_2\lambda_2)+\mathrm{h.c.} \end{aligned} \tag{19.84}$$

Moving around on the accessible part of the Coulomb branch of the $\mathcal{N}$=2 theory corresponds to giving a vacuum expectation value (vev) to $\varphi_3 = \phi_5 + i\phi_6$, which is the plane $\theta = \pi/2$.

The Coulomb branch of the moduli space of the $\mathcal{N}=2$ $SU(N)$ gauge theory is parametrised by the vevs of the complex adjoint scalar φ_3 which set the potential $\mathrm{Tr}[\phi_3, \phi_3^\dagger]^2$ to zero. This generically breaks the theory to $U(1)^{N-1}$. This moduli space is of course an $N-1$ complex dimensional space, but we are just focusing on the one-complex dimensional subspace corresponding to $SU(N-1)\times U(1)$. The low energy effective action of the theory is described in terms of a low energy field u with an effective complex coupling $\tau(u)$:

$$\tau(u) = \tau_{\mathrm{c}} + \frac{\theta}{2\pi} + i\frac{4\pi}{g_{\mathrm{YM}}^2}, \tag{19.85}$$

where the classical value is $\tau_{\mathrm{c}} = \theta_{\mathrm{s}}/2\pi + i/g_{\mathrm{s}}$ in our case. The quantities θ_{s} and g_{s} are of course set by the R–R scalar $C_{(0)}$ and the dilaton Φ. Recall that $C_{(0)}$ couples to $F\wedge F$ on the D3-brane world volume, contributing to the θ-angle in the $\mathcal{N}=2$ effective low energy theory.

19.4.1 The five dimensional solution

As before, at $r\to\infty$, the various functions in the solution have the following asymptotic behaviour[322]:

$$\rho(r)\to 1,\ \ \chi(r)\to 0,\ \ A(r)\to r/\ell. \tag{19.86}$$

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For arbitrary r, the values of the functions are determined by the following flow equations:

$$\begin{aligned}
\frac{d\alpha}{dr} &= \frac{1}{\ell}\frac{\partial W}{\partial \alpha} = \frac{1}{3\ell}\left(\frac{1}{\rho^2} - \rho^4\cosh(2\chi)\right) \\
\frac{d\chi}{dr} &= \frac{1}{\ell}\frac{\partial W}{\partial \chi} = -\frac{1}{2\ell}\rho^4\sinh(2\chi) \\
\frac{dA}{dr} &= -\frac{2}{3\ell}W = \frac{2}{3\ell}\left(\frac{1}{\rho^2} + \frac{1}{2}\rho^4\cosh(2\chi)\right),
\end{aligned} \tag{19.87}$$

where the function

$$W = -\left(\frac{1}{\rho^2} + \frac{1}{2}\rho^4\cosh(2\chi)\right),$$

can be used to construct the potential via:

$$V = \frac{4}{\ell^2}\left[\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial W}{\partial \varphi_i}\right)^2 - \frac{4}{3}W^2\right] = \frac{1}{3\ell^2}\left(\frac{\partial W}{\partial \alpha}\right)^2 + \frac{2}{\ell^2}\left(\frac{\partial W}{\partial \chi}\right)^2 - \frac{16}{3\ell^2}W^2. \tag{19.88}$$

The functions W and V are plotted as contour maps in figure 19.6, and as three dimensional figures in figure 19.7.‡

By using the middle equation of (19.87), we can write expressions for $d\alpha/d\chi$ and $dA/d\chi$, which we can integrate (with some manipulation) to give:

$$\begin{aligned}
e^A &= k\frac{\rho^2}{\sinh(2\chi)} \\
\rho^6 &= \cosh(2\chi) + \sinh^2(2\chi)\left(\gamma + \log\left[\frac{\sinh\chi}{\cosh\chi}\right]\right).
\end{aligned} \tag{19.89}$$

Here, k is a constant we shall fix later, while γ is a constant whose values characterise a family of different solutions for $(\rho(r), \chi(r))$ representing different flows to the $\mathcal{N}=2$ gauge theory in the IR. See figure 19.8.

- For $\gamma < 0$, equation (19.89) yields a *finite* value, $\chi_0 = \frac{1}{2}\cosh^{-1} c_0$, of χ in the IR, while ρ goes to *zero*. The supergravity solution has a naked singularity as a result.
- For $\gamma = 0$, χ *diverges* in the IR and again ρ goes to zero. Supergravity again has singular behaviour, coming from both the divergence and the zero.
- For $\gamma > 0$ both χ and ρ diverge, and the supergravity is singular.

‡ As mentioned before, the reader should not take the small scale variations of the contours near the fixed points seriously. They are due to loss of numerical accuracy.

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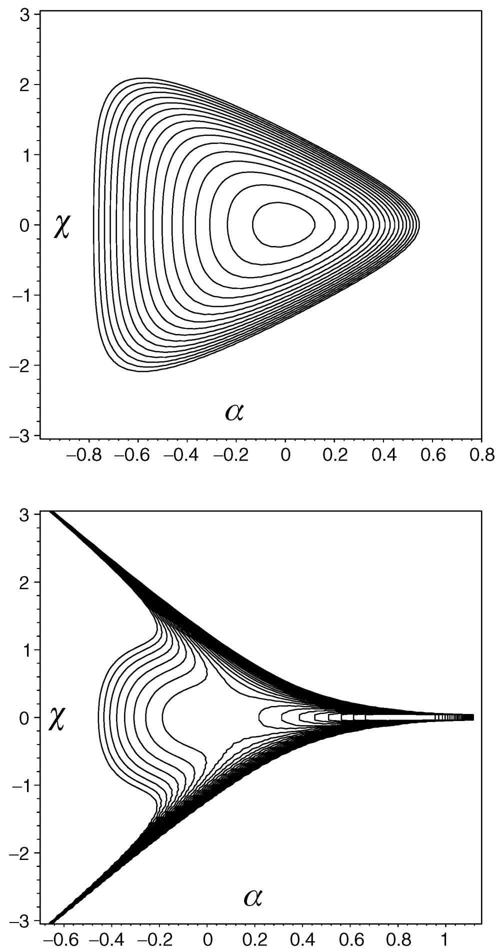

Fig. 19.6. Contour plots of the superpotential and potential, W and V, as functions of the scalars α, χ, for the RG flow to an $\mathcal{N} = 2$ gauge dual. The flows depicted in figure 19.8 are centred on the ridges to the left, the case $\gamma = 0$ being precisely along the ridge.

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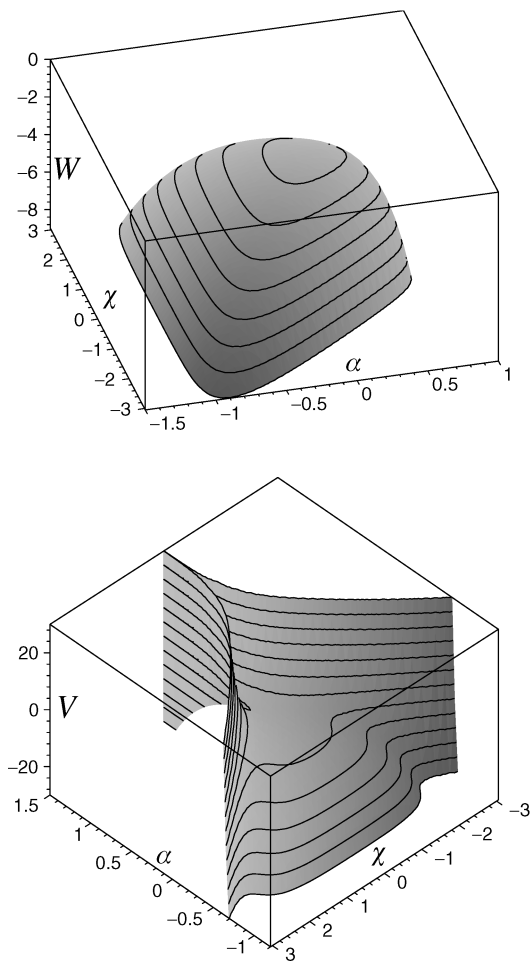

Fig. 19.7. Three dimensional figures of the superpotential and potential, W and V, as functions of the scalar α, χ, for the RG flow to an $\mathcal{N} = 2$ gauge dual.

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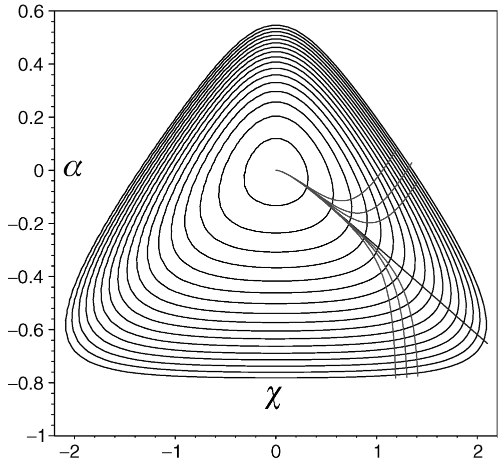

Fig. 19.8. The families of (χ, α) curves for differing γ, given by equation (19.89), superimposed on the contours of the superpotential W. There are three classes of curves. The middle curve is $\gamma = 0$, the $\gamma < 0$ curves are below it, and the $\gamma > 0$ curves are above. The flow from UV to IR along each curve is to the right.

All of our intuition gathered in this chapter and the previous one points towards there being sensible physics concerning the Coulomb branch of the expected $\mathcal{N} = 2$ dual gauge theory to be found at the end of the flow. We see that instead, the supergravity solution flows to regions which produce unphysical singularities. Somehow, this must be obscuring actual physical information.

This is where it is useful again to study the ten dimensional lift of the solution and probe it with a D3-brane. Following the wisdom of the previous two examples, we might find that the probe has a better handle on what are the right variables to use for the extraction of meaningful physics.

19.4.2 The ten dimensional solution

The ten dimensional solution written in the form (19.14), with[327, 328]:

$$ds_5^2 = \ell^2 \frac{\Omega^2}{\rho^2}\left[\frac{d\theta^2}{c} + \rho^6 \cos^2\theta\left(\frac{\sigma_1^2}{cX_2} + \frac{\sigma_2^2 + \sigma_3^2}{X_1}\right) + \sin^2\theta \frac{d\phi^2}{X_2}\right], \quad (19.90)$$

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where $c = \cosh(2\chi)$, and

$$\begin{aligned}\Omega^2 &= \frac{(cX_1X_2)^{1/4}}{\rho}\\ X_1 &= \cos^2\theta + \rho^6\cosh(2\chi)\sin^2\theta\\ X_2 &= \cosh(2\chi)\cos^2\theta + \rho^6\sin^2\theta.\end{aligned} \tag{19.91}$$

It is easily seen that the non-trivial radial dependences of $\rho(r)$ and $\chi(r)$ deform the supergravity solution from $\mathrm{AdS}_5 \times S^5$ at $r = \infty$ where there is an obvious $SO(6)$ symmetry (the round S^5 is restored), to a spacetime which only has an $SU(2) \times U(1)^2$ symmetry, which is manifest in the metric (19.90).

There are also explicit solutions for the R–R two-form potential, $C_{(2)}$, and the NS–NS two-form potential $B_{(2)}$. We will not need them here. The fields $(\Phi, C_{(0)})$ are gathered into a complex scalar field which we shall denote as $\lambda = C_{(0)} + ie^{-\Phi}$, and the solution for them is as follows:

$$\lambda = i\left(\frac{1-B}{1+B}\right), \tag{19.92}$$

with

$$B = \left[\frac{b^{1/4} - b^{-1/4}}{b^{1/} + b^{-1/4}}\right]e^{2i\phi}, \quad \text{where } b \equiv c\frac{X_1}{X_2}. \tag{19.93}$$

We shall extract the specific form for the dilaton, which we shall need, a bit later.

We will need the explicit form for the R–R four-form potential $C_{(4)}$, to which the D3-brane naturally couples. It is

$$C_{(4)} = e^{4A}\frac{X_1}{g_s\rho^2}dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3. \tag{19.94}$$

As is clear from the behaviour displayed in figure 19.8, it is evident that in the IR the supergravity becomes singular. This makes it hard to interpret the physics which is supposed be telling us about a dual gauge theory. Again, it is prudent to probe the geometry with a D3-brane to see if we can determine more about the physics.

19.4.3 Probing with a D3-brane

Following on what we did before, it is again straightforward to probe the geometry[332, 333], and the reader is urged to carry out the computation.

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The result is an effective Lagrangian $\mathcal{L} = T - V$, where:

$$\begin{aligned}
T &= \frac{\mu_3}{2g_s}\Omega^4 e^{2A} \\
&\quad \times \left\{ \dot{r}^2 + \frac{\ell^2}{\rho^2}\left(\frac{\dot{\theta}^2}{c} + \rho^6 \cos^2\theta \left(\frac{v_1^2}{c\bar{X}_2} + \frac{v_2^2+v_3^2}{\bar{X}_1} \right) + \sin^2\theta \frac{\dot{\phi}^2}{\bar{X}_2} \right) \right\}, \\
&= \frac{\mu_3}{2g_s}\frac{\ell^2 \left(c\bar{X}_1\bar{X}_2\right)^{1/2}}{(c^2-1)} \\
&\quad \times \left\{ \frac{\dot{c}^2}{\rho^6(c^2-1)^2} + \left(\frac{\dot{\theta}^2}{c} + \rho^6 \cos^2\theta \left(\frac{v_1^2}{c\bar{X}_2} + \frac{v_2^2+v_3^2}{\bar{X}_1} \right) + \sin^2\theta \frac{\dot{\phi}^2}{\bar{X}_2} \right) \right\}, \\
V &= \frac{\mu_3}{g_s} e^{4A}\left(\Omega^4 - \frac{\bar{X}_1}{\rho^2} \right) = \frac{\mu_3}{g_s}\frac{k^4\rho^6}{(c^2-1)^2}\left(\sqrt{c\bar{X}_1\bar{X}_2} - \bar{X}_1 \right),
\end{aligned} \tag{19.95}$$

where the v_i are the natural velocities associated to the one-forms σ_i given in insert 7.4 (p. 180), and in the last line we have used the first of the results in equations (19.89). The penultimate line was arrived at by using the fact that the second flow equation in (19.87) allows us to replace $\dot{r}^2$ by $\dot{c}^2\ell^2/[\rho^8(c^2-1)^2]$.

19.4.4 The moduli space

In order to make the potential vanish, there are two independent conditions: $c\bar{X}_2 = \bar{X}_1$, which means $\theta = \pi/2$, or $\rho = 0$. Notice that for the cases of $\gamma > 0$, the second situation does not exist, since (as is clear from figure 19.8) $\rho \to \infty$ and $\chi \to \infty$, while for $\gamma < 0$, the flow assigns a specific value, $\chi_0 = \frac{1}{2}\cosh^{-1} c_0$, for χ while $\rho \to 0$. The moduli space is parameterised by the coordinates (θ, ϕ), with metric:

$$ds^2_{\mathcal{M}2}(\gamma < 0) = \frac{\mu_3}{2g_s}\frac{\ell^2}{(c_0^2-1)}\left(\cos^2\theta d\theta^2 + \sin^2\theta d\phi^2 \right).$$

Notice that at $\gamma = 1$, the value of c_0 diverges, and so the metric vanishes. In the first situation there is a sensible metric for all classes of γ. It is parameterised by the (c, ϕ) space and the metric is:

$$ds^2_{\mathcal{M}1}(\gamma) = \frac{\mu_3}{2g_s}\frac{\ell^2 ck^2}{(c^2-1)}\left(\frac{dc^2}{(c^2-1)^2} + d\phi^2 \right). \tag{19.96}$$

As discussed earlier, the $\gamma < 0$ flows lead to $\rho = 0$ and some finite value of c, which we call c_0. The supergravity is singular there, but the probe metric is perfectly smooth there. This situation is similar to ones we have

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encountered before, and is suggestive of a the edge of a disklike D3-brane source.

In the case $\gamma \geq 0$, c diverges, and the probe metric vanishes again. This is a signal of an enhançon-like locus, which we encountered in chapter 15. It appears here to be a circle, as would appear in the case of wrapped D7-branes, but we must be careful before we interpret this in gauge theory. As in the previous two examples that we have studied in this chapter, we must be careful to ensure that we are using the right coordinates.

We have two scalars, c and ϕ, but we must recall that these should be the components of a complex scalar, the adjoint scalar in the low energy effective low energy $U(1)$ action on the brane. So they should have the same coefficient[332, 333]. So we must find a complex coordinate z in which the metric is conformal to $dzd\bar{z}$. This is achieved by finding a new radial coordinate v such that

$$\frac{dc^2}{(c^2-1)^2} = \frac{dv^2}{v^2},$$

which has solution

$$v = \sqrt{\frac{c+1}{c-1}},$$

and so our putative enhançon circle at $c \to \infty$ on the $\gamma \geq 0$ branches is at $z = 1$. We can write the metric as:

$$ds^2_{\mathcal{M}1}(\gamma) = \frac{\mu_3}{2g_{\rm s}} \frac{\ell^2 ck^2}{(c+1)^2} dzd\bar{z}. \tag{19.97}$$

In the low energy theory, the scalar field Y, being part of the $\mathcal{N}=2$ gauge multiplet on the brane's world-volume, should have the same functional dependence for the kinetic term that the $U(1)$ gauge field on the probe has[333]. This translates into a kinetic term for Y:

$$ds^2 = \frac{\mu_3}{2} e^{-\Phi} dY d\bar{Y}, \tag{19.98}$$

where the dilaton may be extracted from the equation (19.93) as:

$$e^{-\Phi} = \frac{c}{g_{\rm s}|\cos\phi + ic\sin\phi|}.$$

We must therefore change variables to the complex coordinates Y. Writing

$$dzd\bar{z} = dYd\bar{Y}\frac{\partial z}{\partial Y}\frac{\partial \bar{z}}{\partial \bar{Y}},$$

we get an equation

$$\left|\frac{\partial Y}{\partial z}\right|^2 = k^2\ell^2 \left|\frac{\cos\phi + ic\sin\phi}{c+1}\right|^2 = \frac{k^2\ell^2}{4}\left|1+\frac{1}{z^2}\right|,$$

and therefore

$$Y = \frac{k\ell}{2}\left(z + \frac{1}{z}\right). \tag{19.99}$$

We should write our complex coupling τ in terms of this coordinate. Some substitution gives the holomorphic result:

$$\tau = \frac{i}{g_{\rm s}}\left(\frac{Y^2}{Y^2 - k^2\ell^2}\right)^{1/2} + \frac{\theta_{\rm s}}{2\pi}. \tag{19.100}$$

We have a branch cut forming a segment[333] of the real line: $-k\ell \leq Y \leq k\ell$. We see from the change of variables in equation (19.99) that this branch cut is the circle $z = 1$, which is the enhançon, appearing in the $\gamma \leq 0$ flows. The $\gamma > 0$ flows are currently believed to be unphysical.

So we see that in fact the singular behaviour of the supergravity was hiding valuable physics which we uncovered by probing with a D3-brane. Just as in chapter 15, we find a region of the moduli space of large N gauge theory with eight supercharges where the constituent D-branes have spread out into a locus which we call the enhançon. Just as there, were this not to have happened (as the naive supergravity would allow), the constituent branes would have attained negative values for their tension. In the dual gauge theory, this negativity is a negative value for the kinetic term in the low energy action one moduli space, or alternatively, a negative value for the effective squared gauge coupling $g^2_{\rm eff}$. In any of those pictures, this would be unphysical, and the brane physics protects itself against this case by moving the constituent branes to quantum corrected positions. In the language of the gauge theory, this is of course a large N manifestation of the Seiberg–Witten locus[240], which owes its origin to the same positivity requirements[§].

19.5 Beyond gravity duals

The last example is a situation where a supergravity solution, in attempting to reveal the physics about highly non-trivial behaviour of the dual gauge theory, needs to be supplemented with information about the string theory. We found this by probing with D3-branes by hand.

This is the expected sign that our ability to extract useful information about dual gauge theories will rely on our success in understanding more about the full string theory in the background. The D3-brane probe method, while powerful in its own right, is only a hybrid method, and

§ See also section 16.1.12, for examples where quantum corrections to brane geometry in F-theory correlate with underlying Seiberg–Witten theory.

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much more progress will be made when some way of computing in the full string theory for these backgrounds is found. The difficulty here is that one of the most crucial features of the solution is that it is supported by N units of R–R flux. This cannot be described as a small perturbation of an NS–NS background, and so the string theory must be phrased directly in terms of the R–R data. It should have been apparent, however, from the many studies that we have carried out in this book that it is in fact difficult to describe fully the strings propagating in such backgrounds. Looking back, it should be clear that we have only ever described string propagation in these backgrounds in the supergravity limits. The full conformal field theories that we described or alluded to were only ever for propagation in non-trivial NS–NS fields (like K3 geometry, or the NS5-brane's core). For the R–R p-branes, or the non-trivial F-theory or other such fascinating backgrounds, we were never in a position to present a world-sheet model (like a σ-model of chapter 2) which corresponded to the full string theory in the background, even perturbatively. The problem is that in the formalism described in chapter 7, the vertex operators corresponding to R–R states introduce world-sheet branch cuts in the presence of the superconformal generators, making them non–local with respect to each other[1], and hence outside the realm of the local conformal field theories that we have been studying.

Tools for the description of string theory propagating in R–R backgrounds need to be developed further, with some urgency. Results in this area will be especially interesting in view of the variety of physical phenomena that we have learned about from D-branes throughout this book. We learned all of this by indirect arguments combined with powerful technology in various limits (such as open strings, conformal field theory, and supergravity). Imagine what we might learn, and what useful tools we could develop if we could formulate things more directly.

A tantalising glimpse in this direction has been obtained recently. In addition to Minkowski space and $\mathrm{AdS}_5\times\mathrm{S}^5$, it has been realised[344] that a certain type of pp-wave (see pp. 422–423) with R–R flux is also maximally supersymmetric. Furthermore, the pp-wave can be obtained[345,347] as a certain limit of $\mathrm{AdS}_5\times\mathrm{S}^5$ that focuses on trajectories with large angular momentum in the S^5. String propagation in this pp-wave is exactly solvable, despite the R–R flux, in the light-cone gauge [346].

Remarkable, a class of gauge theory operators from the original dual CFT with R-charge going as $\sqrt{N}$ as $N\to\infty$ can be directly identified with the full tower of string states[347]. Properties of the light-cone string can be reconstructed from the gauge theory, and vice versa. This is an exciting development that will undoubtedly be explored further.