BULL. AUSTRAL. MATH. SOC. 16A62
VOL. 12 (1975), 171-177.

A note on divisible and codivisible dimension

Paul E. Bland

In this paper the right global divisible dimension and the right global codivisible dimension of a ring R are studied relative to a torsion theory of mod$R$. The main result shows that if (A, B) is a central splitting torsion theory on mod$R$, then the right global divisible dimension of R with respect to (B, A) is equal to the right global codivisible dimension of R with respect to (A, B).

Throughout this paper R will denote an associative ring with identity and our attention will be confined to the category mod$R$ of unital right R-modules. The reader is referred to [8] and [10] for the general results and terminology on torsion theories.

If (A, B) is a torsion theory on mod$R$, then an R-module M is said to be divisible (codivisible), if given an exact sequence $0 \to L \to X \to N \to 0$, where N is torsion (L is torsion free), the induced map $\hom_R(X, M) \to \hom_R(L, M)$ $\left(\hom_R(M, X) \to \hom_R(M, N)\right)$ is an epimorphism. By taking X to be projective (injective), we see that M is divisible (codivisible) if and only if $\operatorname{ext}_R^1(N, M) = 0$ for every torsion module N $\left(\operatorname{ext}_R^1(M, L) = 0\right.$ for every torsion free module $\left.L\right)$. Divisible modules are due to Lambek [8] while codivisible modules were introduced in [3].

In [9], Rangaswamy defined divisible and codivisible dimension for modules and a global divisible and a global codivisible dimension for

Received 12 November 1974.

https://doi.org/10.1017/S0004972700023790 Published online by Cambridge University Press

rings. Briefly, if (A, B) is a torsion theory on mod$R$ and M is an R-module, then one can build an exact sequence

$$(*) \qquad 0 \to M \xrightarrow{\alpha_0} D_0 \xrightarrow{\alpha_1} D_1 \to \dots \xrightarrow{\alpha_n} D_n \to \dots ,$$

where each D_i is divisible and cokerα_i is torsion for $i \geq 0$. (Note that the D_i's are all torsion for $i \geq 1$.) Such a sequence is called a divisible resolution of M. The divisible dimension of M is then defined to be the smallest integer n such that there exists a divisible resolution of M of the form $(*)$ with Imα_n divisible. If no such integer exists, then we say that the divisible dimension of M is ∞. If div.d(M) denotes the divisible dimension of M, then standard arguments [6], *mutatis mutandis*, show that div.d(M) is independent of the divisible resolution of M. The right global divisible dimension of R, written (A, B)-r.gl.div.d(R), is now defined to be $\sup\{\text{div.d}(M) \mid M \in \text{mod}R\}$. Dually, one may define a codivisible resolution of a module M to be an exact sequence

$$\dots \to C_n \xrightarrow{\beta_n} \dots \to C_1 \xrightarrow{\beta_1} C_0 \xrightarrow{\beta_0} M \to 0 ,$$

where each C_i is codivisible and kerβ_i is torsion free for $i \geq 0$. The codivisible dimension of a module and the right global codivisible dimension of a ring are then defined in the obvious way. (A, B)-r.gl.cod.d(R) will denote the right global codivisible dimension of R. If $(0, M)$ $((M, 0))$ denotes the torsion theory on mod$R$ in which every module is torsion free (torsion), then

$$(0, M)\text{-r.gl.cod.d}(R) = \text{right global projective dimension of } R =$$
$$= \text{right global injective dimension of } R = (M, 0)\text{-r.gl.div.d}(R) .$$

It seems worth pointing out that this is true for every central splitting torsion theory on mod$R$. That is, if (A, B) is a central splitting torsion theory on mod$R$, then we will show that

$$(A, B)\text{-r.gl.cod.d}(R) = (B, A)\text{-r.gl.div.d}(R) .$$

It has been shown in [9] that (A, B)-r.gl.cod.d$(R) \neq (A, B)$-r.gl.div.d(R).

If A is a TTF class and (A, B) and (C, A) are the associated torsion theories with torsion functors T and S respectively, then Jans [7] has shown that the following are equivalent:

(1) $M = T(M) \oplus S(M)$ for all $M \in \text{mod}R$;

(2) $R = T(R) \oplus S(R)$ (ring direct sum);

(3) $B = C$;

(4) $T(S(M)) = 0$ and $S(M/T(M)) = M/T(M)$ for all $M \in \text{mod}R$.

Under the above conditions Bernhardt [2] has called (A, B) central splitting. Hereafter, unless stated otherwise, we will assume that (A, B) is a central splitting torsion theory on $\text{mod}R$. T and S will denote the torsion functors relative to (A, B) and (B, A) respectively. Since (A, B) is central splitting it is not difficult to show that $MT(R) = T(M)$ for every module M where $MT(R) = \left\{\sum m_i t_i \mid m_i \in M \text{ and } t_i \in T(R)\right\}$. Similarly, $MS(R) = S(M)$.

LEMMA 1. *For any torsion theory* (A, B) *on* $\text{mod}R$ *,* $M/T(M)$ *is an injective* $R/T(R)$*-module if and only if* $M/T(M)$ *is an injective* R*-module.*

Proof. Suppose that $M/T(M)$ is an injective $R/T(R)$-module and let I be a right ideal of R . Consider the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \xrightarrow{i} & T \\ & & {\scriptstyle f}\big\downarrow & & \\ & & M/T(M) & & \end{array} ,$$

where i is the canonical injection. This yields a diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & I/T(R) & \xrightarrow{i'} & R/T(R) \\ & & {\scriptstyle f'}\big\downarrow & & \\ & & M/T(M) & & \end{array} ,$$

where $i'(x+T(R)) = x + T(R)$ and $f'(x+T(R)) = f(x)$ are $R/T(R)$-linear. Note that f' is well defined, for if $x \in I \cap T(R) = T(I)$, then $f(x) \in f(T(I)) \subseteq T(M/T(M)) = 0$. Thus there is a mapping $g' : R/T(R) \to M/T(M)$ such that $f' = g' \circ i'$. But then if $\eta : R \to R/T(R)$ is the canonical projection and we set $g = g' \circ \eta$, then $g \circ i = f$. Hence $M/T(M)$ is R-injective by Baer's criterion [1, Theorem

1]. The converse is obvious.

The following proposition is the key for proving the main result of this paper.

PORPOSITION 2. *M is divisible with respect to (B, A) if and only if $S(M)$ is an injective $S(R)$-module.*

Proof. Since $M = T(M) \oplus S(M)$, then for any $M \in B$ we have that $\mathrm{ext}_R^1(B, M) \cong \mathrm{ext}_R^1(B, T(M)) \oplus \mathrm{ext}_R^1(B, S(M))$. But $\mathrm{ext}_R^1(B, T(M)) = 0$ since (A, B) is splitting [11, Lemma 1.2]. Hence $\mathrm{ext}_R^1(B, M) \cong \mathrm{ext}_R^1(B, S(M))$. Notice next that since $MT(R) = T(M)$, $M \in B$ if and only if $MT(R) = 0$ and so the image of the inclusion functor $F : \mathrm{mod}S(R) \hookrightarrow \mathrm{mod}R$ is exactly B. Also by using Lemma 1 we can show that

$$\mathrm{ext}_R^1(B, S(M)) \cong \mathrm{ext}_{S(R)}^1(B, S(M)) .$$

Hence $\mathrm{ext}_R^1(B, M) \cong \mathrm{ext}_{S(R)}^1(B, S(M))$ and so the proposition follows.

We can now prove our main result. In what is to follow r.gl.proj.d(R) and r.gl.inj.d(R) will stand for the right global projective dimension and the right global injective dimension of R respectively.

PROPOSITION 3. *The following are equal:*

(a) r.gl.proj.d $S(R)$;

(b) (A, B)-r.gl.cod.d(R) ;

(c) (B, A)-r.gl.div.d(R) .

Proof. That *(a)* equals *(b)* follows from [9, Theorem 14]. Since r.gl.proj.d$(S(R))$ = r.gl.inj.d$(S(R))$ we will show *(a)* equals *(c)* by showing that (B, A)-r.gl.div.d(R) = r.gl.inj.d$(S(R))$. Let

$$0 \to M \xrightarrow{\alpha_0} D_0 \xrightarrow{\alpha_1} D_1 \to \dots \xrightarrow{\alpha_n} D_n \to \dots$$

be a divisible resolution of M with respect to (B, A). (Note that $S(D_i) = D_i$ for all $i \geq 1$.) Since S is an exact functor [11, Theorem 3.1] we see via Proposition 2 that

$$0 \to S(M) \xrightarrow{S(\alpha_0)} S(D_0) \xrightarrow{S(\alpha_1)} S(D_1) \to \dots \xrightarrow{S(\alpha_n)} S(D_n) \to \dots$$

is an $S(R)$-injective resolution of the $S(R)$-module $S(M)$, where $S(\alpha_i) = \alpha_i|S(D_{i-1})$ for $i \geq 0$ with $D_{-1} = M$. Now $\operatorname{Im} S(\alpha_i) \cong S(\operatorname{Im}\alpha_i)$ for $i \geq 0$. Hence it follows that if the $S(R)$-injective dimension of $S(M)$ is n, then $S(\operatorname{Im}\alpha_n)$ is an injective $S(R)$-module and so, by Proposition 2, $\operatorname{Im}\alpha_n$ is divisible with respect to (B, A). Therefore (B, A)-r.gl.div.d$(R) \leq$ r.gl.inj.d$(S(R))$.

On the other hand, let M be an $S(R)$-module and suppose that

$$0 \to M \to E_0 \to E_1 \to \dots \to E_n \to \dots$$

is an $S(R)$-injective resolution of M. Since $S(E_i) = E_i$ for each $i \geq 0$, we see (again by Proposition 2) that E_i is divisible with respect to (B, A) for each $i \geq 0$. Thus it follows that

$$0 \to M \to E_0 \to E_1 \to \dots \to E_n \to \dots$$

is a divisible resolution of M with respect to (B, A). It now follows easily that r.gl.inj.d$(S(R)) \leq (B, A)$-r.gl.div.d(R).

If A is replaced by B, B by A, and $S(R)$ by $T(R)$ in the proposition above, then the resulting proposition is true. Thus we have

PROPOSITION 4. *The following are equal:*

(a) r.gl.proj.d(R) ;

(b) $\sup\{(A, B)$-r.gl.cod.d(R), (B, A)-r.gl.cod.d$(R)\}$;

(c) $\sup\{(A, B)$-r.gl.div.d(R), (B, A)-r.gl.div.d$(R)\}$.

Proof. Since $R = T(R) \oplus S(R)$ (ring direct sum), then r.gl.proj.d$(R) = \sup\{$r.gl.proj.d$(T(R))$, r.gl.proj.d$(S(R))\}$.

Rangaswamy has shown in [9] that for any torsion theory (A, B) on mod$R$ every submodule of a codivisible module is codivisible if and only if $R/T(R)$ is right hereditary. Under the assumption of central splitting the following proposition should now be evident.

PROPOSITION 5. *The following are equivalent:*

https://doi.org/10.1017/S0004972700023790 Published online by Cambridge University Press

(a) *with respect to* (A, B) *every submodule of a codivisible module is codivisible;*

(b) *with respect to* (B, A) *every factor module of a divisible module is divisible;*

(c) $S(R)$ *is right hereditary.*

Faith and Walker [5, Theorem 5.3] have shown that a ring R is QF (quasi-Frobenius) if and only if every injective R-module is projeccive while Faith [4, Theorem A] obtained the dual characterization that R is QF if and only if every projective R-module is injective. Since M is codivisible with respect to any torsion theory (A, B) on mod$R$ if and only if $M/MT(R)$ is a projective R-module [9, Theorem 8], these observations along with Proposition 2 yield the following

PROPOSITION 6. *The following are equivalent:*

(a) *every module which is divisible with respect to* (B, A) *is codivisible with respect to* (A, B) *;*

(b) *every module which is codivisible with respect to* (A, B) *is divisible with respect to* (B, A) *;*

(c) $S(R)$ *is* QF.

References

[1] Reinhold Baer, "Abelian groups that are direct summands of every containing abelian group", *Bull. Amer. Math. Soc.* **46** (1940), 800-806.

[2] Robert L. Bernhardt, "Splitting hereditary torsion theories over semi-perfect rings", *Proc. Amer. Math. Soc.* **22** (1969), 681-687.

[3] Paul E. Bland, "Perfect torsion theories", *Proc. Amer. Math. Soc.* **41** (1973), 349-355.

[4] Carl Faith, "Rings with ascending condition on annihilators", *Nagoya Math. J.* **27** (1966), 179-191.

[5] Carl Faith and Elbert A. Walker, "Direct-sum representations of injective modules", *J. Algebra* **5** (1967), 203-221.

[6] Irving Kaplansky, *Fields and rings*, 2nd ed. (Chicago Lectures in Mathematics Series. University of Chicago Press, Chicago and London, 1972).

[7] J.P. Jans, "Some aspects of torsion", *Pacific J. Math.* **15** (1965), 1249-1259.

[8] Joachim Lambek, *Torsion theories, additive semantics, and rings of quotients* (Lecture Notes in Mathematics, **177**. Springer-Verlag, Berlin, Heidelberg, New York, 1971).

[9] K.M. Rangaswamy, "Codivisible modules", *Comm. Algebra* (to appear).

[10] Bo Stenström, *Rings and modules of quotients* (Lecture Notes in Mathematics, **237**. Springer-Verlag, Berlin, Heidelberg, New York, 1971).

[11] Mark L. Teply, "Homological dimension and splitting torsion theories", *Pacific J. Math.* **34** (1970), 193-205.

Department of Mathematics,
Eastern Kentucky University,
Richmond,
Kentucky,
USA.

https://doi.org/10.1017/S0004972700023790 Published online by Cambridge University Press