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CALCULUS OF VARIATIONS IN MEAN AND CONVEX LAGRANGIANS, II

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We prove the Legendre Necessary Condition of the Calculus of Variations in Mean in an arbitrary finite dimension. When the Lagrangian is convex, we establish that if the Euler–Lagrange equation possesses an almost periodic solution then it possesses periodic and constant solutions. We deduce from this fact various consequences on the structure of the set of almost periodic solutions.

1. INTRODUCTION

The notation is as in [**3**]. $AP^k(\mathbb{R}^n)$ denotes the space of u.a.p. (uniformly almost periodic, or Bohr almost periodic) functions from $\mathbb{R}$ into $\mathbb{R}^n$ of which all the derivatives, up to order k, are u.a.p. If f is u.a.p., $\mathrm{Mod}(f)$ denotes the $\mathbb{Z}$-module of frequencies of f. $Q^{pk}(\omega;\mathbb{R}^n)$ is the subspace of $A^{pk}(\mathbb{R}^n)$ of all the functions f such that $\mathrm{Mod}(f)$ is generated by $\omega = (\omega_1, \ldots, \omega_s)$. The mean value of an u.a.p. function f is denoted by $\mathcal{M}\{f\}$, and its Fourier–Bohr coefficients by $a(f;\lambda) := \mathcal{M}\{f(t)e^{-i\lambda t}\}_t$, with $\lambda \in \mathbb{R}$. The translation operation is denoted, for $r \in \mathbb{R}$, by $\tau_r f(t) := f(t+r)$.

If E is either $AP^1(\mathbb{R}^n)$ or $QP^1(\omega;\mathbb{R}^n)$, on a lagrangian $L \in C^r(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$, $r \geqslant 1$, we construct a nonlinear functional $\phi\colon E \to \mathbb{R}$, $\phi(x) := \mathcal{M}\{L(x, \dot{x})\}$ where $\dot{x} = dx/dt$, and we formulate a variational problem in mean

(P) minimise ϕ on E.

In [**2, 3**] we have established that ϕ is Fréchet - C^1 and that $\phi'(x) = 0$ if and only if x is an u.a.p. solution of the Euler–Lagrange equation:

(E) $L_x(x, \dot{x}) = (d/dt)\, L_{\dot{x}}(x, \dot{x})$.

2. SECOND VARIATION

In [**3**, part C] we have established the Legendre Necessary Condition of the second order in the case where the dimension n is equal to 1; we prove it here for an arbitrary positive integer n.

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THEOREM 1. *Let E be either $AP^1(\mathbf{R}^n)$ or $QP^1(\omega; \mathbf{R}^n)$, let $L \in C^2(\mathbf{R}^n \times \mathbf{R}^n, \mathbf{R})$ and $\phi(x) := \mathcal{M}\{L(x, \dot{x})\}$ for $x \in E$. If x realises a local minimum of ϕ on E, then for all $t \in \mathbf{R}$ $L_{\dot{x}\dot{x}}(x(t), \dot{x}(t)) \geqslant 0$ (a nonnegative definite bilinear form).*

PROOF: We prove the case with $E = AP^1(\mathbf{R}^n)$, the case of $QP^1(\omega, \mathbf{R}^n)$ being similar. From results of [**2, 4**] we know that $\phi \in C^2(E, \mathbf{R})$ and if $x, h, k \in E$ we have:

$$\phi''(x)(h,k) = \mathcal{M}\{L_{xx}(x,\dot{x})(h,k) + L_{x\dot{x}}(x,\dot{x})\left(h,\dot{k}\right) + L_{\dot{x}x}(x,\dot{x})\left(\dot{h},k\right) + L_{\dot{x}\dot{x}}(x,\dot{x})\left(\dot{h},\dot{k}\right)\}$$

If x realises a local minimum of ϕ on E, then the usual Necessary Condition of the second order is $\phi''(x)(h, h) \geqslant 0$ for all $h \in E$.

By the same calculation as in the one-dimensional case, see the proof of Theorem 2 of [**3**], for $f \in E$, $\lambda \in \mathbf{R}$ taking in turn $h(t) = \cos\lambda t.f(t)$, $h(t) = \sin\lambda t.f(t)$ we obtain:

$$0 \leqslant \mathcal{M}\{L_{xx}.(f, f) + 2L_{x\dot{x}}\left(f, \dot{f}\right) + L_{\dot{x}\dot{x}}.\left(\dot{f}, \dot{f}\right) + \lambda^2 L_{\dot{x}\dot{x}}.(f, f)\}.$$

This inequality must be valid for all $\lambda \in \mathbf{R}$. When λ is sufficiently large, we see that

$$0 \leqslant \mathcal{M}\{L_{\dot{x}\dot{x}}(x, \dot{x})(f, f)\}, \text{ for every } f \in E.$$

Because E is dense in $AP^0(\mathbf{R}^n)$, this last inequality is valid for every $f \in AP^0(\mathbf{R}^n)$. We fix now $f \in E$, and write $u = L_{\dot{x}\dot{x}}(x, \dot{x})(f, f) \in AP^0(\mathbf{R})$. It is clear that $u^+, u^- \in AP^0(\mathbf{R})$, and so $\sqrt{u^-} \in AP^0(\mathbf{R})$. For every $\alpha \in AP^0(\mathbf{R})$, $\alpha f \in AP^0(\mathbf{R}^n)$ and we have:

$$0 \leqslant \mathcal{M}\{L_{\dot{x}\dot{x}}(x, \dot{x})(\alpha f, \alpha f)\} = \mathcal{M}\{u\alpha^2\},$$

therefore $0 \leqslant \mathcal{M}\{u^-\alpha^2\} \leqslant \mathcal{M}\{u^+\alpha^2\}$. Also if $\alpha = \sqrt{u^-}$ then we have $0 \leqslant \mathcal{M}\{(u^-)^2\} \leqslant \mathcal{M}\{u^+u^-\} = \mathcal{M}\{0\} = 0$. Therefore, (see [**1**, p.19]), $u^- = 0$ and so $u = u^+$, that is $u \geqslant 0$. Hence

$$L_{\dot{x}\dot{x}}(x(t), \dot{x}(t))(f(t), f(t)) \geqslant 0 \text{ for all } t \in \mathbf{R}.$$

It is sufficient to take $f = z \in \mathbf{R}^n$, and so we have $L_{\dot{x}\dot{x}}(x(t), \dot{x}(t))(z, z) \geqslant 0$ for all $t \in \mathbf{R}$, and all $z \in \mathbf{R}^n$. ∎

3. CONVEX LAGRANGIANS

In this section we consider a lagrangian L which is convex on $\mathbf{R}^n \times \mathbf{R}^n$, and so ϕ is convex. Therefore when $x \in AP^1(\mathbf{R}^n)$, x solves (P) if and only if x is an u.a.p. solutions of (E) (see [**3**, part D]). The existence of an u.a.p. solution of (E) implies the existence of many others.

PROPOSITION 1. *Let $L \in C^2(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$ be a convex function. Let $x \in AP^1(\mathbb{R}^n)$ be an u.a.p. solution of (E). Then for every $(z, w) \in \overline{co}(x, \dot{x})(\mathbb{R})$ there exists $y \in AP^1(\mathbb{R}^n)$ such that y solves (E), $y(0) = z$, $\dot{y}(0) = w$, $\mathcal{M}\{y\} = \mathcal{M}\{x\}$ and* $\mathrm{Mod}(y) \subset \mathrm{Mod}(x)$.

PROOF: By a theorem of Bochner, $\{\tau_r(x, \dot{x}) : \tau \in \mathbb{R}\}$ is relatively compact in $AP^0(\mathbb{R}^n \times \mathbb{R}^n)$. Then by a theorem of Mazur (see, [**5**, p.416]) $\overline{co}\{\tau_r(x, \dot{x}) | r \in \mathbb{R}\}$ is compact in $AP^0(\mathbb{R}^n \times \mathbb{R}^n)$. Let $(z, w) \in \overline{co}(x, \dot{x})(\mathbb{R})$ in $\mathbb{R}^n \times \mathbb{R}^n$, then there exists a sequence:

$$\left(\sum_{\alpha=1}^{m_v} \lambda_{v,\alpha}(x(t_{v,\alpha}), \dot{x}(t_{v,\alpha}))\right)_{v \in \mathbb{N}}, \text{ where } \lambda_{v,\alpha} \geqslant 0$$

and

$$\sum_{\alpha=1}^{m_v} \lambda_{v,\alpha} = 1, \text{ that converges to } (z, w) \text{ in } \mathbb{R}^n \times \mathbb{R}^n.$$

But the sequence $\left(\sum_{\alpha=1}^{m_v} \lambda_{v,\alpha}\, \tau_{t_{v,\alpha}}(x, \dot{x})\right)_{v \in \mathbb{N}}$ takes its values in $\overline{co}\{\tau_r(x, \dot{x}) : r \in \mathbb{R}\}$. Therefore this sequence admits a subsequence that converges uniformly towards an element $(y, y_1) \in AP^0(\mathbb{R}^n \times \mathbb{R}^n)$. It is easy to verify that $y_1 = \dot{y}$ and that y has the required properties. $\square$

In [**3**, Part D], we proved that if (E) admits periodic solutions with incommensurable periods, then necessarily (E) admit a nonperiodic u.a.p. solution. The main result of the present paper is a kind of converse: if (E) admits an u.a.p. solution then (E) admits periodic solutions. Explicitly we have

THEOREM 2. *Let $L \in C^2(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$ be a convex lagrangian. We assume that (E) possesses an u.a.p. solution $x \in AP^1(\mathbb{R}^n)$. Then*

(i) *(E) posseses a constant solution equal to $\mathcal{M}\{x\}(\mathcal{M}\{x\} \in \overline{co}x(\mathbb{R}))$;*
(ii) *let $T \in \mathbb{R}$, $T \neq 0$; if there exists an integer $k \neq 0$ such that $a(x; (2\pi/T)k) \neq 0$, then*

(E) posseses a non constant T-periodic solution x^T. Moreover $\mathcal{M}\{x^T\} = \mathcal{M}\{x\}$, $(x^T, \dot{x}^T)(\mathbb{R}) \subset \overline{co}(x, \dot{x})(\mathbb{R})$ in $\mathbb{R}^n \times \mathbb{R}^n$, $(x^T, \dot{x}^T) \in \overline{co}\{\tau_r(x, \dot{x}) | r \in \mathbb{R}\}$ in $AP^1(\mathbb{R}^n \times \mathbb{R}^n)$, and for all $m \in \mathbb{Z}$ $a(x^T; (2\pi/T)m) = a(x, (2\pi/T)m)$.

PROOF: If $v \in \mathbb{N}$, $v \geqslant 1$, $t \in \mathbb{R}$, $T \neq 0$, we define the function $C_{v,T}(x) := 1/v \sum_{k=0}^{v-1} \tau_{kT} x \in AP^1(\mathbb{R}^n)$. We remark that $(d/dt)C_{v,T}(x) = C_{v,T}(\dot{x})$, and from [**1**, p.44], we know that there exists a T-periodic function x^T, of class C^1, such that $(C_{v,T}(x))_{v \in \mathbb{N}}$ converges to x^T in AP^1-norm. Because x is an u.a.p. solution of (E), x solves (P), and because L is autonomous and $\mathcal{M}$ is invariant under translations, we have $\phi(\tau_{kT} x) = \phi(x) = \operatorname{Inf} \phi$. The convexity of ϕ implies:

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$\phi(C_{v,T}(x)) \leqslant \sum_{k=0}^{v-1} 1/v \ \phi(\tau_{kT}x) = \operatorname{Inf}\phi$, therefore $\phi(C_{v,T}(x)) = \operatorname{Inf}\phi$, and by the continuity of ϕ, we obtain $\phi(x^T) = \lim_v \phi(C_{v,T}(x)) = \operatorname{Inf}\phi$. Thus x^T is a solution of (E). Since $(C_{v,T}(x), C_{v,T}(\dot{x}))$ is an element of $co\{\tau_r(x, \dot{x}) | r \in \mathbb{R}$ we see that $(x^T, \dot{x}^T) \in \overline{co}\{\tau_r(x, \dot{x}) | r \in \mathbb{R}\}$ in $AP^0(\mathbb{R}^n \times \mathbb{R}^n)$, therefore $(x^T \dot{x}^T)(\mathbb{R}) \subset \overline{co}(x, \dot{x})(\mathbb{R})$. By [**1**, p.45], we know that the Fourier series of x^T is exactly the T-periodic part of the Fourier–Bohr series of x. Writing $\Lambda_T = \{\lambda \in \mathbb{R} \mid \lambda \in 2\pi/T\mathbb{Z}\}$ we have $x^T(t) \sim \sum_{\lambda \in \Lambda_T} a(x, \lambda)e^{i\lambda t}$. From this relation we deduce that $\mathcal{M}\{x^T\} = \mathcal{M}(x)$ and that for all $m \in \mathbb{Z}$ $a(x^T; (2\pi/T)\, m) = a(x; (2\pi/T)\, m)$.
We have shown that, for each $T \neq 0$, x^T is a T-periodic solution of (E), but we must refine the argument to assertain whether or not x^T is constant.

It is well known that $\{\lambda \in R \mid a(x; \lambda) \neq 0\}$ is a countable set. Note that $\Lambda_{T_1} \cap \Lambda_{T_2} \neq \{0\}$ implies $T_1/T_2 \in \mathbb{Q}$, and that $\bigcup_{T \neq 0} \Lambda_T = \mathbb{R}$. Consequently there exists a real number $T_0 \neq 0$ such that $\Lambda_{T_0} = \{0\}$, so $x^T = a(x, 0) = \mathcal{M}(x)$. This fact justifies assertion (i).

It there exists an integer $k \neq 0$ such that $a(x; (2\pi/T)k) \neq 0$, then the Fourier series of x^T contains at least the term $a(x; (2\pi/T)k)e^{i(2\pi/T)kt}$, therefore x^T is non constant. That justifies (ii). ∎

Remark. Assertion (i) has already been established in [**3**, Theorem 4], but with a radically difference method based on a Fixed Point theorem.

From Theorem 2 we can deduce two general principles for the study of the Euler–Lagrange equations with convex lagrangian:

COROLLARY. *Let $L \in C^2(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$ be a convex lagrangian.*

(1) *If (E) does not possess any constant solution then (E) possesses neither u.a.p. solutions nor periodic solutions.*
(2) *If (E) does not posses any periodic solution, or if all the periods of its periodic solutions are rational multiples of one fixed real number, then (E) possesses no nonperiodic u.a.p. solution.*

We can give some simple and useful indications on the problem of u.a.p. solutions of (E). We shall assume that the lagrangians satisfy a condition of differentiability and convexity:

(D.C.) $\qquad L \in C^2(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R})$ and L is a convex function .

If $x_0 \in \mathbb{R}^n$, we remark that x_0 is a constant solution of (E) if and only if $L_x(x_0, 0) = 0$. And so by the Corollary, when $0 \notin L_x(\mathbb{R}^n \times 0)$ we are sure that (E) does not admit any u.a.p. solution. Therefore the interesting case is that when

$0 \in L_x(\mathsf{R}^n \times 0)$. Consequently we consider the condition:

(Z) there exists $x_0 \in \mathsf{R}^n$ such that $L_x(x_0, 0) = 0$.

PROPOSITION 2. *Under the conditions (D.C.) and (Z) we have:*

(i) *ϕ is bounded from below on $AP^1(\mathsf{R}^n)$, even if L is not bounded from below on $\mathsf{R}^n \times \mathsf{R}^n$, and* $\operatorname{Min}\phi(AP^1(\mathsf{R}^n)) = \operatorname{Min} L(\mathsf{R}^n \times 0) = L(x_0, 0)$.

(ii) *If in addition we assume that $L_{xx}(x_0, 0) > 0$ then all the u.a.p. solutions of (E) admit x_0 as mean value.*

PROOF: Because x_0 solves (E), x_0 solves (P), therefore $\operatorname{Min}\phi = \phi(x_0) = L(x_0, 0)$. The convexity of $x \to L(x, 0)$ on R^n and (Z) ensure that $L(x_0, 0) = \operatorname{Min} L(\mathsf{R}^n \times 0)$. For (ii), the assumption on $L_{xx}(x_0, 0)$ implies that x_0 is the sole vector of R^n that satisfies $L_x(x_0, 0) = 0$; that is, x_0 is the sole constant solution of (E). We conclude by using the assertion (i) of Theorem 2. ∎

We divide the condition (Z) into two subconditions:

(Z1) There exists $x_1 \in \mathsf{R}^n$, such that $L_x(x_1, 0) = 0$ and $L_{\dot{x}}(x_1, 0) \neq 0$.

(Z2) There exists $x_2 \in \mathsf{R}^n$ such that $L_x(x_2, 0) = 0$ and $L_{\dot{x}}(x_2, 0) = 0$.

LEMMA. *Under condition (D.C.) we have:*

(1) *(Z1) implies that, for $x \in \mathsf{R}^n$, if $L_x(x, 0) = 0$ then $L_{\dot{x}}(x, 0) \neq 0$;*

(2) *(Z2) implies that, for $x \in \mathsf{R}^n$, if $L_x(x, 0) = 0$ then $L_{\dot{x}}(x, 0) = 0$.*

PROOF: For (2), if $L_x(x, 0) = 0$ then $L(x, 0) = \operatorname{Min} L(\mathsf{R}^n \times 0)$. But $L(x_2, 0) = \operatorname{Min} L(\mathsf{R}^n \times 0) = \operatorname{Min} L(\mathsf{R}^n \times \mathsf{R}^n)$, therefore $L(x, 0) = \operatorname{Min} L(\mathsf{R}^n \times \mathsf{R}^n)$, and so $L'(x, 0) = 0$, and then $L_x(x, 0) = 0$. Then (1) is a consequence of (2). ∎

This lemma shows that the conditions (Z1) and (Z2) induce a partition of the set of lagrangians that satisfy (D.C.) and (Z). In particular, we cannot have (Z1) and (Z2) simultaneously. We can characterise the two conditions

PROPOSITION 3. *Under the assumptions (D.C.) and (Z) we have:*

(i) *(Z1) is equivalent to:* $\operatorname{Min}\phi(AP^1(\mathsf{R}^n)) > \operatorname{Inf} L(\mathsf{R}^n \times \mathsf{R}^n)$.

(ii) *(Z2) is equivalent to:* $\operatorname{Min}\phi(AP^1(\mathsf{R}^n)) = \operatorname{Inf} L(\mathsf{R}^n \times \mathsf{R}^n)$.

PROOF: We remark that (i) is a consequence of (ii). If (Z2) is valid then $\operatorname{Min} L(\mathsf{R}^n \times 0) = \operatorname{Inf} L(\mathsf{R}^n \times \mathsf{R}^n)$ because of the convexity of L, and by assertion (i) of Proposition 2, we deduce that $\operatorname{Min}\phi = \operatorname{Inf} L(\mathsf{R}^n \times \mathsf{R}^n)$. Conversely, if $\operatorname{Min}\phi = \operatorname{Inf}(\mathsf{R}^n \times \mathsf{R}^n)$, by assumption (Z) and assertion (i) of Proposition 2, $L(x_0, 0) = \operatorname{Min}\phi$, therefore $L(x_0, 0) = \operatorname{Inf} L(\mathsf{R}^n \times \mathsf{R}^n)$, and consequently $L'(x_0, 0) = 0$, and (Z2) is satisfied. ∎

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PROPOSITION 4. *Under the assumptions (D.C.) and (Z2) we have:*

(i) *let* $x \in AP^1(\mathbf{R}^n)$; *then* x *is an u.a.p. solution of (E) if and only if* $L(x(t), \dot{x}(t)) = \operatorname{Min} L(\mathbf{R}^n \times \mathbf{R}^n)$ *for all* $t \in \mathbf{R}$;

(ii) *if* $\operatorname{Argmin} L := L^{-1}(\operatorname{Inf} L(\mathbf{R}^n \times \mathbf{R}^n))$ *is included in* $\mathbf{R}^n \times 0$, *then all the u.a.p. solutions of (E) are constant; precisely they are the* $x \in \mathbf{R}^n$ *such that* $L_x(x, 0) = 0$;

(iii) *if* $L''(x_2, 0) > 0$ *then the constant* x_2 *is the single u.a.p. solution of (E);*

(iv) *if there exists a non-trivial linear subspace* S *of* $\mathbf{R}^n$ *such that* $[Int_{(S+0)\times(S+0)}(\operatorname{Argmin} L \cap (S+0) \times (S+0))] \cap (\mathbf{R}^n \times 0) \neq \emptyset$ *then, for each* $\mathbf{Z}$*-module of real numbers* M *there exists an u.a.p. solution of (E) which the module of frequences is* M. *In particular, for each real* $T \neq 0$ *there exists a non constant* T*-periodic solution of (E).*

PROOF: If $x \in AP^{-1}(\mathbf{R}^n)$ is a solution of (E) then $\mathcal{M}\{L(x, \dot{x})\} = \operatorname{Min} \phi = L(x_2, 0)$, therefore $\mathcal{M}\{L(x, \dot{x}) - L(x_2, 0)\} = 0$.
We know that $L(x_2, 0) = \operatorname{Min} L(\mathbf{R}^n \times \mathbf{R}^n)$, and so $L(x, \dot{x}) - L(x_2, 0) \geqslant 0$. By a classical theorem, (see [1, p.20]), we have necessarily $L(x, \dot{x}) - L(x_2, 0) = 0$, and consequently $L(x, \dot{x}) = \operatorname{Min} L(\mathbf{R}^n \times \mathbf{R}^n)$. Conversely if $x \in AP^1(\mathbf{R}^n)$ satisfies $L(x, \dot{x}) = \operatorname{Min} L(\mathbf{R}^n \times \mathbf{R}^n)$ then $L(x, \dot{x}) = L(x_2, 0)$ and $\phi(x) = L(x_2, 0) = \operatorname{Min} \phi$, therefore x solves (P) and (E). Assertion (i) is proved.

We remark that we can formulate the condition of assertion (i) in the following way: $(x(t), \dot{x}(t)) \in \operatorname{Argmin} L$ for all $t \in \mathbf{R}$. When $\operatorname{Argmin} L \subset \mathbf{R}^n \times 0$ we have necessarily $\dot{x} = 0$, therefore x is constant. This justifies (ii).
For (iii), because of the convexity of $L, (x_2, 0)$ is the single vector of $\mathbf{R}^n \times \mathbf{R}^n$ that realises the minimum of L. If x is a T-periodic function, of class C^1, with values of $\mathbf{R}^n$, then by the Formula of the Mean Value there exists $\hat{t} \in [0, T]$ such that $1/T \int_0^T L(x(t), \dot{x}(t))dt = L(x(\hat{t}), \dot{x}(\hat{t})) > L(x_2, 0)$, therefore $\phi(x) > \phi(x_2)$. And so (E) cannot possess a nonconstant periodic solution. By assertion (2) of the Corollary (E) cannot possess any nonconstant u.a.p. solution. Since $L''(x_2, 0) > 0$ implies $L_{xx}(x_2, 0) > 0$, we can conclude by using assertion (ii) of Proposition 2.
For (iv), the additional assumption implies the existence of some $c \in S$ and of an $\varepsilon > 0$ such that for all $z, w \in X$ $|c - z| < \varepsilon$ and $|w| < \varepsilon$ imply that $(z, w) \in \operatorname{Argmin} L$.

By the non triviality of S there exists some $x \in AP^1(S)$ such that $\operatorname{Mod}(x) = M$; $(x, \dot{x})(\mathbf{R})$ is bounded in $S \times S$ and so there exists $\rho > 0$ such that $|\rho x(t)| < \varepsilon$, $|\rho \dot{x}(t)| < \varepsilon$ for all $t \in \mathbf{R}$. We have $c + \rho \in AP^1(S)$, $\operatorname{Mod}(c + \rho x) = M$ and $|c - (c + \rho x)(t)| < c$, $|d/dt\,(c + \rho x)(t)| = |\rho \dot{x}(t)| < \varepsilon$, therefore $(c + \rho x(t), d/dt\,(c + \rho x)(t)) \in \operatorname{Argmin} L$, and then $c + \rho x$ solves (E) because of assertion (i). ∎

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Remarks. $A = \operatorname{Argmin} L$ is a closed convex subset of $\mathbb{R}^n \times \mathbb{R}^n$ and the condition of assertion (i) is $(x(t), \dot{x}(t)) \in A$ for all $t \in \mathbb{R}$. In introducing $A(x) = \{y \in \mathbb{R}^n \mid (x, y) \in A\}$, we can formulate this condition in terms of differential inclusion $\dot{x}(t) \in A(x(t))$ for all $t \in \mathbb{R}$, where the multifunction A has a closed convex graph.

A particular case of the additional assumption of (iv) is when $(\operatorname{Int} \operatorname{Argmin} L) \cap (\mathbb{R}^n \times 0) \neq \emptyset$.

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