This book is a very useful reference to the basic properties of and the main results in modern cyclic theory and it goes far beyond Loday's monograph (J.-L. Loday, *Cyclic homology*, Grundlehren der Mathematischen Wissenschaften, Volume 301 (Springer, 1992)). Although proofs are either not provided at all or are only sketched, there are ample references to the literature where details can be found.

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Hélein, F. *Harmonic maps, conservation laws and moving frames* (Cambridge University Press, 2002), 290 pp, 0 521 81160 0 (hardback), £55.

A harmonic map $u : M \to N$ between two Riemannian manifolds (M, g) and (N, h) is a solution (in local coordinates) of the system of elliptic equations

$$\Delta_g u^i + g^{\alpha\beta}(x)\Gamma^i_{jk}(u(x))\frac{\partial u^j}{\partial x^\alpha}\frac{\partial u^k}{\partial x^\beta} = 0,$$

where Δ_g is the Laplacian corresponding to the metric g on the source manifold (M, g) and the Γ^i_{jk} are the Christoffel symbols of the target manifold (N, h). These are the Euler–Lagrange equations corresponding to the energy or Dirichlet integral

$$E[u] = \frac{1}{2}\int_M |\mathrm{d}u(x)|^2 \,\mathrm{d}v_g.$$

Harmonic maps arise in a variety of different situations in geometry and physics. A submanifold M of an affine Euclidean space has constant mean curvature if and only if its Gauss map is a harmonic map. A submanifold M of a manifold N is minimal if and only if the immersion of M into N is harmonic. Harmonic maps with values in a sphere have been used in condensed matter physics to model nematic liquid crystals. Harmonic maps between surfaces and Lie groups are studied in theoretical physics because of their close relation to anti self-dual Yang–Mills connections. For a review of the theory of harmonic maps up to 1988 the reader is referred to [**3**, **4**].

The book under review provides a self-contained, readable and accessible introduction to the analytical aspects of the theory of harmonic maps as well as an exposition of some recent results due to the author. Only a few basic facts from differential geometry and the calculus of variations are assumed and these can be found in, for example, [**6**].

Chapter 1 contains introductory material on the geometric and analytic setting of harmonic maps. First, the notion of the Laplacian associated with a Riemannian metric is defined, and then a discussion of smooth harmonic maps between two Riemannian manifolds follows. The variational framework is then developed, in which harmonic maps are thought of as critical points of the Dirichlet integral. Noether's theorem is proved followed by a section on Sobolev spaces and various notions of weakly harmonic maps.

Chapter 2 is devoted to harmonic maps with symmetries and in particular to harmonic maps between a surface and a sphere, a special case that arises frequently both in differential geometry and in physics. Here the conformal transformations of the domain manifold combined with the symmetries of the target create a rich setting for the study of harmonic maps. The chapter ends with a discussion of weak compactness and regularity results in two dimensions.

Chapter 3 discusses compensation phenomena in which certain quadratic expressions of first derivatives of functions have more regularity than expected. This is related to the compensated compactness method of Murat and Tartar [**11**, **14**] as well as later work by Müller [**9**] and Coifman, Lions, Meyer and Semmes [**2**] on compensation phenomena using Hardy spaces. A related phenomenon in the setting of hyperbolic partial differential equations has been discovered by Klainerman and Machedon [**7**]. Regularity results for wave maps into spheres have been proved by Tao [**12**, **13**] and for wave maps into symmetric spaces by Klainerman and Rodnianski [**8**]. The chapter starts with a discussion of Wente's inequality following the presentation in [**1**], continues with two sections on Hardy and Lorentz spaces, and closes with an application of these ideas to the regularity of weakly stationary maps due to Evans [**5**].

In Chapter 4 the symmetry assumptions on the target manifold are dropped. The new tool introduced here is that of an orthonormal moving frame, a notion that goes back to Darboux and Cartan. One passes from one frame to another using the action of a gauge group. Coulomb frames turn out to be particularly useful in the regularity theory presented in the first three sections and the chapter ends with a discussion of the compactness of weakly harmonic maps in the weak topology.

In Chapter 5 the author studies the space of conformal parametrizations of surfaces with second fundamental form in L^2 and proves a compactness result for this space. The results of this chapter are related to earlier work by Toro [**15**, **16**] and Müller and Šverák [**10**].

The book ends with an extensive bibliography.

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N. BOURNAVEAS

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Schilling, R. *Measures, integrals and martingales* (Cambridge University Press, 2005), xi + 352 pp., 0 521 61525 9 (paperback), £24.99, 0 521 85015 0 (hardback), £50.

Whereas measure theory and integration are of central importance in modern analysis, no single treatment of this topic has become standard in the undergraduate curriculum. The outstanding issues are whether to emphasize Lebesgue or Riemann integration, whether or not to teach integration alongside probability, and whether to define integrals from measures or vice versa. In each case, Schilling advocates the first option.

In this attractive textbook, Schilling presents the theory in a style consistent with the historical development due to Lebesgue, Kolmogorov and others. The mathematical level of this book is appropriate for undergraduates at levels 4 or 3, and these students could attempt the accompanying exercises, for which solutions appear on the author's web page. Generally, the author strikes the right balance when stating results, making them short enough to be memorable, while sufficiently detailed as to be precise.

The author wisely starts with elementary material on counting and sets, before considering measures as countably additive set functions on σ-algebras. Carathéodory's extension theorem leads directly to the existence of Lebesgue measure on $\mathbb{R}^n$ and thence to the integrals of simple functions; the full definition of the Lebesgue integral and the convergence theorems follow.

In a style consistent with multivariable calculus, Jacobi's transformation formula has a rigorous treatment which follows naturally from the discussion of product measures and induced measures. The chapters on Hilbert space and L^p have an elegant conciseness, benefiting most those students who have some previous familiarity with these topics.

After a detailed discussion of uniform integrability, martingales emerge midway through the book, and the martingale convergence theorem is used to prove Kolmogorov's strong law of large numbers and the Borel–Cantelli lemmas. Martingales also find employment in harmonic analysis, as in the Calderón–Zygmund decomposition theorem. The presentation of the Hardy–Littlewood maximal theorem on $\mathbb{R}^n$ features a universal constant and hence seems elaborate; some lecturers might prefer to present first the one-dimensional case, where Riesz's sunrise lemma [**4**] gives a simpler proof. There is one exercise on rearrangements, a topic which is central to Hardy and Littlewood's presentation of maximal functions, as in their example in [**2**]: 'a batsman's total satisfaction for the season is a maximum, for a given stock of innings, when the innings are played in decreasing order'.

Bourbaki's [**1**] approach to integration starts with the space $C(\Omega)$ of continuous bounded functions on a locally compact Hausdorff space and emphasizes Riesz's representation theorem for bounded linear functionals on $C(\Omega)$, a result which is virtually suppressed in the book under review. For probabilists, a significant drawback of Bourbaki's approach to integration is the unsatisfactory treatment of conditional probability when Ω is not necessarily a Polish space. Overcoming such problems, Schilling introduces conditional expectations via orthogonal projections onto subspaces of L^2 associated with σ-algebras, a natural approach which admits further development into Markov processes and non-commutative integration. The coverage of independence here is conceptually similar, although less systematic, than that by Itô [**3**]; in particular, there is no zero–one law.

To illustrate the theory, the author provides examples including Friedrichs mollifiers, Bernoulli random variables and the Haar wavelet. In the discussion of classical function systems such as Laguerre polynomials, the author could have mentioned that their orthogonality follows easily from differential equations. Brownian motion provides the most advanced application of the theory, and here it is introduced by Haar orthogonal series with Gaussian coefficients; one can pursue this route and introduce the stochastic integral in L^2 relatively easily.

The appendices feature a concise summary of point-set topology and a construction of sets that are not Lebesgue measurable. For the benefit of those still interested, there is a summary of Riemann integration. There are good indices of notation and subjects, and ample references.