# Functions with Dominant Polar Singularities on the Circle of Convergence

By R. Wilson

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1. *Introduction.* In an earlier paper[1] the function

(1) $$f(z) = c_0 + c_1 z + c_2 z^2 + \dots .$$

was considered, having on its circle of convergence, taken to be $|z| = 1$, only isolated essential points of finite exponential order, situated at the points $e^{i\alpha_\nu}$, $(\nu = 1, 2, \dots, k)$. It was there proved,[2] in the cases $k = 2$, $k = 3$, that the upper density of small coefficients is positive only if the points $e^{i\alpha_\nu}$ are situated at some of the vertices of a regular polygon inscribed in the circle of convergence and if the singularities are virtually identical or linearly related. In this case the sequence of small coefficients possesses a *density.* The coefficients $c_n$ can be interpolated in the form

(2) $$c_n = \sum_{\nu=1}^{k} G_\nu(n) e^{in\alpha_\nu}$$

where the $G_\nu(z)$ are integral functions of order less than 1, so that the result stated may be regarded as expressing an arithmetical property of a certain type of integral function at positive integer points.

If we take the very special case in which the $G_\nu(z)$ are replaced by polynomials $P_\nu(z)$ an improved result might be expected owing to the simplicity in behaviour of polynomials in comparison with that of integral functions. This situation arises when the essential points are replaced by poles. If there are three simple poles on the circle of convergence and no other singularities in the entire plane or at infinity then the existence of three vanishing coefficients is sufficient to ensure the given result. In the case of three multiple poles our method supposes the existence of an infinite sequence of $n$ such that $c_n$, $c_{n+p}$, $c_{n+q}$ are all small. This condition enables us, however, to ease the restrictions on the function which now need have only three poles as dominant singularities on its circle of convergence and is otherwise unrestricted. "Small" coefficients in the case of poles means coefficients which are small in relation to those arising from

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[1] 1. [2] 1: 69, Theorem 1; 77, Theorem 5.

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the poles, a finer distinction than in the original theorem where the term has a less precise significance.

2. *Simple Poles.* Let $f(z)$ have simple poles at the points $e^{i\alpha_\nu}$ $(\nu = 1, 2, 3)$, and be otherwise regular in the finite part of the plane and at infinity. Then

$$f(z) = c + \sum_{\nu=1}^{3} \frac{A_\nu}{1 - ze^{-i\alpha_\nu}}. \tag{3}$$

Thus, in (1),

$$c_n = \sum_{\nu=1}^{3} A_\nu e^{-in\alpha_\nu} \qquad (n > 0),$$

and if $c_n$ vanishes for three values $n,\ n+p,\ n+q$,

$$\begin{aligned} a_1 + a_2 + a_3 &= 0 \\ a_1 e^{-ip\alpha_1} + a_2 e^{-ip\alpha_2} + a_3 e^{-ip\alpha_3} &= 0 \\ a_1 e^{-iq\alpha_1} + a_2 e^{-iq\alpha_2} + a_3 e^{-iq\alpha_3} &= 0, \end{aligned} \tag{4}$$

where $a_\nu = A_\nu e^{-i\alpha_\nu}$, $(\nu = 1, 2, 3)$, and none of the $a_\nu$ are zero.

It follows that

$$\begin{vmatrix} 1 & 1 & 1 \\ e^{-ip\alpha_1} & e^{-ip\alpha_2} & e^{-ip\alpha_3} \\ e^{-iq\alpha_1} & e^{-iq\alpha_2} & e^{-iq\alpha_3} \end{vmatrix} = 0, \tag{5}$$

and, from an elementary argument,[1] that each $\alpha_\mu - \alpha_\nu$ is an integral multiple of either $2\pi/p$, $2\pi/q$, $2\pi/(p-q)$ or $2\pi/s$ if $p$ and $q$ have a highest common factor $s$. In the first three cases the ratios of $a_1, a_2, a_3$ are determined from (4), in the last case we get only

$$a_1 + a_2 + a_3 = 0.$$

It is clear that the relations (4) will be repeated at regular intervals and that the sequence of zero coefficients will therefore form a set of *positive* density. We can relax the condition of regularity at infinity to the extent that (3) becomes

$$f(z) = P(z) + \sum_{\nu=1}^{3} \frac{A_\nu}{1 - ze^{-i\alpha_\nu}}$$

where $P(z)$ is a polynomial of degree $h$. For this case, however, the index $n$ in the set of relations (4) must exceed $h$. Since $h$ is arbitrary this requires the postulation of an infinite sequence of vanishing coefficients.

[1] 1, 76, *f.n.*

3. *Multiple poles.* We say that $f(z)$ has $k$ "dominant" poles of order $m$ on $|z| = 1$ if it can be written in the form

$$(6) \qquad f(z) = \sum_{\nu=1}^{k} \sum_{\mu=1}^{m} \frac{A_{\mu\nu}}{(1 - ze^{-i\alpha_\nu})^\mu} + \sum_{0}^{\infty} c_n^* z^n$$

where $c_n^* = o(n^{m-1})$. A sequence of coefficients $c_n$ which are $o(n^{m-1})$ will be described as a sequence of small coefficients. For the function (6) the coefficients take the form

$$c_n = \sum_{\nu=1}^{k} P_\nu(n)\, e^{-in\alpha_\nu} + c_n^*$$

wl.... the $P_\nu(n)$ are polynomials of degree $m - 1$.

We now prove the following theorem:

THEOREM. *If $f(z)$ has on its circle of convergence three dominant poles and there exists an infinite sequence of $n$ such that $c_n$, $c_{n+p}$, $c_{n+q}$ are small, then the dominant poles are situated at three of the vertices of some regular polygon inscribed in the circle of convergence and the dominant elements of these poles are virtually identical or at least linearly related. In this case the small coefficients are regularly distributed in a sequence of positive density.*

In this hypothesis $p$ and $q$ are constants. Let $n$ run through the sequence. Then, in place of (4) we have

$$(7) \qquad \begin{aligned} &Q_1 + Q_2 + Q_3 = o(n^{m-1}) \\ &Q_1 e^{-ip\alpha_1} + Q_2 e^{-ip\alpha_2} + Q_3 e^{-ip\alpha_3} = o(n^{m-1}) \\ &Q_1 e^{-iq\alpha_1} + Q_2 e^{-iq\alpha_2} + Q_3 e^{-iq\alpha_3} = o(n^{m-1}) \end{aligned}$$

where $Q_\nu = P_\nu(n) e^{in\alpha_\nu}$. For

$$P_\nu(n + p) = P_\nu(n) + o(n^{m-1}), \qquad P_\nu(n + q) = P_\nu(n) + o(n^{m-1}).$$

On solving (7) we see that, unless the determinant of the system vanishes, each $Q_\nu$ and therefore each $P_\nu(n)$ will be $o(n^{m-1})$, contrary to hypothesis. The rest of the argument follows as in § 2, except that we can only conclude that the dominant elements of the dominant poles are virtually identical or at least linearly related.

4. *Mandelbrojt's theorem.* The results of §§ 2 and 3 assume a small number of vanishing or small coefficients, as the case may be, $f(z)$ having polar singularities on its circles of convergence. Mandelbrojt has discussed the *maximum* number of vanishing coefficients in the following theorem:[1]

[1] 2, 420 ; 3, 36.

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THEOREM. *If in the series* (1) *there is an infinite sequence of more than* $k-1$ *consecutive vanishing coefficients then either* $f(z)$ *has at least* $k+1$ *poles on its circle of convergence or it possesses, on this circle, singularities other than poles.*

The corresponding problem for essential singularities, which includes poles, has been dealt with in a previous paper.[1] We prove below the following extension of Mandelbrojt's theorem.

THEOREM. *If* $f(z)$ *has exactly* $k$ *dominant poles on its circle of convergence then not more than* $k-1$ *consecutive coefficients can be small for an infinite sequence of* $n$.

Suppose that there exists an infinite sequence of $k$ consecutive small coefficients of the function $f(z)$ of (6); then for a value $n$ belonging to the sequence we have, as in (7),

$$\sum_{\nu=1}^{k} P_\nu(n+\mu)e^{-i(n+\mu)a_\nu} = o(n^{m-1}) \qquad (\mu = 0, 1, \ldots, k-1),$$

which may be written

$$\sum_{\nu=1}^{k} P_\nu(n)e^{-i(n+\mu)a_\nu} = o(n^{m-1}) \qquad (\mu = 0, 1, \ldots, k-1).$$

On writing $Q_\nu$ for $P_\nu(n)e^{-ina_\nu}$ we get the system of $k$ equations

$$\sum_{1}^{k} Q_\nu e^{-i\mu a_\nu} = o(n^{m-1}) \qquad (\mu = 0, 1, \ldots, k-1).$$

On solving for the $Q_\nu$ we see that unless the determinant of the system vanishes each $Q_\nu$ and therefore each $P_\nu(n)$ is $o(n^{m-1})$. Now this determinant has the value $\Pi(e^{-ia_\mu} - e^{-ia_\nu})$, $(\mu \neq \nu)$, and can therefore vanish only if at least two of the polar singularities coincide, which is contrary to the hypothesis. This proves the theorem.

[1] 1, 79, Theorem 6.

## REFERENCES.

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2. S. Mandelbrojt, *Annales de l'École Normale supérieure*, 40 (1923).
3. S. Mandelbrojt, *Mémorial des Sciences Mathématiques*, Fasc. LIV.

UNIVERSITY COLLEGE,
SWANSEA.