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AN ANALOGUE OF THE SCHUR–SIEGEL–SMYTH TRACE PROBLEM

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Abstract

By analogy with the trace of an algebraic integer α with conjugates $\alpha_1 = \alpha, \dots, \alpha_d$, we define the G-*measure* $\mathrm{G}(\alpha) = \sum_{i=1}^{d}(|\alpha_i| + 1/|\alpha_i|)$ and the *absolute* G-*measure* $\mathrm{g}(\alpha) = \mathrm{G}(\alpha)/d$. We establish an analogue of the Schur–Siegel–Smyth trace problem for totally positive algebraic integers. Then we consider the case where α has all its conjugates in a sector $|\arg z| \le \theta$, $0 < \theta < 90^\circ$. We compute the greatest lower bound $c(\theta)$ of the absolute G-measure of α, for α belonging to 11 consecutive subintervals of $]0, 90[$. This phenomenon appears here for the first time, conforming to a conjecture of Rhin and Smyth on the nature of the function $c(\theta)$. All computations are done by the method of explicit auxiliary functions.

2020 *Mathematics subject classification*: primary 11R06; secondary 11C08, 11R80, 11Y40.

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1. Introduction

Let α be a nonzero algebraic integer of degree $d \ge 1$ with conjugates $\alpha_1 = \alpha, \dots, \alpha_d$. By analogy with the trace, $\mathrm{Tr}(\alpha) = \sum_{i=1}^{d} \alpha_i$, we define the G-*measure* of α by

$$\mathrm{G}(\alpha) = \sum_{i=1}^{d}(|\alpha_i| + 1/|\alpha_i|)$$

and the *absolute* G-*measure* of α by $\mathrm{g}(\alpha) = \mathrm{G}(\alpha)/d$. It is obvious that $\mathrm{g}(\alpha) \ge 2$, with equality if and only if α is a root of unity. Indeed, if $g(\alpha) = 2$, then $|\alpha_i| = 1$ for $i = 1, \dots, d$. By Kronecker's theorem, we deduce that α is a root of unity.

The *absolute trace* of α is $\mathrm{tr}(\alpha) = \mathrm{Tr}(\alpha)/d$. The Schur–Siegel–Smyth trace problem is formulated as follows: fix $\rho < 2$ and show that all but finitely many totally positive algebraic integers α have $\mathrm{tr}(\alpha) > \rho$. This problem were studied by Schur [13], Siegel [14] and Smyth [15]. In 2016, we solved it for $\rho < 1.792812$ [5]. Recently, in 2021, Wang *et al.* solved it for 1.793145 [17]. On the other hand, Serre [1] showed that the method of explicit auxiliary functions used in all attacks on the problem since [15] does not give such an inequality for any ρ larger than 1.8983021 . . . . Therefore, this method cannot be used to prove that 2 is the smallest limit point of the set of numbers $\{\mathrm{tr}(\alpha) : \alpha \text{ is a totally positive algebraic integer}\}$.

The aim of this paper is to search for the supremum, say s, of all $\rho > 0$ such that all but finitely many totally positive algebraic numbers satisfy $\mathrm{g}(\alpha) > \rho$. The following theorem, using the algebraic numbers β_n defined by Smyth [15], proves that $s \leq 4$.

THEOREM 1.1. *Let β_n^2 be a totally positive algebraic integer of degree 2^n, defined by*

$$\beta_0^2 = 1, \quad \beta_n^2 = \beta_{n+1}^2 + \beta_{n+1}^{-2} - 2.$$

Then $\lim_{n\to+\infty} \mathrm{g}(\beta_n^2) = 4$.

Let $\mathcal{G} = \{\mathrm{g}(\alpha) : \alpha \text{ is a totally positive algebraic integer}\}$. Theorem 1.1 shows that 4 is a limit point of this set. Thus, the analogue of the Schur–Siegel–Smyth trace problem for the G-measure is: fix $\rho < 4$ and prove that all but finitely many totally positive algebraic numbers satisfy $\mathrm{g}(\alpha) > \rho$. The following theorem solves the problem for $\rho \leq 3.024561$.

THEOREM 1.2. *If α is a nonzero totally positive algebraic integer whose minimal polynomial is different from $x-1$, $x-2$, x^2-3x+1, x^2-4x+2, x^2-5x+5, x^3-6x^2+9x-3, $x^3-7x^2+14x-7$, $x^4-9x^3+27x^2-31x+11$ and $x^6-13x^5+64x^4-151x^3+177x^2-96x+19$, then* $\mathrm{g}(\alpha) \geq 3.024561$. *Moreover, the first five points of $\mathcal{G}$ are:*

$$\begin{aligned}
2 \quad &= g(x-1),\\
2.5 \quad &= g(x-2),\\
2.954545 \quad &= g(x^4-9x^3+27x^2-31x+11),\\
3 \quad &= g(x^2-3x+1) = g(x^2-4x+2) = g(x^2-5x+5)\\
&= g(x^3-6x^2+9x-3) = g(x^3-7x^2+14x-7),\\
3.008771 \quad &= g(x^6-13x^5+64x^4-151x^3+177x^2-96x+19).
\end{aligned}$$

From now on, we consider a nonzero algebraic integer α, not a root of unity, all of whose conjugates lie in a sector $S_\theta = \{z \in \mathbb{C} : |\arg z| \leq \theta\}$, $0 < \theta < 90$. We first recall a result of Langevin [10] on the Mahler measure M: there is a function $c(\theta)$ on $[0, 180°)$, always greater than 1, such that if all the conjugates of α lie in S_θ, then $\mathrm{M}(\alpha)^{1/d} \geq c(\theta)$ where d denotes the degree of α. In 1995, Rhin and Smyth [11] were the first to make this result effective. More precisely, they succeeded in finding the exact value of $c(\theta)$ for θ in nine subintervals of $[0, 120°]$. They used the method of explicit auxiliary functions with polynomials found by heuristic search. They conjectured that $c(\theta)$ is a 'staircase' decreasing function of θ, which is constant except for finitely many left discontinuities in any closed subinterval of $[0, 180°)$. In 2004, thanks to Wu's algorithm [18], Rhin and Wu [12] gave the exact value of $c(\theta)$ for four new subintervals of $[0, 140°]$ and extended four existing subintervals. In 2013, the author and Rhin [9] found for the first time a complete subinterval and a 14th subinterval. A complete subinterval is an interval on which the function $c(\theta)$ describing the minimum on the sector $|\arg z| \leq \theta$ is constant, with jump discontinuities at each end. These improvements came from our *recursive algorithm*, based on Wu's algorithm but where

TABLE 1. Intervals where $c(\theta)$ is known exactly.

i	$c(\theta)$	θ_i	θ'_i	P_i
1	2.5	0	22.386177	$x-2$
2	2.489363	22.386177	24.841669	$x^4-7x^3+19x^2-23x+11$
3	2.469953	24.841669	25.472522	$x^6-9x^5+34x^4-67x^3+73x^2-42x+11$
4	2.418970	25.472522	30	$x^4-6x^3+14x^2-14x+6$
5	2.309401	30	36.243112	x^2-3x+3
6	2.281940	36.243112	44.837281	$x^4-5x^3+11x^2-11x+5$
7	2.263377	44.837281	45.309984	$x^{10}-10x^9+50x^8-156x^7+334x^6-509x^5+560x^4-439x^3+236x^2-79x+13$
8	2.221900	45.309984	65.737119	$x^4-4x^3+8x^2-7x+3$
9	2.191644	65.737119	68.598457	$x^4-2x^3+5x^2-3x+3$
10	2.174972	68.598457	84.537478	$x^6-3x^5+8x^4-10x^3+11x^2-5x+3$
11	2.159893	84.537478	86.24	$x^7-2x^6+7x^5-10x^4+13x^3-13x^2+6x-4$

the polynomials are found by induction. We applied this method to measures such as the trace [2], the length [3] and the house [4], as well as to unusual measures (see [6–8]). Here we prove the following result.

THEOREM 1.3. *There exist a left discontinuous, strictly positive, staircase function h on* $[0, 90^\circ)$ *and a positive, continuous, monotonically decreasing function f on* $[0, 90^\circ)$ *such that*

$$\min(f(\theta), h(\theta)) \le c(\theta) \le h(\theta).$$

Moreover, the exact value of $c(\theta)$ *is known on* 11 *subintervals of* $[0, 90^\circ)$.

The function $h(\theta)$ is the smallest value of $g(\alpha)$ that could be found for α having all its conjugates in $|\arg z| \le \theta$. The function $f(\theta)$ is given by $f(\theta) = \max_{1\le i\le 11}(f_i(\theta))$, where the functions $f_i(\theta)$ are defined by

$$f_i(\theta) = \min_{z\in S_\theta}\Big(|z| + \frac{1}{|z|} - \sum_{1\le j\le J} c_{ij}\log|Q_{ij}(z)|\Big).$$

Table 1 summarises our results. One can read off the 11 intervals $[\theta_i, \theta'_i)$ where $f(\theta) > g(\theta)$ so that $c(\theta) = h(\theta) = h(\theta_i)$ for θ in these intervals. Also $c(\theta) = c(\theta_i) = g(P_i)$. The real numbers c_{ij} and the polynomials Q_{ij} are given in Table 3.

What is remarkable here and appears for the first time is that we are able to find complete and consecutive subintervals from 0 to 86.24°. This supports the Rhin–Smyth conjecture that $c(\theta)$ is a 'staircase' decreasing function of θ, which is constant except for finitely many left discontinuities in any closed subinterval of $[0, 90^\circ)$.

2. Proof of Theorem 1.1

The sequence of totally positive algebraic integers $(\beta_n^2)_{n\ge 0}$ is defined by

$$\beta_0^2 = 1, \quad \beta_n^2 = \beta_{n+1}^2 + \beta_{n+1}^{-2} - 2.$$

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By induction, we can easily prove that

$$\mathrm{Tr}(\beta_n^2) = 2^{n+1} - 1.$$

On the other hand,

$$\begin{aligned}\mathrm{G}(\beta_n^2) &= \sum_{i=1}^{2^n}(\beta_{n,i}^2 + \beta_{n,i}^{-2}) = 2\sum_{i=1}^{2^{n-1}}(\beta_{n,i}^2 + \beta_{n,i}^{-2}) = 2\sum_{i=1}^{2^{n-1}}(\beta_{n-1,i}^2 + 2)\\ &= 2\mathrm{Tr}(\beta_{n-1,i}^2) + 2^{n+1} = 2(2^n - 1) + 2^{n+1} = 2^{n+2} - 1.\end{aligned}$$

Therefore, $\mathrm{g}(\beta_n^2) = 4 - 2^{-n}$. This proves Theorem 1.1.

3. Proof of Theorem 1.2

3.1. The principle of auxiliary functions. The auxiliary function involved in the study of the G-measure is

$$f(x) = x + \frac{1}{x} - \sum_{1\le j\le J} c_j \log|Q_j(x)| \quad \text{for } x > 0,$$

where the c_j are positive real numbers and the polynomials Q_j are nonzero polynomials in $\mathbb{Z}[x]$.

Let m denote the minimum of the function f and P the minimal polynomial of α. If P does not divide any Q_j, then

$$\sum_{i=1}^{d} f(\alpha_i) \ge md,$$

that is,

$$\mathrm{G}(\alpha) \ge md + \sum_{1\le j\le J} c_j \log\left|\prod_{i=1}^{d} Q_j(\alpha_i)\right|.$$

Since P is monic and does not divide any Q_j, it follows that $\prod_{i=1}^{d} Q_j(\alpha_i)$ is a nonzero integer because it is the resultant of P and Q_j. Hence, if α is not a root of Q_j, then

$$\mathrm{g}(\alpha) \ge m.$$

REMARK 3.1. In order to ensure a better convergence in our program with Pascal, we have reduced the size of the coefficients of the polynomials involved in the auxiliary function by working on $[-1, \infty)$. Thus, the auxiliary function becomes

$$f(x) = x + 1 + \frac{1}{x+1} - \sum_{1\le j\le J} c_j \log|Q_j(x)| \quad \text{for } x > -1, \tag{3.1}$$

where the polynomials Q_j and the real numbers c_j are given in Table 2.

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TABLE 2. Coefficients and polynomials involved in Theorem 1.2.

c_j	$Q_j(x)$
0.5872584799	x
0.3240774699	$x-1$
0.04301779133	$x-2$
0.02489357462	x^2-x-1
0.06118017624	x^2-2x-1
0.02597132015	x^2-3x+1
0.02056091724	x^3-3x^2+1
0.01042864328	x^3-4x^2+3x+1
0.008913963067	$x^4-5x^3+5x^2+x-1$
0.005057484791	$x^4-5x^3+5x^2+2x-1$
0.04610960921	$x^4-5x^3+6x^2-1$
0.003516260097	$x^5-7x^4+15x^3-9x^2-2x+1$
0.01648697997	$x^6-7x^5+14x^4-5x^3-7x^2+2x+1$
0.0003993670660	$x^6-7x^5+15x^4-6x^3-9x^2+2x+1$
0.004172742794	$x^6-7x^5+15x^4-8x^3-6x^2+3x+1$
0.0001188583080	$2x^6-11x^5+17x^4-2x^3-9x^2+x+1$
0.004874387122	$x^7-8x^6+22x^5-22x^4-x^3+11x^2-x-1$
0.0002150795560	$x^7-9x^6+27x^5-28x^4-x^3+11x^2-x-1$
0.0008555104000	$x^7-9x^6+27x^5-29x^4+13x^2-x-1$
0.001587985433	$x^8-10x^7+36x^6-53x^5+17x^4+25x^3-13x^2-3x+1$
0.001364196657	$x^8-10x^7+36x^6-54x^5+22x^4+18x^3-11x^2-2x+1$
0.003483780418	$x^8-10x^7+36x^6-55x^5+26x^4+14x^3-12x^2+1$
0.004752996662	$x^8-10x^7+37x^6-59x^5+29x^4+17x^3-14x^2-x+1$
0.0007175262240	$x^{10}-11x^9+45x^8-79x^7+34x^6+62x^5-60x^4-8x^3+16x^2-1$
0.0005260961560	$x^{10}-12x^9+54x^8-108x^7+72x^6+50x^5-69x^4-6x^3+16x^2-1$
0.005738782521	$x^{10}-12x^9+55x^8-116x^7+94x^6+30x^5-77x^4+10x^3+17x^2-2x-1$
0.0001911119380	$x^{10}-12x^9+55x^8-117x^7+99x^6+23x^5-76x^4+12x^3+17x^2-2x-1$
0.002928853691	$x^{11}-13x^{10}+67x^9-170x^8+202x^7-43x^6-124x^5+78x^4+21x^3-18x^2-x+1$
0.0007971698040	$2x^{11}-24x^{10}+113x^9-257x^8+259x^7-8x^6-178x^5+82x^4+30x^3-20x^2-x+1$
0.0004529337020	$x^{12}-15x^{11}+91x^{10}-283x^9+459x^8-302x^7-115x^6+252x^5-55x^4-51x^3+17x^2+3x-1$
0.002607019825	$x^{12}-15x^{11}+91x^{10}-283x^9+460x^8-308x^7-105x^6+252x^5-66x^4-46x^3+19x^2+2x-1$
0.00005528121	$x^{13}-15x^{12}+90x^{11}-269x^{10}+383x^9-102x^8-362x^7+322x^6+70x^5-139x^4+4x^3+21x^2-x-1$
0.001180256932	$x^{13}-17x^{12}+121x^{11}-465x^{10}+1027x^9-1239x^8+558x^7+365x^6-477x^5+66x^4+84x^3-22x^2-4x+1$
0.001337485201	$x^{14}-17x^{13}+120x^{12}-450x^{11}+936x^{10}-954x^9+80x^8+733x^7-460x^6-142x^5+166x^4+7x^3-22x^2+1$
0.003168622821	$x^{14}-17x^{13}+120x^{12}-451x^{11}+946x^{10}-991x^9+137x^8+720x^7-517x^6-99x^5+181x^4-11x^3-23x^2+2x+1$
0.0003609579220	$x^{14}-17x^{13}+120x^{12}-451x^{11}+946x^{10}-991x^9+139x^8+707x^7-489x^6-118x^5+174x^4-x^3-23x^2+x+1$
0.001958131607	$x^{14}-17x^{13}+120x^{12}-451x^{11}+946x^{10}-992x^9+145x^8+697x^7-491x^6-101x^5+165x^4-4x^3-21x^2+x+1$
0.0003505691830	$x^{14}-17x^{13}+120x^{12}-451x^{11}+946x^{10}-992x^9+146x^8+691x^7-479x^6-109x^5+164x^4-2x^3-21x^2+x+1$
0.001990883626	$x^{15}-19x^{14}+154x^{13}-691x^{12}+1849x^{11}-2895x^{10}+2177x^9+304x^8-1783x^7+870x^6+308x^5-304x^4+33x^2-2x-1$
0.0004854896110	$x^{15}-19x^{14}+154x^{13}-691x^{12}+1850x^{11}-2906x^{10}+2225x^9+200x^8-1674x^7+841x^6+267x^5-272x^4-2x^3+29x^2-x-1$
0.0008248955840	$x^{15}-19x^{14}+154x^{13}-691x^{12}+1850x^{11}-2906x^{10}+2225x^9+201x^8-1681x^7+859x^6+248x^5-268x^4+3x^3+27x^2-x-1$
0.0007192476320	$x^{16}-20x^{15}+172x^{14}-828x^{13}+2419x^{12}-4280x^{11}+4057x^{10}-692x^9-2501x^8+2099x^7+21x^6-653x^5+160x^4+73x^3-25x^2-3x+1$
0.0001696531250	$x^{17}-21x^{16}+191x^{15}-980x^{14}+3078x^{13}-5915x^{12}+6186x^{11}-1333x^{10}-4347x^9+4092x^8+156x^7-1628x^6+315x^5+286x^4-60x^3-26x^2+3x+1$
0.0001190382390	$x^{18}-22x^{17}+210x^{16}-1136x^{15}+3796x^{14}-7906x^{13}+9420x^{12}-3717x^{11}-5344x^{10}+7327x^9-1313x^8-2764x^7+1357x^6+398x^5-318x^4-19x^3+30x^2-1$
0.0001056112650	$x^{18}-22x^{17}+212x^{16}-1171x^{15}+4059x^{14}-9010x^{13}+12222x^{12}-7984x^{11}-1986x^{10}+7192x^9-3354x^8-1452x^7+1485x^6+37x^5-267x^4+14x^3+25x^2-x-1$

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TABLE 3. The polynomials Q_j and their coefficients c_j involved in the functions f_i, $1 \le i \le 11$.

f_1	c_j	Q_j
	0.73888608	$x - 1$
	0.39043052	$x - 2$
	0.04192943	$x^2 - 3x + 3$
	0.02267925	$x^2 - 4x + 5$
	0.01211818	$x^4 - 7x^3 + 19x^2 - 23x + 11$
	0.00944891	$x^5 - 9x^4 + 33x^3 - 60x^2 + 53x - 17$
f_2	c_j	Q_j
	0.70761747	$x - 1$
	0.14194593	$x^2 - 3x + 3$
	0.05040292	$x^2 - 4x + 5$
	0.02205576	$x^4 - 7x^3 + 19x^2 - 23x + 11$
	0.01262594	$x^5 - 9x^4 + 33x^3 - 60x^2 + 53x - 17$
	0.00061889	$x^6 - 9x^5 + 34x^4 - 67x^3 + 73x^2 - 42x + 11$
f_3	c_j	Q_j
	0.70892626	$x - 1$
	0.13697112	$x^2 - 3x + 3$
	0.05619086	$x^2 - 4x + 5$
	0.02480448	$x^4 - 7x^3 + 19x^2 - 23x + 11$
	0.01617321	$x^6 - 9x^5 + 34x^4 - 67x^3 + 73x^2 - 42x + 11$
f_4	c_j	Q_j
	0.76705405	$x - 1$
	0.11120526	$x^2 - 3x + 3$
	0.05427695	$x^2 - 4x + 5$
	0.01410761	$x^4 - 6x^3 + 14x^2 - 14x + 6$
	0.03158715	$x^4 - 7x^3 + 19x^2 - 23x + 11$
f_5	c_j	Q_j
	0.69764717	$x - 1$
	0.17276632	$x^2 - 3x + 3$
	0.04309305	$x^4 - 5x^3 + 11x^2 - 11x + 5$
f_6	c_j	Q_j
	0.00218431	$x - 1$
	0.16574248	$x^2 - 2x + 2$
	0.01192929	$x^4 - 3x^3 + 5x^2 - 3x + 1$
	0.10100115	$x^4 - 4x^3 + 8x^2 - 7x + 3$
	0.00778105	$x^4 - 5x^3 + 11x^2 - 11x + 5$
	0.05789443	$x^6 - 6x^5 + 18x^4 - 29x^3 + 28x^2 - 15x + 4$
	0.00331505	$x^8 - 8x^7 + 32x^6 - 76x^5 + 117x^4 - 117x^3 + 75x^2 - 28x + 5$
	0.00174856	$x^8 - 9x^7 + 40x^6 - 105x^5 + 178x^4 - 196x^3 + 139x^2 - 58x + 12$
	0.00052318	$x^9 - 9x^8 + 40x^7 - 106x^6 + 184x^5 - 212x^4 + 162x^3 - 77x^2 + 21x - 2$
	0.00385077	$x^{10} - 11x^9 + 60x^8 - 203x^7 + 467x^6 - 756x^5 + 872x^4 - 708x^3 + 391x^2 - 134x + 23$

Continued

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TABLE 3. Continued

f_7	c_j	Q_j
	0.00095472	$x - 1$
	0.16770496	$x^2 - 2x + 2$
	0.01122625	$x^4 - 3x^3 + 5x^2 - 3x + 1$
	0.10056818	$x^4 - 4x^3 + 8x^2 - 7x + 3$
	0.05757477	$x^6 - 6x^5 + 18x^4 - 29x^3 + 28x^2 - 15x + 4$
	0.00232352	$x^8 - 8x^7 + 32x^6 - 76x^5 + 117x^4 - 117x^3 + 75x^2 - 28x + 5$
	0.00009956	$x^8 - 9x^7 + 40x^6 - 105x^5 + 178x^4 - 196x^3 + 139x^2 - 58x + 12$
	0.00083995	$x^9 - 9x^8 + 40x^7 - 106x^6 + 184x^5 - 212x^4 + 162x^3 - 77x^2 + 21x - 2$
	0.00696407	$x^{10} - 10x^9 + 50x^8 - 156x^7 + 334x^6 - 509x^5 + 560x^4 - 439x^3 + 236x^2 - 79x + 13$
	0.00644360	$x^{10} - 11x^9 + 60x^8 - 203x^7 + 467x^6 - 756x^5 + 872x^4 - 708x^3 + 391x^2 - 134x + 23$

f_8	c_j	Q_j
	0.00008104	$x - 1$
	0.17595994	$x^2 - x + 1$
	0.09680963	$x^2 - x + 2$
	0.01972259	$x^4 - 2x^3 + 4x^2 - 2x + 1$
	0.01270024	$x^4 - 2x^3 + 5x^2 - 3x + 3$
	0.00173352	$x^4 - 3x^3 + 6x^2 - 6x + 3$
	0.00051087	$x^4 - 4x^3 + 8x^2 - 7x + 3$

f_9	c_j	Q_j
	0.00275034	$x - 1$
	0.17257924	$x^2 - x + 1$
	0.11909230	$x^2 - x + 2$
	0.02008775	$x^4 - 2x^3 + 5x^2 - 3x + 2$
	0.01275003	$x^4 - 2x^3 + 5x^2 - 3x + 3$
	0.00458509	$x^5 - 2x^4 + 5x^3 - 4x^2 + 4x - 1$
	0.00544333	$x^6 - 3x^5 + 8x^4 - 10x^3 + 11x^2 - 5x + 3$

f_{10}	c_j	Q_j
	0.00000904	$x - 1$
	0.00001815	$x^2 - x + 1$
	0.11538720	$x^2 + 1$
	0.04577509	$x^3 - x^2 + 2x - 1$
	0.05422539	$x^4 - x^3 + 4x^2 - 2x + 3$
	0.01954309	$x^6 - x^5 + 6x^4 - 4x^3 + 10x^2 - 3x + 4$
	0.00063457	$x^6 - 3x^5 + 8x^4 - 10x^3 + 11x^2 - 5x + 3$
	0.00419345	$x^7 - 2x^6 + 7x^5 - 10x^4 + 13x^3 - 13x^2 + 6x - 4$

f_{11}	c_j	Q_j
	0.00072540	$x - 1$
	0.00209131	$x^2 - x + 1$
	0.13468134	$x^2 + 1$
	0.04432202	$x^3 - x^2 + 2x - 1$
	0.07117836	$x^4 - x^3 + 4x^2 - 2x + 3$
	0.00041641	$x^7 - 2x^6 + 7x^5 - 10x^4 + 13x^3 - 13x^2 + 6x - 4$

3.2. Auxiliary functions and generalised integer transfinite diameter. This and the following sections reproduce the corresponding sections of [5].

Let K be a compact subset of $\mathbb{C}$. The *transfinite diameter* of K is defined by

$$t(K) = \liminf_{\substack{n\geq 1\\ n\to\infty}} \inf_{\substack{P\in\mathbb{C}[X]\\ P \text{ monic}\\ \deg(P)=n}} |P|_{\infty,K}^{1/n},$$

where $|P|_{\infty,K} = \sup_{z\in K} |P(z)|$ for $P \in \mathbb{C}[X]$. The *integer transfinite diameter* of K is defined by

$$t_{\mathbb{Z}}(K) = \liminf_{\substack{n\geq 1\\ n\to\infty}} \inf_{\substack{P\in\mathbb{Z}[X]\\ \deg(P)=n}} |P|_{\infty,K}^{1/n}.$$

Finally, if φ is a positive function defined on K, the *φ-generalised integer transfinite diameter of K* is defined by

$$t_{\mathbb{Z},\varphi}(K) = \liminf_{\substack{n\geq 1\\ n\to\infty}} \inf_{\substack{P\in\mathbb{Z}[X]\\ \deg(P)=n}} \sup_{z\in K} (|P(z)|^{1/n}\varphi(z)).$$

In the auxiliary function (3.1), we replace the coefficients c_j by rational numbers a_j/q where q is a positive integer such that qc_j is an integer for $1 \leq j \leq J$. Then, for $x > -1$,

$$f(x) = x + 1 + \frac{1}{x+1} - \frac{t}{r}\log|Q(x)| \geq m, \tag{3.2}$$

where $Q = \prod_{j=1}^{J} Q_j^{a_j} \in \mathbb{Z}[X]$ is of degree $r = \sum_{j=1}^{J} a_j \deg Q_j$ and $t = \sum_{j=1}^{J} c_j \deg Q_j$ (this formulation was introduced by Serre). Thus we seek a polynomial $Q \in \mathbb{Z}[X]$ such that

$$\sup_{x>-1} |Q(x)|^{t/r} e^{-(x+1+1/(x+1))} \leq e^{-m}.$$

If we suppose that t is fixed, this is equivalent to finding an effective upper bound for the weighted integer transfinite diameter over the interval $[-1,\infty[$ with the weight $\varphi(x) = e^{-(x+1+1/(x+1))}$, that is,

$$t_{\mathbb{Z},\varphi}([-1,\infty)) = \liminf_{\substack{r\geq 1\\ r\to\infty}} \inf_{\substack{P\in\mathbb{Z}[X]\\ \deg(P)=r}} \sup_{x>-1} (|P(x)|^{t/r}\varphi(x)).$$

Even though we have replaced the compact set K by the infinite interval $[-1,\infty[$, the weight φ ensures that the quantity $t_{\mathbb{Z},\varphi}([-1,\infty))$ is finite.

3.3. Construction of an auxiliary function. The main point is to find a set of 'good' polynomials Q_j to give the best possible value for m. Until 2003, the polynomials were found heuristically. In 2003, Wu [18] developed an algorithm that allows a systematic search for 'good' polynomials. For an auxiliary function as defined by (3.1), we fix a set E_0 of control points, uniformly distributed on the real interval $I = [-1, A]$ where A is 'sufficiently large'. Using the LLL algorithm, we find a polynomial Q small on E_0 in terms of the quadratic norm. We test this polynomial in the auxiliary function and keep only the factors of Q which have a nonzero exponent. The convergence of this new function gives local minima that we add to the set of points E_0 to get a new set of control points E_1. We use the LLL algorithm again with the set E_1 and the process is repeated.

In 2006, we made two improvements to Wu's algorithm in the use of the LLL algorithm. The first is, at each step, to take into account not only the new control points but also the new polynomials of the best auxiliary function. The second is the introduction of a corrective coefficient t. The idea is to get good polynomials Q_j by induction. Thus, we call this algorithm the *recursive algorithm*. The first step is the optimisation of the auxiliary function $f_1 = x + 1 + 1/(x + 1) - t \log x$. We take $t = c_1$ where c_1 gives the best function f_1. We suppose that we have some polynomials Q_1, $Q_2, \ldots, Q_J$ and a function f as good as possible for this set of polynomials in the form (3.2). We seek a polynomial $R \in \mathbb{Z}[x]$ of degree k such that

$$\sup_{x\in I} |Q(x)R(x)|^{t/(r+k)} e^{-(x+1+1/(x+1))} \leq e^{-m}$$

where $Q = \prod_{j=1}^{J} Q_j$. We want the quantity

$$\sup_{x\in I} |Q(x)R(x)| \exp\Big(\frac{-(x + 1 + 1/(x + 1))(r + k)}{t}\Big)$$

to be as small as possible. We apply the LLL algorithm to the linear forms

$$Q(x_i)R(x_i) \exp\Big(\frac{-(x_i + 1 + 1/(x_i + 1))(r + k)}{t}\Big).$$

The x_i are control points uniformly distributed on the interval I to which we have added points where f has local minima. Thus we find a polynomial R whose irreducible factors R_j are good candidates to enlarge the set $\{Q_1, \ldots, Q_J\}$. We only keep the factors R_j that have a nonzero coefficient in the newly optimised auxiliary function f. After optimisation, some of the previous polynomials Q_j may have a zero exponent and so are removed.

We applied this recursive algorithm for $k = \deg R$ running from 4 to 25. We stop there because at that point the minimum m only varies in the fifth decimal place.

3.4. Optimization of the c_j. We have to solve a problem of the following form: find

$$\max_{C} \min_{x\in X} f(x, C)$$

where $f(x, C)$ is a linear form with respect to $C = (c_0, c_1, \ldots, c_k)$ (c_0 is the coefficient of x and is equal to 1), X is a compact domain of $\mathbb{C}$ and the maximum is taken over $c_j \geq 0$ for $j = 0, \ldots, k$. A classical solution involves taking very many control points $(x_i)_{1\leq i\leq N}$ and solving the standard problem of linear programming:

$$\max_{C} \min_{1\leq j\leq N} f(x_i, C).$$

The result then depends on the choice of the control points.

The idea of semi-infinite linear programming (introduced into number theory by Smyth [16]) involves repeating the previous process, adding new control points at each

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step and verifying that this process converges to m, the value of the linear form for an optimum choice of C. The algorithm is as follows.

(1) Choose an initial value C^0 for C and calculate $m'_0 = \min_{x\in X} f(x, C^0)$.
(2) Choose a finite set X_0 of control points belonging to X and set

$$m'_0 \le m \le m_0 = \min_{x\in X_0} f(x, C^0).$$

(3) Add to X_0 the points where $f(x, C^0)$ has local minima to get a new set X_1 of control points.
(4) Solve the usual linear programming problem $\max_C \min_{x\in X_1} f(x, C)$. We get a new value for C denoted by C^1 and a result from the linear programming problem equal to $m'_1 = \min_{x\in X} f(x, C^1)$. Then

$$m'_0 \le m'_1 \le m \le m_1 = \min_{x\in X_1} f(x, C^1) \le m_0.$$

(5) Repeat steps 2–4, giving two sequences (m_i) and (m'_i) which satisfy

$$m'_0 \le m'_1 \le \cdots \le m'_i \le m \le m_i \le \cdots \le m_1 \le m_0.$$

We stop when there is a good enough convergence, for example when $m_i - m'_i \le 10^{-6}$. If p iterations are sufficient then we take $m = m'_p$.

4. Proof of Theorem 1.3

We assume that α is an algebraic integer all of whose conjugates $\alpha_1 = \alpha, \ldots, \alpha_d$ lie in S_θ. The auxiliary functions f_i, $1 \le i \le 11$, are of the form

$$f_i(z) = |z| + 1/|z| - \sum_{1\le j\le J} c_{ij} \log |Q_{ij}(z)| \quad \text{for all } z \in S_\theta,$$

where the coefficients c_{ij} are positive real numbers and the polynomials Q_{ij} are nonzero in $\mathbb{Z}[z]$.

Since the function f_i is invariant under complex conjugation, we can limit ourselves to $0 \le \arg z \le \theta$. Moreover, the function f_i is harmonic outside the union of arbitrary small discs around the roots of the polynomials Q_{ij}, so the minimum is taken on the upper edge of S_θ where $z = xe^{i\theta}$ with $x > 0$.

The auxiliary function on the half line $R_\theta = \{z \in \mathbb{C}, z = xe^{i\theta}, x > 0\}$ is

$$f_i(z) = x + 1/x - \sum_{1\le j\le J} c_j \log |Q_j(z)|.$$

We proceed as in the Section 3.3. For several values of k, we search for a polynomial $R(z) = \sum_{l=0}^{k} a_l z^l \in \mathbb{Z}[z]$ such that

$$\sup_{x>0} |Q(z)R(z)| \exp\left(\frac{-(x + 1/x)(r + k)}{t}\right)$$

is as small as possible. But, here, $R(z)$ is not a real linear form in the unknown coefficients a_i. So, we replace it by its real part and its imaginary part. Then, we apply

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LLL to the two linear forms

$$|Q(z_n)| \cdot \mathrm{Re}(R(z_n)) \cdot \exp\Big(\frac{-(x_n + 1/x_n)(r+k)}{t}\Big),$$

$$|Q(z_n)| \cdot \mathrm{Im}(R(z_n)) \cdot \exp\Big(\frac{-(x_n + 1/x_n)(r+k)}{t}\Big),$$

where $z_n = x_n e^{i\theta}$. The x_n are suitable control points in $[0, 50]$, including the points where f_i has its least local minima. Then we proceed as described above.

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