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COMPACTNESS AND STRONG SEPARATION

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Two point sets H and K are said to be strongly separated if there exist two mutually exclusive domains D_H and D_K containing H and K respectively such that either $\bar{D}_H$ and $\bar{D}_K$ are mutually exclusive or $\bar{D}_H \cdot \bar{D}_K$ is $\bar{H} \cdot \bar{K}$. R. L. Moore has shown [**2**, Theorem 153, Chapter I] that if S is a normal Moore space and H and K are two mutually separated point sets then H and K are strongly separated. In this paper it is shown that if S is a Moore space, (1) H and K are two mutually separated point sets and (2) the closure of the set of all boundary points of H which do not belong to $\bar{K}$ is compact, then H and K are strongly separated. It is further shown that the above proposition does not remain true in every separable space satisfying Moore's Axioms 0-6 if condition (2) of its hypothesis is replaced by either (a) H and K are conditionally compact, there exists a domain containing H whose closure does not intersect $\bar{K}$, and there exists a domain containing K whose closure does not intersect $\bar{H}$ or (b) H is a conditionally compact domain, $\bar{H}$ does not intersect $\bar{K}$, and K is conditionally compact. It is also shown that if S is connected, M separates H from K in S and the boundary of M is compact then some closed subset of M separates H from K.

A Moore space is one satisfying Axiom 0 and the first three parts of Axiom 1 of [**2**]. It is assumed throughout that S is a Moore space. The terms compact and conditionally compact are as in [**3**] and other definitions and notation are as in [**2**].

THEOREM 1. *Suppose H and K are two mutually separated point sets such that $\bar{H}$ is compact. Then H and K are strongly separated.*

Proof. Suppose $\bar{H}$ and $\bar{K}$ are mutually exclusive. Then the conclusion follows from [**2**, Theorem 16, Chapter I].

Suppose $\bar{H}$ and $\bar{K}$ are not mutually exclusive. Then $\bar{H} \cdot \bar{K}$ is compact. With the aid of [**2**, Theorem 155, Chapter I] it can be shown that there exists a sequence $D_1, D_2, D_3, \ldots$ of domains such that (1) for each n, D_n contains $\bar{D}_{n+1}$, (2) $\bar{H} \cdot \bar{K}$ is the common part of all of the domains of this sequence, (3) every domain that contains $\bar{H} \cdot \bar{K}$ contains some domain of this sequence, (4) $S - \bar{D}_1$ contains a point of H and a point of K, and (5) there exists a

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sequence $P_1, P_2, P_3, \ldots$ of points such that for each n, P_n is a point of $H \cdot (D_n - \bar{D}_{n+1})$.

Let H_1 denote $(\bar{H} + D_1) - D_1$ and, for each positive integer n, let H_{n+1} denote $\bar{H} \cdot (\bar{D}_n - D_{n+1})$. For each n, P_n belongs to H_{n+1}, H_n and $\bar{K}$ are mutually exclusive and closed, H_n is compact, and H_{n+2} lies in $\bar{D}_{n+1}$. Therefore there exists a sequence $E_1, E_2, E_3, \ldots$ such that for each n, E_n is a domain containing H_n, $\bar{E}_n$ does not intersect $\bar{K}$, and $\bar{E}_{n+2}$ is a subset of D_n. Let E denote $\bar{H} \cdot \bar{K} + \sum \bar{E}_n$. E contains no point of K.

Suppose there is a limit point Q of E that does not belong to E. There exists a sequence $Q_1, Q_2, Q_3, \ldots$ of distinct points of E converging to Q. Since for each positive integer n, Q is not a limit point of $\bar{E}_n$ and since Q is not a limit point of $\bar{H} \cdot \bar{K}$, there exists an increasing sequence $n_1, n_2, n_3, \ldots$ of integers and a subsequence $Q_1', Q_2', Q_3', \ldots$ of the sequence $Q_1, Q_2, Q_3, \ldots$ such that $n_1 > 2$ and for each j, Q_j' is a point of $\bar{E}_{n_j}$. $D_{n_1-2}, D_{n_2-2}, D_{n_3-2}, \ldots$ is a sequence of domains such that $\bar{H} \cdot \bar{K}$ is their common part, each of them contains the next one and for each j, Q_j' is a point of $(D_{n_j-2}) - (\bar{H} \cdot \bar{K})$. It follows from [**2**, Theorem 151, Chapter I] that $Q_1' + Q_2' + Q_3' + \ldots$ has a limit point belonging to $\bar{H} \cdot \bar{K}$. Since Q is the only limit point of this point set, Q belongs to $\bar{H} \cdot \bar{K}$. This involves a contradiction. Hence E is closed. Therefore $S - E$ is a domain D_K containing K.

There exists a sequence $F_1, F_2, F_3, \ldots$ such that for each n, F_n is a domain containing H_n and $\bar{F}_n$ is a subset of E_n. Let D_H denote $\sum F_n$. D_H contains H.

Suppose D_H and D_K have some point P in common. For some positive integer i, P is a point of F_i and for each positive integer n, P is a point of $S - (\bar{E}_n + \bar{H} \cdot \bar{K})$. Thus P is not a point of $\bar{H} \cdot \bar{K}$ but P belongs to $S - \bar{E}_i$. This involves a contradiction. Therefore D_H and D_K are two mutually exclusive domains containing H and K respectively.

Suppose X is a point of $\bar{H} \cdot \bar{K}$, then X belongs neither to H nor to K but is a limit point of both of them. D_H contains H and D_K contains K, thus X is a limit point of both D_H and D_K. Therefore $\bar{D}_H \cdot \bar{D}_K$ contains $\bar{H} \cdot \bar{K}$.

Suppose some point Y of $\bar{D}_H \cdot \bar{D}_K$ does not belong to $\bar{H} \cdot \bar{K}$. Since D_H and D_K are mutually exclusive domains, Y is a limit point of each of them but belongs to neither of them. Suppose Y is a limit point of F_n for some positive integer n. F_n is a subset of E_n and E_n is a domain containing Y but containing no point of D_K. This involves a contradiction, thus Y is not a limit point of F_n for any positive integer n. There exists an increasing sequence $n_1, n_2, n_3, \ldots$ of positive integers and a sequence $Y_1, Y_2, Y_3, \ldots$ of distinct points of D_H converging to Y such that $n_1 > 2$ and for each positive integer j, Y_j is a point of F_{n_j}. $D_{n_1-2}, D_{n_2-2}, D_{n_3-2}, \ldots$ is a sequence of domains such that $\bar{H} \cdot \bar{K}$ is their common part, each of them contains the next one and for each positive integer j, Y_j is a point of $(D_{n_j-2}) - (\bar{H} \cdot \bar{K})$. It follows from [**2**, Theorem 151, Chapter I] that the point set $Y_1 + Y_2 + Y_3 + \ldots$ has a limit point in $\bar{H} \cdot \bar{K}$. Since Y is the only limit point of this point set, Y belongs to $\bar{H} \cdot \bar{K}$. Therefore $\bar{H} \cdot \bar{K}$ is $\bar{D}_H \cdot \bar{D}_K$.

THEOREM 2. *Suppose H and K are two mutually separated point sets such that if M denotes the set of all boundary points of H that do not belong to $\bar{K}$, $\bar{M}$ is compact. Then H and K are strongly separated.*

Proof. There are two cases to be considered.

Case 1. Suppose H is a subset of M. Since $\bar{M}$ is compact, $\bar{H}$ is compact, and the conclusion of this theorem follows from Theorem 1.

Case 2. Suppose H is not a subset of M. Every boundary point of H is a point of $M + \bar{K}$. Since no point of H is a point of $\bar{K}$, there is a point of H which is not a boundary point of H. Let D denote the domain $(H + M) - M$. Since M and K are two mutually separated point sets and $\bar{M}$ is compact, it follows from Theorem 1 that there exist two mutually exclusive domains D_M and D_K' containing M and K respectively such that either $\bar{D}_M$ and $\overline{D_K'}$ are mutually exclusive or $\bar{D}_M \cdot \overline{D_K'}$ is $\bar{M} \cdot \bar{K}$.

Let D_K denote $(D_K' + \bar{H}) - \bar{H}$. D_M and D_K are two domains containing M and K respectively such that either $\bar{D}_M$ and $\bar{D}_K$ are mutually exclusive or $\bar{D}_M \cdot \bar{D}_K$ is $\bar{M} \cdot \bar{K}$. Let D_H denote the domain $D + D_M$.

Suppose D_H contains some point P of D_K. P does not belong to D since D is a subset of H and $D_K = D_K' - \bar{H} \cdot D_K'$. P does not belong to D_M since D_M and D_K' are mutually exclusive. Therefore, D_H does not intersect D_K.

There are two subcases to be considered.

Case 2A. Suppose that $\bar{H}$ and $\bar{K}$ are mutually exclusive. $\bar{M}$ and $\bar{K}$ are mutually exclusive since $\bar{M}$ is a subset of $\bar{H}$. Thus $\bar{D}_M$ and $\overline{D_K'}$ are mutually exclusive and therefore $\bar{D}_M$ does not intersect $\bar{D}_K$. Suppose some point P belongs to $\bar{D}$ and $\bar{D}_K$. Since D and D_K are mutually exclusive domains, P is a boundary point of each of them. Since D is a subset of H and no point of D_K is a point of $\bar{H}$, P is a boundary point of H and since $\bar{H}$ and $\bar{K}$ are mutually exclusive, P is not a point of $\bar{K}$. Therefore P is a point of M. But M is a subset of $\bar{D}_M$. This involves a contradiction. Therefore $\bar{D}$ and $\bar{D}_K$ are mutually exclusive and since $\bar{D}_H = \bar{D} + \bar{D}_M$, it follows that $\bar{D}_H$ and $\bar{D}_K$ are mutually exclusive.

Case 2B. Suppose $\bar{H}$ intersects $\bar{K}$ and P is a point of $\bar{H} \cdot \bar{K}$. Since D is $(H + M) - M$, D_M contains M and $D + M$ contains H, it follows that $\bar{D}_H$ contains $\bar{H}$ and therefore P is in $\bar{D}_H$. K is a subset of $D_K' \cdot (S - \bar{H})$, therefore $\bar{D}_K$ contains $\bar{K}$ and P is in $\bar{D}_K$. Hence $\bar{H} \cdot \bar{K}$ is a subset of $\bar{D}_H \cdot \bar{D}_K$.

Suppose some point Q of $\bar{D}_H \cdot \bar{D}_K$ does not belong to $\bar{H} \cdot \bar{K}$. $\bar{D}_H$ is $\bar{D} + \bar{D}_M$. $\bar{D}_K$ is the closure of $(D_K' - \bar{H} \cdot D_K')$. $\overline{D_K'}$ contains $\bar{D}_K$ and Q is in $\bar{D}_H \cdot \overline{D_K'}$. Therefore Q belongs to either $\bar{D} \cdot \overline{D_K'}$ or $\bar{D}_M \cdot \overline{D_K'}$. Suppose Q is a point of $\bar{D} \cdot \overline{D_K'}$. $\bar{H}$ contains $\bar{D}$, thus Q belongs to $\bar{H}$. D contains no point of $\bar{D}_K$, thus Q is a limit point of D. If Q is not a limit point of K, Q is a point of M and therefore Q is in either $\bar{H} \cdot \bar{K}$ or $\bar{D}_M$. Suppose Q is in $\bar{D}_M \cdot \bar{D}_K$. Then Q belongs to $\bar{D}_M \cdot \overline{D_K'}$ and hence to $\bar{M} \cdot \bar{K}$. $\bar{M}$ is a subset of $\bar{H}$, therefore Q is in $\bar{H} \cdot \bar{K}$.

$\bar{H} \cdot \bar{K}$ contains $\bar{D}_H \cdot \bar{D}_K$ and is a subset of it, thus $\bar{H} \cdot \bar{K}$ is $\bar{D}_H \cdot \bar{D}_K$. Therefore either $\bar{D}_H$ and $\bar{D}_K$ are mutually exclusive or $\bar{H} \cdot \bar{K}$ is $\bar{D}_H \cdot \bar{D}_K$.

THEOREM 3. *If H and K are two mutually separated point sets and one of them has a compact boundary then H and K are strongly separated.*

Proof. Suppose H has a compact boundary β. Suppose β is a subset of $\bar{K}$. Since H and K are mutually separated, H is a domain. $S - \bar{H}$ is a domain D containing K. $\bar{H}$ contains β and since $\bar{D}$ contains $\bar{K}$, it contains β. Therefore β is a subset of $\bar{H} \cdot \bar{K}$ and $\bar{H} \cdot \bar{D}$. Each point of $\bar{H} \cdot \bar{K}$ is in β, thus $\bar{H} \cdot \bar{D}$ contains $\bar{H} \cdot \bar{K}$. Each point of $\bar{H} \cdot \bar{D}$ is in β, thus $\bar{H} \cdot \bar{K}$ is $\bar{H} \cdot \bar{D}$ and the domains H and D satisfy the conclusion of this theorem. Now suppose β is not a subset of $\bar{K}$. Since β is closed and compact, the closure of the set of all points of β that do not belong to $\bar{K}$ is compact and it follows from Theorem 2 that H and K are strongly separated.

THEOREM 4. *If S is connected, the point set M separates the point set H from the point set K in S, and the boundary of M is compact, then some closed subset of M separates H from K.*

Proof. $S - M$ is the sum of two mutually separated point sets U and V containing H and K respectively. Since S is connected, U and $M + V$ are not mutually separated. Since U and V are mutually separated, either U contains a limit point of M or M contains a limit point of U, thus the boundary of U exists and is a subset of the boundary of M and hence is compact. It follows from Theorem 3 that there exist two domains D_U and D_V containing U and V respectively such that either $\bar{D}_U$ and $\bar{D}_V$ are mutually exclusive or $\bar{D}_U \cdot \bar{D}_V$ is $\bar{U} \cdot \bar{V}$. D_U and D_V are mutually exclusive domains containing H and K respectively and S is connected, therefore $S - (D_U + D_V)$ is a closed subset of M which separates H from K.

THEOREM 5. *If the hypothesis of the continuum is true, there exists a separable space which satisfies Moore's Axioms* 0 *and* 1-6 [**2**] *and contains two conditionally compact point sets H and K such that*

(1) *$\bar{H}$ and $\bar{K}$ are mutually exclusive,*

(2) *there exists a domain containing H whose closure does not intersect $\bar{K}$ and there exists a domain containing K whose closure does not intersect $\bar{H}$, and*

(3) *H and K are not strongly separated.*

Proof. Let F denote the collection of all nondecreasing infinite sequences of positive integers. Let M denote the set of all points in a Cartesian plane E whose coordinates are positive integers. Let $\mathscr{H}$ denote the collection of subsets of M to which h belongs if and only if the points of h are the points of a sequence $P_1, P_2, P_3, \ldots$ such that P_1 is $(1, 1)$ and for each positive integer n, P_{n+1} is at a distance of 1 from P_n and either above it or to the right of it. For each element h of $\mathscr{H}$, let G_h denote the collection of all images of h in M under a translation in E throwing $(0, 0)$ into $(k, -k)$ for some integer k.

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If the hypothesis of the continuum is true, it follows from [**1**, Theorem 1] that there exists an uncountable subcollection $\mathscr{H}'$ of $\mathscr{H}$ such that

(1) if h and h' are two elements of $\mathscr{H}'$, no point set of G_h contains infinitely many points of any one point set of $G_{h'}$ and

(2) if M' is an infinite subset of M, there exists a point set h of $\mathscr{H}'$ such that some element of G_h contains infinitely many points of M'.

Let $\mathscr{G}$ denote the collection to which g belongs if and only if g is in G_h for some h in $\mathscr{H}'$. There exists a reversible transformation T from F to $\mathscr{H}'$. If A and B are two points of E, let AB denote the straight line interval having end points A and B and let $L(AB)$ denote its length.

Suppose P is a point (x, y) of M, k is a number between 0 and 1, and g is an element of $\mathscr{G}$ containing P. Let A_P and A_P' denote two points of a line having slope -1 and containing P such that (1) A_P is above A_P', (2) P is the mid-point of A_PA_P', and (3) the length of this interval is $(x + y)^{-1}$. Let B_P and B_P' denote the points of A_PA_P' which lie in the order $A_PB_PPB_P'A_P'$ such that B_PB_P' is the middle third interval of A_PA_P'. For some element h of $\mathscr{H}'$, g is an element of G_h and for some positive integer i, P is the ith point of g. Let v denote the ith term of the sequence f of F such that $T(f) = h$. Let C_P, D_P, E_P, E_P', D_P', and C_P' denote the points of B_PB_P' in the order indicated from B_P to B_P' such that each of the intervals B_PC_P, C_PD_P, D_PE_P, $E_P'D_P'$, $D'C'$, and $C'B'$ has length $[3(v + 6)(x + y)]^{-1}$.

Let A_{Pk} denote the point of A_PB_P such that $L(B_PA_{Pk}) = k[L(A_PB_P)]$. Let C_{Pk} denote the point of B_PC_P such that $L(B_PC_{Pk}) = k[L(B_PC_P)]$. Let D_{Pk} denote the point of D_PE_P such that $L(D_PD_{Pk}) = k[L(D_PE_P)]$. Let E_{Pk} denote the point of PE_P such that $L(PE_{Pk}) = k[L(PE_P)]$. Let A_{Pk}', C_{Pk}', D_{Pk}', and E_{Pk}' denote the images of A_{Pk}, C_{Pk}, D_{Pk}, and E_{Pk} respectively under a rotation of A_PA_P' about P which throws A_P into A_P'.

Let $P_1, P_2, P_3, \ldots$ denote the points of g. Let s_{g0} denote $\sum P_jP_{j+1}$. Let t_{g0} denote $\sum B_{P_j}B_{P_{j+1}}$ and let t_{g0}' denote $\sum B_{P_j}'B_{P_{j+1}}'$. Let s_{g1} denote

$$\sum (E_{P_j}E_{P_{j+1}} + E_{P_j}'E_{P_{j+1}}').$$

Let s_{gk}, s_{gk}', t_{gk}, and t_{gk}' denote the point sets obtained by substituting E, D, A, and A' for X and E', D', C, and C' for Y respectively in the expression

$$\sum (X_{P_jk}X_{P_{j+1}k} + Y_{P_jk}Y_{P_{j+1}k}).$$

Suppose g is a point set of the collection $\mathscr{G}$, k is a number between 0 and 1, n is a positive integer, and z is some one of the symbols s, t, and t'. Let ${}_zR_{gkn}$ denote the set to which w belongs if and only if either

(1) w is z_{g0},

(2) for some number k' between 0 and k, w is $z_{gk'}$, or

(3) w is a point of E which is separated from $(0, 0)$ by the sum of the point sets z_{gk} and the interval A_PA_P' for the nth point P of g.

Let ${}_uR_{gkn}$ denote the set to which w belongs if and only if either

(1) w is s_{g1},

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(2) for some number k' between k and 1, w is either $s_{gk'}$ or $s_{gk'}'$, or

(3) w is a point of E which is separated from $(0, 0)$ by the sum of s_{gk}, s_{gk}', $D_{Pk}E_{Pk}$, and $D_{Pk}'E_{Pk}'$ for the nth point P of g.

Suppose g is in G, k_1 and k_2 are numbers between 0 and 1, n is a positive integer, and z is some one of the symbols s, s', t, and t'. Let ${}_zR_{gk_1k_2n}$ denote the set to which w belongs if and only if either

(1) for some number k between k_1 and k_2, w is z_{gk} or

(2) w is a point of E which is separated from $(0, 0)$ by the sum of z_{gk_1}, z_{gk_2}, and the intervals $X_{Pk_1}X_{Pk_2}$ for the nth point P of g where X is some one of the symbols A, C, D, E, A', C', D', and E'.

Let Σ_1 denote a space such that

(1) X is a point in Σ_1 if and only if either X is a point of E or X is s_{gk}, s_{gk}', t_{gk}, or t_{gk}' for some g and k and

(2) R is a region in Σ_1 if and only if either R is the interior of a circle in E, R is ${}_zR_{gkn}$ for some z, g, k, and n, or R is ${}_zR_{gk_1k_2n}$ for some z, g, k_1, k_2, and n.

The space Σ_1 satisfies Axioms 0 and 1-6 and is separable.

Let H denote the point set to which h belongs if and only if for some point P of M, h is B_P. Let K denote the set of all points B_P'. Let D_U denote the point set to which the point w belongs if and only if for some point P of M, w is a point of the interior of a square with center B_P and one vertex P. Let D_V denote a similar set but with B_P' as the center of each square. D_U and D_V are domains containing H and K respectively. Since $\bar{H} - H$ is the set of all points t_{g0} of Σ_1 and those points of $\bar{D}_V - D_V$ which do not lie in E lie in a subset of the set of all points s_{gk}, s_{gk}', and t_{gk}' and since $E \cdot (\bar{D}_V - D_V)$ does not intersect $\bar{H}$, $\bar{H}$ and $\bar{D}_V$ are mutually exclusive. Similarly $\bar{K}$ and $\bar{D}_U$ are mutually exclusive.

Suppose D_H is a domain containing H and D_K is a domain containing K. For each point P of M, there exists a number r_P such that each point of the interval PB_P nearer to B_P than r_P is in D_H and each point of the interval PB_P' nearer to B_P' than r_P is in D_K. There exists a sequence $r_1, r_2, r_3, \ldots$ such that for each positive integer n, $r_n < r_P$ for every point $P = (x, y)$ of M such that $x + y < n + 2$. There exists an increasing sequence of integers $k_1, k_2, k_3, \ldots$ such that for each positive integer n, $[(k_n + 6)(n + 1)]^{-1} < r_n$. This sequence is a member f of the collection F. $T(f)$ is a point set h of the collection $\mathscr{H}$. Let $P_1, P_2, P_3, \ldots$ denote the points of h and for each positive integer n, let x_n and y_n denote the abscissa and ordinate respectively of P_n. $x_n + y_n = n + 1$ and

$$L(B_{P_n}E_{P_n}) = [(k_n + 6)(n + 1)]^{-1}$$

thus E_{P_n} is a point of D_H. Similarly, E_{P_n}' belongs to D_K. Each of the sequences $E_{P_1}, E_{P_2}, E_{P_3}, \ldots$ and $E_{P_1}', E_{P_2}', E_{P_3}', \ldots$ converges to the point s_{h1}. Therefore $\bar{D}_H$ intersects $\bar{D}_K$.

Theorem 6. *If the hypothesis of the continuum is true, there exists a separable* *space satisfying Axioms* 0 *and* 1-6 [**2**] *and containing two conditionally compact point sets* H *and* K *such that*

(1) $\bar{H}$ *and* $\bar{K}$ *are mutually exclusive,*

(2) K *is a domain, and*

(3) H *and* K *are not strongly separated.*

Proof. Notation not introduced here will be that of the previous theorem. Suppose P is a point (x, y) of M, k is a number between 0 and 1, and g is a point set of the collection $\mathscr{G}$ containing P. Let E_P'' denote the midpoint of the interval $A_P'B_P'$. Let $P_1, P_2, P_3, \ldots$ denote the points of g.

Let s_{gk} denote the point set to which w belongs if and only if for some positive integer j, w is either (1) a point of the straight line interval $E_{P_jk}E_{P_{j+1}k}$ or (2) a point of the straight line interval whose endpoints are the point of $P_jE_{P_j}''$ whose distance from P_j is $k[L(P_jE_{P_j}'')]$ and the point of $P_{j+1}E_{P_{j+1}}''$ whose distance from P_{j+1} is $k[L(P_{j+1}E_{P_{j+1}}'')]$. Let s_{gk}' denote the point set to which w belongs if and only if for some positive integer j, w is either (1) a point of the straight line interval $D_{P_jk}D_{P_{j+1}k}$ or (2) a point of the straight line interval whose endpoints are the point of $A_{P_j}'E_{P_j}''$ whose distance from A_{P_j}' is $k[L(A_{P_j}'E_{P_j}'')]$ and the point of $A_{P_{j+1}}'E_{P_{j+1}}''$ whose distance from $A_{P_{j+1}}'$ is $k[L(A_{P_{j+1}}'E_{P_{j+1}}'')]$. Let s_{g1} denote the sum of all the straight line intervals $E_{P_j}E_{P_{j+1}}$ and $E_{P_j}''E_{P_{j+1}}''$. s_{g0}, t_{g0}, and t_{gk} are as described in the previous theorem.

Let ${}_sR_{gkn}$, ${}_tR_{gkn}$, ${}_uR_{gkn}$, ${}_sR_{gk_1k_2n}$, ${}_{s'}R_{gk_1k_2n}$, and ${}_tT_{gk_1k_2n}$ denote the set described in the preceding theorem taking into account the new definitions of s_{gk}, s_{gk}', and s_{g1} in the paragraph above.

Let Σ_2 denote a space such that (1) X is a point in Σ_2 if and only if either X is a point of E or X is s_{gk}, s_{gk}', or t_{gk} for some g and k and (2) R is a region in Σ_2 if and only if either R is the interior of a circle in E, R is ${}_zR_{gkn}$ for some z, g, k, and n, or R is ${}_zR_{gk_1k_2n}$ for some z, g, k_1, k_2, and n.

The space Σ_2 is separable and satisfies Axioms 0 and 1-6.

Let H denote the set of all points B_P. Let K denote the point set to which k belongs if and only if for some point P of M, k is a point of the interior of a square with center B' and a vertex at P.

Suppose D_H is a domain containing H. For each point P of M, there exists a number r_P such that each point of the interval PB_P nearer to B_P than r_P is in D_H. There exists, as in the preceding example, a point set h of the collection $\mathscr{H}'$ such that if $P_1, P_2, P_3, \ldots$ denote the points of h, then for each positive integer n, E_{P_n} is a point of D_H. But, for each positive integer n, E_{P_n}'' is a point of $\bar{K}$. The sequences $E_{P_1}, E_{P_2}, E_{P_3}, \ldots$ and $E_{P_1}'', E_{P_2}'', E_{P_3}'', \ldots$ converge to the point s_{h1}, thus if D_K is a domain containing K, $\bar{D}_H$ intersects $\bar{D}_K$.

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