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CONTINUOUS TRACE C^*-ALGEBRAS WITH GIVEN DIXMIER-DOUADY CLASS

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Abstract

We give an explicit construction of a continuous trace C^*-algebra with prescribed Dixmier-Douady class, and with only finite-dimensional irreducible representations. These algebras often have non-trivial automorphisms, and we show how a recent description of the outer automorphism group of a stable continuous trace C^*-algebra follows easily from our main result. Since our motivation came from work on a new notion of central separable algebras, we explore the connections between this purely algebraic subject and C^*-algebras.

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Let A be a continuous trace C^*-algebra with paracompact spectrum T. Dixmier and Douady [4] constructed a cohomology class $\delta(A) \in H^3(T, \mathbf{Z})$, now known as the Dixmier-Douady class of A, which vanishes exactly when A is the C^*-algebra defined by a continuous field of Hilbert spaces over T [3, 10.7.15]. This invariant has attracted considerable attention in recent years since in the case of separable algebras $\delta(A)$ determines A up to stable isomorphism. (This can easily be deduced from, for example, [2, Théorème 2], [8, Lemma 1.11] and [3, 10.8.4].)

Dixmier and Douady also showed in [4] that every class in $H^3(T, \mathbf{Z})$ is $\delta(A)$ for some A. Their proof of this uses Zorn's lemma and the fact that when H is an infinite-dimensional Hilbert space the sheaf of germs of $U(H)$-valued functions is

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soft, and all the irreducible representations of the resulting algebra are infinite-dimensional. We present here an explicit construction of a continuous trace C^*-algebra with prescribed Dixmier-Douady class. The irreducible representations of the C^*-algebra we construct are all finite-dimensional; as it is easy to see that $n\delta(A) = 0$ when A is n-homogeneous [5, Proposition 1.4] it follows that our algebra is in general far from homogeneous.

This construction is the content of our first section. In Section 2 we discuss the automorphism groups of the algebras we have built. For any continuous trace C^*-algebra A with spectrum T and any automorphism $\alpha \in \operatorname{Aut}_{C(T)} A$ there is a cohomology class $\zeta(\alpha) \in H^2(T, \mathbf{Z})$ which vanishes when α is implemented by a multiplier, and the main theorem of [8] asserts that when A is stable and separable every class in $H^2(T, \mathbf{Z})$ arises this way. The proof of this in [8] is modelled on the surjectivity argument of Dixmier-Douady, and is not constructive; however, for suitable algebras of the type in Section 1 we can write down automorphisms corresponding to given elements of $H^2(T, \mathbf{Z})$, and we use this to give a short proof of the theorem in [8].

The construction we describe here arose in connection with work on a notion of central separable algebra which does not require that the algebra have an identity [14], [11], and when the spectrum T is compact our C^*-algebras are also central separable algebras in this sense. In our third section we discuss the relationship between these central separable algebras and continuous trace C^*-algebras.

Our notation concerning continuous trace C^*-algebras will more or less conform to that of [3, Chapter 10]. If H is a continuous field of Hilbert spaces over T with continuous sections $\Gamma(H)$, we denote the corresponding field of elementary C^*-algebras by $\mathfrak{A}(H)$ and write $\Gamma(\mathfrak{A}(H))$ for the C^*-algebra defined by H. In Section 3 it will be crucial that we are working with purely algebraic tensor products, so we shall always write $\overline{\otimes}$ when we mean to take a completion.

1. The construction of continuous trace C^*-algebras with prescribed Dixmier-Douady class

Let T be a paracompact space, and let $\mathscr{R}$, $\mathscr{S}$ respectively denote the sheaves of germs of continuous real and S^1-valued functions on T. Then the covering map $t \to \exp 2\pi i t$ gives a short exact sequence

$$0 \to \mathbf{Z} \to \mathscr{R} \to \mathscr{S} \to 0$$

of sheaves, which in turn gives a long exact sequence of cohomology

$$\ldots \to H^2(T, \mathscr{R}) \to H^2(T, \mathscr{S}) \to H^3(T, \mathbf{Z}) \to H^3(T, \mathscr{R}) \to \ldots.$$

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The sheaf $\mathscr{R}$ is fine, so $H^n(T, \mathscr{R}) = 0$ for $n > 0$, and the middle map is an isomorphism. The Dixmier-Douady class of a continuous trace C^*-algebra A with paracompact spectrum T is by definition the image of a class in $H^2(T, \mathscr{S})$ [3, Section 10.7], so we may as well start with a 2-cocycle $\lambda_{ijk}: N_{ijk} \to S^1$ relative to a locally finite open cover $\{N_i: i \in I\}$. We can always replace the cover by another $\{M_i: i \in I\}$ with $\overline{M_i} \subset N_i$, so we may also suppose that each λ_{ijk} is defined on the closure $\overline{N_{ijk}}$.

THEOREM 1. *Let $\{N_i: i \in I\}$ be a locally finite cover of a locally compact paracompact space T by relatively compact open sets, and suppose that $\lambda_{ijk}: \overline{N_{ijk}} \to S^1$ is a 2-cocycle. Let*

$$A_1 = \left\{ \sum_{j,k \in I} \phi_{jk} e_{jk} \middle| \phi_{jk} \in C(T), \phi_{jk} \equiv 0 \textit{ outside } N_{jk} \right\}$$

have the obvious structure as a $C(T)$-module, multiplication defined by

$$(\phi_{jk} e_{jk})(\psi_{lm} e_{lm}) = \delta_{kl} \psi e_{jm},$$

where δ_{kl} is the Kronecker delta and ψ is given by

$$\psi(t) = \begin{cases} \overline{\lambda_{jkm}(t)}\, \phi_{jk}(t) \psi_{km}(t) & \textit{for } t \in N_{jkm}, \\ 0 & \textit{for } t \notin N_{jkm}, \end{cases}$$

and involution defined by

$$(\phi_{jk} e_{jk})^* = \overline{\phi_{jk}}\, e_{kj}.$$

For $t \in T$ let $I(t) = \{i \in I: t \in N_i\}$: note that $n_t = |I(t)|$ is finite. If $t \in N_{ij}$ then for the usual $n_t \times n_t$ matrix norm we have

$$\left\| \left(\lambda_{ikl}(t) \phi_{kl}(t)\right)_{k,l \in I(t)} \right\| = \left\| \left(\lambda_{jkl}(t) \phi_{kl}(t)\right)_{k,l \in I(t)} \right\|,$$

so that for each t we have a semi-norm $\|\cdot\|_t$ on A_1. Let A be the set of $a \in A_1$ such that $t \to \|a\|_t$ vanishes at infinity, and set $\|a\| = \sup \|a\|_t$. Then A is a continuous trace C^*-algebra with spectrum T whose Dixmier-Douady class $\delta(A) \in H^3(T, \mathbf{Z}) = H^2(T, \mathscr{S})$ is represented by the cocycle $\{N_i, \lambda_{ijk}\}$. The dimension of the irreducible representation corresponding to $t \in T$ is n_t.

PROOF. Simple calculations using the cocycle identity show that A is a $*$-algebra with the above operations, and that for $t \in N_i$

$$\pi_{i,t}\left(\sum \phi_{jk} e_{jk}\right) = \left(\overline{\lambda_{ijk}(t)}\, \phi_{jk}(t)\right)_{j,k \in I(t)}$$

defines a $*$-representation of A into $M_{n_t}(\mathbf{C})$. If $t \in N_{ij}$ and $D(\mu_k)$ is the diagonal matrix with entries μ_k, then the cocycle identity yields

$$D\left(\overline{\lambda_{ijk}(t)}\right)\left[\left(\overline{\lambda_{ikl}(t)}\, \phi_{kl}(t)\right)_{k,l \in I(t)}\right] D\left(\lambda_{ijl}(t)\right) = \left[\left(\overline{\lambda_{jkl}(t)}\, \phi_{kl}(t)\right)_{k,l \in I(t)}\right].$$

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The diagonal matrices are unitary, so we deduce that the norms of the matrices $\pi_{i,t}(a)$ and $\pi_{j,t}(a)$ are always equal, and we have well-defined semi-norms

$$\|a\|_t = \|\pi_{i,t}(a)\| \quad \text{for } t \in N_i,$$

as claimed. The norm on A satisfies the C^*-condition $\|aa^*\| = \|a\|^2$ since each $\|\cdot\|_t$ does, and it is not hard to see that A is complete, so A is a C^*-algebra.

For each $t \in T$ we define an ideal in A_1 by

$$J_t = \left\{ \sum_{j,k} \phi_{jk} e_{jk} \in A \,|\, \phi_{jk}(t) = 0 \text{ for all } j, k \right\}.$$

We denote the quotient C^*-algebra A/J_t by $A(t)$, and we write $a(t)$ for the image of $a \in A_1$ in $A(t)$. Note that if $t \in N_i$ the representation $\pi_{i,t}$ induces an isomorphism of $A(t)$ onto M_{n_t}, so each $A(t)$ is an elementary C^*-algebra. In fact, $\mathfrak{A} = \{A(t), A_1\}$ is a continuous field of elementary C^*-algebras over T such that A is the C^*-algebra of continuous sections vanishing at infinity. For by definition $\{a(t)\colon a \in A\}$ is all of $A(t)$, and the continuity of

$$t \to \|a(t)\| = \|\pi_{i,t}(a)\|$$

follows from the continuity of the matrix norm (note that if $t_\alpha \to t$ then $I(t_\alpha)$ eventually contains $I(t)$). Further, if $x = (x(t)) \in \prod A(t)$ then there are unique scalars $\nu_{jk}(t)$ such that

$$\pi_{i,t}(x(t)) = \left(\overline{\lambda_{ijk}(t)}\, \nu_{jk}(t)\right)_{j,k \in I(t)}.$$

If x is locally uniformly approximable by elements of A_1, then standard arguments show that the ν_{jk} are continuous and vanish off N_{jk}, so that x is the section defined by $\sum \nu_{jk} e_{jk} \in A$. Thus $\mathfrak{A}$ is a continuous field as asserted and $A = \Gamma_0(\mathfrak{A})$ has spectrum T. It is easy to see that $\mathfrak{A}$ satisfies Fell's condition (for example, if $t \in N_i$, $\rho \equiv 1$ near t and $\rho \equiv 0$ off N_i, then $(\rho e_{ii})(s)$ is a rank one projection for s near t) and hence A has continuous trace.

To compute the Dixmier-Douady class of A we build fields H_i of Hilbert spaces over $\overline{N_i}$ and isomorphisms of the associated fields of elementary C^*-algebras $\mathfrak{A}(H_i)$ onto $\mathfrak{A}|_{\overline{N_i}}$. For $t \in \overline{N_i}$ we define

$$H_i(t) = \left\{ \sum_{k \in I(t)} \lambda_k e_k \colon \lambda_k \in \mathbf{C} \right\},$$

with the usual inner product $(e_k|e_l) = \delta_{kl}$, and take as our space of continuous sections

$$\Gamma(H_i) = \left\{ \sum_{k \in I_i} \phi_k e_k \colon \phi_k \in C(\overline{N_i}), \phi \equiv 0 \text{ off } N_k \right\},$$

where $I_i = \{k\colon N_k \cap \overline{N_i} \neq \{\varnothing\}\}$. It is routine to check that this does define a continuous field of Hilbert spaces. The corresponding field $\mathfrak{A}_i = \mathfrak{A}(H_i)$ of

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elementary C^*-algebras is that generated by fields of the form $e \otimes \bar{f}$ for $e, f \in \Gamma(H_i)$, where $x \otimes \bar{y}$ denotes the rank one operator $z \to (z|y)x$. We define a linear map from $\Gamma(H_i) \otimes \Gamma(\overline{H}_i)$ to $A|_{\overline{N}_i}$ by

$$h_i\left(\phi e_j \otimes \overline{\psi e_k}\right) = \theta e_{jk}, \text{ where } \theta(t) = \begin{cases} \lambda_{ijk}(t)\phi(t)\overline{\psi(t)} & \text{for } t \in \overline{N_{ijk}}, \\ 0 & \text{for } t \in \overline{N}_i \setminus N_{jk}; \end{cases}$$

a standard Urysohn's lemma argument shows that θe_{jk} is in fact the restriction of an element of A. Further, h_i is a $*$-homomorphism, is isometric from the usual norm on $\Gamma(\mathfrak{A}_i)$ to the given one on A, and is easily seen to be surjective; hence it extends to an isomorphism of $\mathfrak{A}_i$ onto $\mathfrak{A}|_{\overline{N}_i}$. (In fact, every θe_{jk} is the image of an elementary tensor so if the index set I is finite h_i defines an isomorphism of the algebraic tensor product $\Gamma(H_i) \otimes_{C(\overline{N}_i)} \Gamma(H_i)$ onto $A|_{\overline{N}_i}$. As the latter algebra is complete so is the algebraic tensor product, which therefore equals $\Gamma(\mathfrak{A}_i)$.) We now define isomorphisms $g_{ij}: H_j|_{\overline{N_{ij}}} \to H_i|_{\overline{N_{ij}}}$ by

$$g_{ij}(t)(\phi_k(t)e_k) = \begin{cases} \lambda_{ijk}(t)\phi_k(t)e_k & \text{if } t \in \overline{N_{ijk}}, \\ 0 & \text{if } t \in \overline{N_{ij}} \setminus N_k. \end{cases}$$

The induced isomorphism Ad g_{ij} of $\mathfrak{A}(H_j)$ into $\mathfrak{A}(H_i)$ is given on elementary tensors by

$$\begin{aligned}(\mathrm{Ad}\, g_{ij})(t)\left(\phi_k(t)e_k \otimes \overline{\psi_l(t)e_l}\right) &= g_{ij}(t)\phi_k(t)e_k \otimes \overline{g_{ij}(t)\psi_l(t)e_l} \\ &= \begin{cases} \lambda_{ijk}(t)\phi_k(t)e_k \otimes \overline{\lambda_{ijl}(t)\psi_l(t)e_l} & \text{if } t \in \overline{N_{ijkl}}, \\ 0 & \text{if } t \in \overline{N_{ij}} \setminus N_{kl}, \end{cases}\end{aligned}$$

so that routine calculations using the cocycle identity give

$$h_i(t) \circ (\mathrm{Ad}\, g_{ij})(t) = h_j(t) \quad \text{for } t \in \overline{N_{ij}}.$$

Thus g_{ij} defines the isomorphism $h_i^{-1}h_j$ as in [3, 10.7.11], and for $t \in \overline{N_{ijk}}$ we have

$$g_{ij}(t)g_{jk}(t) = \lambda_{ijk}(t)g_{ik}(t),$$

so that the class $\delta(A) = \gamma(\mathfrak{A})$ in $H^2(T, \mathscr{S})$ is represented by the cocycle $\{N_i, \lambda_{ijk}\}$ as claimed (see [3, 10.7.12–14]).

REMARKS. 1. If T has covering dimension n, then we can realise any class in $H^2(T, \mathscr{S})$ as a cocycle relative to a cover where at most $n + 1$ different sets intersect. Because the Dixmier-Douady class determines a separable continuous trace C^*-algebra up to stable isomorphism, our theorem implies that every such algebra with spectrum T is stably isomorphic to an algebra whose irreducible representations have dimension $\leqslant n + 1$. This has already been shown by Brown [1, Corollary 2.11] using different reasoning.

2. The last part of the proof could be simplified a bit by constructing the class $\delta(A)$ as in [9, 2.6–2.9] using local rank one projections and intertwining partial isometries rather than fields of Hilbert spaces and isomorphisms. However, in our present proof we also showed that, when T is compact, the algebraic tensor product $\Gamma(H_i) \otimes_{C(T)} \Gamma(\overline{H}_i)$ is complete, and this has some interesting algebraic consequences, which we shall discuss in Section 3.

3. The C^*-algebra A we construct in Theorem 1 can also be viewed as a twisted groupoid C^*-algebra. For simplicity we suppose T is compact and $\{N_i\colon i \in I\}$ is a finite cover. Then we let ψ be the local homeomorphism of the disjoint union $X = \bigcup_i N_i$ onto T, let $\mathscr{R}(\psi)$ be the equivalence relation induced on X by ψ as in [6, Section 4], and let G be the corresponding topological groupoid with left Haar system induced by counting measure on the fibres of ψ (see [12, Section 1.2]). Given a cocycle $\lambda_{ijk}\colon N_{ijk} \to S^1$ we define a 2-cocycle $\sigma\colon G^2 \to S^1$ by

$$\sigma((x, y), (y, z)) = \lambda_{i(x)i(y)i(z)}(\psi(x)),$$

where G^2 denotes the set of composable elements of G and $i\colon X \to I$ is defined by $x \in N_{i(x)}$. We define $\Phi\colon C_c(G) \to A$ by $\Phi f = \sum \phi_{jk} e_{jk}$, where

$$\phi_{jk}(t) = \begin{cases} f(x, y) & \text{if } x \in N_j, y \in N_k \text{ and } \psi(x) = \psi(y) = t, \\ 0 & \text{if } t \notin N_{jk}. \end{cases}$$

It is routine to verify that Φ defines a $*$-monomorphism of the convolution algebra $C_c(G, \sigma)$ (see [12, Section 4.1]) onto a dense subalgebra of A, and hence gives an isomorphism of $C^*(G, \sigma)$ with A.

In particular, when $\lambda_{ijk} \equiv 1$ the algebra A is the C^*-algebra $C^*(\psi)$ associated by Kumjian [6] to the local homeomorphism ψ. This can also be seen directly: his imprimitivity bimodule $l^2(\psi)$ is isomorphic to

$$H = \left\{ \sum_k \phi_k e_k \colon \phi_k \in C(T), \phi_k \equiv 0 \text{ outside } N_k \right\}$$

with $C(T)$-valued inner product given by

$$\left(\sum \phi_k e_k \middle| \sum \psi_l e_l \right) = \sum \overline{\phi_k} \psi_l,$$

and A acts on H by

$$\left(\sum \phi_{jk} e_{jk} \right) \left(\sum \psi_l e_l \right) = \sum_j \left(\sum_k \phi_{jk} \psi_k \right) e_j.$$

This is, of course, the same construction as we carried out locally to prove our theorem, modulo changes in convention regarding inner products.

2. Automorphisms

For any continuous trace C^*-algebra A with paracompact spectrum T there is an exact sequence

$$0 \to \operatorname{Inn} A \to \operatorname{Aut}_{C(T)} A \xrightarrow{\zeta_A} H^2(T, \mathbf{Z}),$$

where $\operatorname{Inn} A$ denotes the group of A determined by multipliers of A (see [13, Section 5]). We shall now investigate the range of the homomorphism ζ_A for the algebra A constructed in Theorem 1.

So let $\{N_i, \lambda_{ijk}\}$ and A be as in Theorem 1, and let $c \in H^2(T, \mathbf{Z})$ be given. If c can be represented by a 1-cocycle $\{N_i, \mu_{ij}\}$ with values in $\mathscr{S}$ relative to the same cover $\{N_i\}$, then we can define an automorphism α of A by

$$\alpha\left(\sum \phi_{jk} e_{jk}\right) = \sum \psi_{jk} e_{jk} \text{ where } \psi_{jk}(t) = \begin{cases} \mu_{jk}(t)\phi_{jk}(t) & \text{if } t \in N_{jk}, \\ 0 & \text{otherwise.} \end{cases}$$

This is easily seen to be a $C(T)$-module automorphism: we compute its class $\zeta(\alpha)$ in $H^1(T, \mathscr{S}) \cong H^2(T, \mathbf{Z})$. Let $\{M_i\}$ be an open cover of T with $\overline{M_i} \subset N_i$, and choose continuous functions $\rho_i\colon T \to [0,1]$ such that $\rho_i \equiv 1$ on M_i and $\rho_i \equiv 0$ off N_i. We can now define multipliers m_i of A by

$$m_i = \sum_{j \in I} \rho_i \overline{\mu_{ij}} e_{jj};$$

note that although $\rho_i \mu_{ij}$ is not defined on all of T, whenever we have $\phi \equiv 0$ off N_{jk} the function $\rho_i \overline{\mu_{ij}} \phi$ does extend to be continuous on T, and simple calculations show that under the usual multiplication rule this gives a multiplier of A. Further, the cocycle identity shows that for $t \in M_i$

$$\alpha\left(\sum \phi_{jk} e_{jk}\right)(t) = \left(m_i\left(\sum \phi_{jk} e_{jk}\right) m_i^*\right)(t),$$

so that m_i implements α over M_i. Moreover, the same cocycle identity also gives

$$\mu_{ij}(t) m_j(t) = m_i(t) \quad \text{for } t \in M_{ij},$$

and we deduce that $\zeta(\alpha)$ is represented by the cocycle $\{M_i, \mu_{ij}\}$ (see [13, Section 5]). This defines the same class as the one we started with, and therefore $\zeta(\alpha) = c$.

Of course, we cannot expect to represent an arbitrary class in $H^2(T, \mathbf{Z})$ relative to a fixed open cover (and we will come back to this question later), but if we start with classes $d \in H^3(T, \mathbf{Z})$ and $c \in H^2(T, \mathbf{Z})$ then we can always represent them as S^1-valued cocycles relative to the same cover. Hence the argument in the preceding paragraph proves the following result:

PROPOSITION 2. *Let T be a locally compact paracompact space, and let $d \in H^3(T, \mathbf{Z})$, $c \in H^2(T, \mathbf{Z})$. Then there are a continuous trace C^*-algebra A with spectrum T, with $\delta(A) = d$, and with only finite-dimensional irreducible representations, and an automorphism $\alpha \in \operatorname{Aut}_{C(T)} A$ such that $\zeta_A(\alpha) = c$.*

REMARK. Alex Kumjian has noticed independently that, given a cocycle μ_{ij}: $N_{ij} \to S^1$, one can write down an automorphism α of the C^*-algebra $C^*(\psi)$ associated to the local homeomorphism ψ: $\bigcup_i N_i \to T$ such that $\zeta(\alpha)$ is represented by $\{N_i, \mu_{ij}\}$. In fact, it was his observation that alerted us to the realisation of A as the twisted groupoid C^*-algebra $C^*(G, \sigma)$ (see Remark (3) in Section 1). The automorphism α can be conveniently viewed in this realisation too: define a continuous 1-cocycle c: $G \to S^1$ by

$$c(x, y) = \mu_{i(x)i(y)}(\psi(x)),$$

and then the automorphism α of $C^*(G, \sigma)$ is defined by

$$\alpha(f)(x, y) = c(x, y)f(x, y) \quad \text{for } f \in C_c(G),$$

as in [12, Proposition II.5.1].

COROLLARY 3 ([8, Theorem 2.1]). *Let A be a separable stable continuous trace C^*-algebra with spectrum T. Then the homomorphism ζ_A is surjective.*

PROOF. Let $c \in H^2(T, \mathbf{Z})$. Then by the proposition there are an algebra B and an automorphism $\alpha \in \operatorname{Aut}_{C(T)} B$ such that $\delta(B) = \delta(A)$ and $\zeta_B(\alpha) = c$. Since the Dixmier-Douady class determines a separable continuous trace C^*-algebra up to stable isomorphism, we have

$$A \cong A \mathbin{\overline{\otimes}} K(H) \cong B \mathbin{\overline{\otimes}} K(H),$$

and we may assume that this isomorphism induces the identity map from $T = \hat{A}$ to $T = \hat{B} = (B \mathbin{\overline{\otimes}} K(H))\hat{}$ (see [10, Lemma 4.3]). If $m \in M(B)$ implements α over N then

$$m \otimes 1 \in M(B) \mathbin{\overline{\otimes}} M(K(H)) \subset M\big(B \mathbin{\overline{\otimes}} K(H)\big)$$

implements $\alpha \otimes \mathrm{id}$ over N, so $\zeta_{B \otimes K(H)}(\alpha \otimes \mathrm{id}) = \zeta_B(\alpha) = c$, and the corresponding automorphism β of A therefore satisfies $\zeta_A(\beta) = c$.

REMARK. Corollary 3 holds for arbitrary C^*-algebras with paracompact spectrum T [10, Corollary 3.12]. However, the proof given there involves establishing surjectivity for $A = C_0(T, K(H))$ first, and this seems to be more complicated than our Proposition 2.

As we have seen above, if A is the C^*-algebra of Theorem 1 corresponding to a cocycle λ_{ijk}: $\overline{N_{ijk}} \to S^1$ then the range of ζ_A contains the subgroup $H^1(\mathscr{N}, \mathscr{S})$ of $H^1(T, \mathscr{S}) \cong H^2(T, \mathbf{Z})$ consisting of those classes realisable on the cover $\mathscr{N} = \{N_i\}$. It is quite easy to see that, while this range need not be all of $H^2(T, \mathbf{Z})$, it may contain more than $H^1(\mathscr{N}, \mathscr{S})$. First of all, take the one set cover $\{T\}$ of a space with $H^2(T, \mathbf{Z}) \neq 0$; then $A \cong C_0(T)$ and the range of ζ_A is $\{0\} \neq H^2(T, \mathbf{Z})$.

This example is more general than it appears, since if Y is a compact set contained in only one member of the cover, then $A|_Y \cong C(Y)$ and no element of $H^2(T, \mathbf{Z})$ whose image in $H^2(Y, \mathbf{Z})$ is non-zero can come from a $C(T)$-automorphism of A. On the other hand, if we take the trivial cover $N_1 = N_2 = T$ consisting of two sets, then $A = C(T, M_2)$ and there can be automorphisms α for which $\zeta(\alpha) \neq 0$ and hence does not belong to $H^1(\mathcal{N}, \mathcal{S})$. This argument is also more general than it first appears, since if Y is a compact subset of $N_1 \cap N_2$ which meets no other N_i, then $A|_Y \cong C(Y, M_2)$ and there could be automorphisms of A which do not trivialise over $N_1 \cap N_2$.

3. Central separable algebras

Let R be a commutative ring with identity, let A be an R-algebra (not necessarily with an identity) and let $Z(A)$ denote the ring of $A - A$ bimodule endomorphisms of A. There is always a natural map i: $R \to Z(A)$ and we call A central if this is an isomorphism. Following [14, Section 2] we say A is separable if $A^2 = A$, A is projective as an $A - A$ bimodule, and for each maximal ideal M of $Z(A)$ we have $MA \neq A$. An immediate property of such algebras is that the multiplication map: $A \otimes_R A \to A$ is split as an $A - A$ bimodule homomorphism. Our main theorem can be strengthened as follows.

PROPOSITION 4. *Let T be a compact Hausdorff space and let $d \in H^3(T, \mathbf{Z})$. Then there is a continuous trace C^*-algebra A with $\delta(A) = d$ which is also a central separable $C(T)$-algebra.*

PROOF. Let $\{N_i\}$ be a finite open cover of T such that d is represented by a cocycle λ_{ijk}: $\overline{N_{ijk}} \to S^1$, and let A be the algebra constructed in Theorem 1. Choose another cover $\{M_i\}$ with $\overline{M_i} \subset N_i$, and functions $\rho_i \in C_0(N_i)$ with $\rho_i \equiv 1$ on $\overline{M_i}$. We define $A - A$ bimodule homomorphisms ω_i on $A \otimes_{C(T)} A$ by interchanging the two copies of $\Gamma(\overline{H_i})$ in

$$A \otimes_{C(T)} A|_{\overline{N_i}} \cong \Gamma(H_i) \otimes_{C(T)} \Gamma(\overline{H_i}) \otimes_{C(T)} \Gamma(H_i) \otimes_{C(T)} \Gamma(\overline{H_i})$$

and multiplying by ρ_i. Let $\tilde{\Omega}$ be the module of $A - A$ homomorphisms generated by the ω_i, and let tr: $A \otimes_{C(T)} \tilde{\Omega} \to Z(A)$ be as in [14, page 174]. We have $Z(A) = C(T)$, since $Z(A)$ is by definition the centre of the multiplier algebra of A. Thus it will follow from [14, Proposition 3.8] that A is central separable if we can prove that the range of tr is not contained in any maximal ideal of $C(T)$. However, a straightforward calculation shows that if $t \in M_i$ then $\operatorname{tr}(\rho_i e_{ii} \otimes \omega_i)$ cannot vanish at t, and the result is proved.

This last result raises an obvious question: what is the relationship between the classes of C^*-algebras with compact spectrum T and central separable algebras over $C(T)$? It is well-known that the central separable $C(T)$-algebras with identity are precisely the locally homogeneous C^*-algebras with spectrum T, but the situation for algebras without identity is rather more complicated. For example, the algebra $FR(H)$ of finite rank operators on a separable (!) Hilbert space H is central separable over $\mathbf{C}$ (we have $FR(H) = H \otimes_{\mathbf{C}} \overline{H}$) but it is not a C^*-algebra unless H is finite-dimensional. Our next proposition gives an answer to this question—and shows, among other things, that Proposition 4 does say more than Theorem 1.

PROPOSITION 5. (1) *Let A be a C^*-algebra with compact Hausdorff spectrum T which is also a central separable $C(T)$-algebra for the natural action of $C(T)$ on A. Then A is a continuous trace C^*-algebra whose irreducible representations have finite, bounded dimensions.*

(2) *There are continuous trace C^*-algebras A with compact spectrum and $\{\dim \pi: \pi \in \hat{A}\}$ bounded which are not central separable algebras.*

The proof of this result will depend on a series of simple lemmas. We begin with a purely algebraic result which is implicit in [14].

LEMMA 6. *Let B be a central separable algebra over a commutative ring R, and suppose that $p \in B$ is a rank one idempotent (i.e., $pBp = Rp$). Then the map $ap \otimes pb \to apb$ induces an isomorphism of $Bp \otimes_R pB$ onto B.*

PROOF. This is a consequence of the proofs of Proposition 4.2 and 4.3 of [14] with $N = Bp$, $M = pB$ and $\lambda: N \otimes_R M \to B$ given by the multiplication in B; the regularity of N, M follows from [14, Propositions 1.1 and 1.6]. The proof of Proposition 4.2 shows that if $A = M \otimes_B N$, then λ induces an isomorphism of $N \otimes_A M$ onto B. However, the last argument in the proof of Proposition 4.3 shows that the multiplication also induces an isomorphism of $A = pB \otimes_B Bp$ onto pBp, which is just Rp since p is rank one. We therefore deduce that $B \cong Bp \otimes_R pB$ as claimed.

LEMMA 7. *Let A be a C^*-algebra which is also a central separable $C(T)$-algebra. Then every closed 2-sided ideal in A is regular, and has the form IA for some 2-sided ideal I in $C(T)$.*

PROOF. The multiplication map $A \otimes_{C(T)} A \to A$ is split and A is therefore a regular 2-sided A-module. If M is a 2-sided ideal in A, then we have $MA = M$

(see, for example, [7, 1.4.5]) and M is regular by [14, Proposition 1.6]. The result now follows from [14, Proposition 3.5].

LEMMA 8. *Let A be a C^*-algebra which is central separable over* $\mathbf{C}$. *Then* $A \cong M_n(\mathbf{C})$.

PROOF. Let $\pi\colon A \to B(H)$ be an irreducible representation; by Lemma 7 A has no non-trivial ideals so π must be faithful. By [14, Proposition 4.8] A must contain a rank one idempotent p: we claim that $P = \pi(p) \in B(H)$ is also rank one. For suppose $\xi \in PH$, $\xi \neq 0$ and $\eta \in P^*H$ satisfies $(\xi|\eta) = 0$. Then for any $a \in A$

$$(\eta|\pi(a)\xi) = (P^*\eta|\pi(a)P\xi) = (\eta|\pi(pap)\xi) \in \mathbf{C}(\eta|\xi) = 0.$$

Since $P^*H = ((I-P)H)^\perp$ this says that

$$\eta \perp \xi, \eta \perp (I-P)H \Rightarrow \eta \perp \pi(A)\xi$$

and because π is irreducible it follows that ξ and $(I-P)H$ span H. Thus $PH = \mathbf{C}\xi$ and P is rank one. The irreducibility of π implies that $\pi(A) \supset K(H)$, and as A has no ideals $\pi(A) = K(H)$. But the latter consists of finite rank operators so H must be finite-dimensional. This will be a $*$-isomorphism if P is chosen so that $P^* = P$.

PROOF OF PROPOSITION 5(1). Let $\pi\colon A \to B(H)$ be irreducible. By Lemma 7 $\ker\pi$ is a regular ideal of the form IA for some ideal I in $C(T)$. The extension $\bar{\pi}$ of π to the multiplier algebra restricts to a representation of $C(T)$ in $\pi(A)' = \mathbf{C}1$, and hence $\ker\bar{\pi}$ is the ideal I_t of functions vanishing at some point t of T. We then have $I_tA \subset \ker\pi$ and the maximality of I_t shows that $I_tA = \ker\pi$. Thus by [14, Propositions 2.7 and 3.5] $A/\ker\pi$ is central separable over $C(T)/I_t \cong \mathbf{C}$, hence isomorphic to $M_n(\mathbf{C})$ by Lemma 8, and π is finite-dimensional.

We now prove that A satisfies Fell's condition. Let $\pi \in \hat{A}$ and choose $a \in A$ such that $\pi(a)$ is a rank one projection. The map $\rho \to \|\rho(a)\|$ is continuous on $\hat{A}$, so $\|\rho(a)^2 - \rho(a)\|$ is small for ρ near π, and if $f \equiv 1$ near 1, $f \equiv 0$ near 0 then $p = f(a)$ will satisfy $\rho(p)^2 = \rho(p) = \rho(p)^*$ for ρ near π. If N is a compact neighbourhood of π then $pAp|_N = A/I_NA$ is central separable over $C(N)$ and has an identity; therefore by shrinking N we may suppose $pAp|_N \cong C(N, M_n)$ for some n. As $\pi(p)$ is rank one, $n = 1$ and $\rho(p)$ is rank one throughout N. Thus A has continuous trace.

Suppose now that $\pi_n \in \hat{A}$ and $\dim\pi_n \geqslant n$ for all n. As $\hat{A}$ is compact, we may assume $\pi_n \to \pi$ (technically, we might have to pass to a subnet, but the idea's the same). Pick $a_n \in A$ with $a_n \geqslant 0$ and rank $\pi_n(a_n) \geqslant n$, and let $a = \sum_{k=1}^{\infty} 2^{-k}a_k$.

Then $a \geqslant 0$ and we have

$$\operatorname{rank} \pi_n(a) = \operatorname{rank}\left(\sum_{k=1}^{\infty} \pi_n(2^{-k}a_k)\right) \geqslant \operatorname{rank} \pi_n(2^{-n}a_n) \geqslant n.$$

By restricting to a compact neighbourhood N of π we may suppose that A has a rank one idempotent p, and then by Lemma 6 multiplication gives an isomorphism μ: $Ap \otimes_{C(N)} pA \to A$. In particular, we can write

$$a = \mu\left(\sum_{i=1}^{m} a_i p \otimes pb_i \right) = \sum_{i=1}^{m} a_i pb_i;$$

but this is impossible since the rank of $\pi_n(\Sigma a_i pb_i)$ is at most m for each n.

The proof of the second part of Proposition 5 consists of building an example. We are grateful to Shaun Disney for providing the following topological lemma.

LEMMA 9. *Let L_n be the canonical complex line bundle over complex projective space $\mathbf{C}P^n$. Then any n sections of L_n have a common zero.*

PROOF. Let $\xi_1, \ldots, \xi_n$ be n sections of $L = L_n$, and suppose they do not simultaneously vanish. Then the direct sum nL of n copies of L has a non-vanishing section, and so can be decomposed as $nL \cong \mathbf{1} \oplus F$, where $\mathbf{1}$ denotes the trivial line bundle. The first Chern class of $\mathbf{1}$ is 0, so

$$c_1(L)^n = c_n(nL) = c_1(\mathbf{1})c_{n-1}(F) = 0.$$

But the cohomology ring $H^*(\mathbf{C}P^n, \mathbf{Z})$ is a truncated polynomial ring generated by $c_1(L)$, and in particular $H^{2n}(\mathbf{C}P^n, \mathbf{Z}) \cong \mathbf{Z}$ is generated by $c_1(L)^n$. We therefore have a contradiction, and the ξ_i must vanish simultaneously.

COROLLARY 10. *Let L_n be the field of Hilbert spaces over $\mathbf{C}P^n$ obtained by putting a Hermitian structure on the canonical line bundle, and let $A_n = \Gamma(\mathfrak{A}(L_n))$ be the C^*-algebra defined by L_n. Then the identity in A_n cannot be written in the form $\sum_{i=1}^{m} \xi_i \otimes \bar{\eta}_i$ for $\xi_i, \eta_i \in \Gamma(L_n)$ unless $m > n$. (In fact $A_n \cong C(\mathbf{C}P^n)$, but this is not important here.)*

PROOF OF PROPOSITION 5(2). Let L_n be as in Corollary 10, and define a field of Hilbert spaces K over the disjoint union $X = \bigcup_{n=1}^{\infty} \mathbf{C}P^n$ by taking $K = L_n$ on $\mathbf{C}P^n$. We now define a field H over the compactification $T = X \cup \{\infty\}$ by

$$H(x) = \mathbf{C} \oplus K(x) \cong \mathbf{C}^2, \qquad H(\infty) = \mathbf{C}, \qquad \Gamma(H) = C(T) \oplus \Gamma_0(K).$$

Let A be the C^*-algebra $\Gamma(\mathfrak{A}(H))$ defined by H; we claim that A is not just $\Gamma(H) \otimes_{C(T)} \Gamma(\overline{H})$. For A contains the closure of $\Gamma_0(K) \otimes_{C(T)} \Gamma_0(K)$, which is the c_0-direct sum of the C^*-algebras A_n. If we define $f \in A_n$ by

$$f(x) = \frac{1}{n} 1_{K(x)} \quad \text{if } x \in \mathbf{C}P^n,$$

then f cannot be written in the form $\sum_{i=1}^m \xi_i \otimes \bar{\eta}_i$ for any finite m, and so does not belong to the algebraic tensor product $\Gamma_0(K) \otimes \Gamma_0(\bar{K})$; this justifies the claim. The algebra A contains the idempotent $p = 1_{C(T)} \otimes 1_{C(T)}$, and it is easy to see that $pAp = C(T)$, so p is rank one, and if A were central separable we would have $A \cong Ap \otimes_{C(T)} pA$ by Lemma 6. However, $Ap = \Gamma(H)$ so we have just shown this is not the case.

Finally, we observe that, although continuous trace C^*-algebras with compact spectrum are not in general central separable, they do always have a dense ideal which is. For any continuous trace C^*-algebra with spectrum T can be constructed from a cover $\{N_i\}$ of T, fields of Hilbert spaces H_i over N_i, and isomorphisms $h_{ij}\colon H_j|_{N_{ij}} \to H_i|_{N_{ij}}$ which satisfy

$$\operatorname{Ad} h_{ij}(t) \circ \operatorname{Ad} h_{jk}(t) = \operatorname{Ad} h_{jk}(t) \quad \text{for } t \in N_{ijk}$$

[3, 10.7.11]. For convenience we suppose N_i is compact. Then the algebraic tensor product $A_i = \Gamma(H_i) \otimes_{C(N_i)} \Gamma(\bar{H}_i)$ is a central separable $C(N_i)$-algebra, and the isomorphisms $\operatorname{Ad} h_{ij}$ map $A_j|_{N_{ij}}$ onto $A_i|_{N_{ij}}$. We can therefore use them to piece together a central separable algebra (cf. the proof of Proposition 4) which is clearly dense in A. Conversely, if A is a C^*-algebra with Hausdorff spectrum T and A contains a dense central separable $C(T)$-subalgebra then as in the proof of Proposition 5 it is not hard to see that A satisfies Fell's condition. Thus this property characterises continuous trace C^*-algebras.

References

[1] L. G. Brown, 'Stable isomorphism of hereditary subalgebras of C^*-algebras', *Pacific J. Math.* **71** (1977), 335–348.

[2] J. Dixmier, 'Champs continus d'espaces hilbertiens et de C^*-algèbres', II, *J. Math. Pures Appl.* **42** (1963), 1–20.

[3] J. Dixmier, *C^*-algebras*, North-Holland, Amsterdam, 1977.

[4] J. Dixmier and A. Douady, 'Champs continus d'espaces hilbertiens et de C^*-algèbres, *Bull. Soc. Math. France* **91** (1963), 227–284.

[5] A. Grothendieck, 'Le groupe de Brauer, I: algèbres d'Azumaya et interprétations diverses', *Séminaire Bourbaki* 1964/65, exposé 290.

[6] A. Kumjian, 'Preliminary C^*-algebras arising from local homeomorphisms', *Math. Scand.* **52** (1983), 269–278.

[7] G. K. Pedersen, *C^*-algebras and their automorphism groups*, Academic Press, London, 1979.

[8] J. Phillips and I. Raeburn, 'Automorphisms of C^*-algebras and second Cech cohomology', *Indiana Univ. Math. J.* **29** (1980), 799–822.
[9] J. Phillips and I. Raeburn, 'Perturbations of C^*-algebras, II', *Proc. London Math. Soc.* (3) **43** (1981), 46–72.
[10] J. Phillips and I. Raeburn, 'Crossed products by locally unitary automorphism groups and principal bundles', *J. Operator Theory* **11** (1984), 215–241.
[11] I. Raeburn and J. L. Taylor, 'The bigger Brauer group and étale cohomology', *Pacific J. Math.*, to appear.
[12] J. Renault, *A groupoid approach to C^*-algebras*, (Lecture Notes in Mathematics, vol. 793), Springer-Verlag, Berlin and New York, 1980.
[13] M. J. Russell, 'Automorphisms and derivations of continuous trace C^*-algebras', *J. London Math. Soc.* (2) **22** (1980), 139–145.
[14] J. L. Taylor, 'A bigger Brauer group', *Pacific J. Math.* **103** (1982), 163–203.

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