108.06 Simple bounds on a sum pertinent to primes

Given a positive integer n and a prime p, there is a unique integer $e = e_p(n)$ such that n is divisible by p^e but not by p^{e+1}. Legendre's identity states that

$$e_p(n!) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \ \dots \tag{1}$$

where $\lfloor x \rfloor$, the integer part of x, indicates the greatest integer that does not exceed x. Clearly, the sum (1) includes only a finite number of non-zero terms.

One quick (and obvious) application of Legendre's identity (see [1] and [2]) is the following theorem on a prime sum considered in Mertens [3]. Note that the proof of the theorem subsumes the essential ideas inherent in Cohen [2] and Mertens [3].

Theorem: Let n be a positive integer. Then

$$\frac{1}{4} \ln n \leqslant \sum_{p \leqslant n} \frac{\ln p}{p} \leqslant 2 \ln n,$$

where the sum is taken over all primes $p \leqslant n$.

Proof: Observe that $\dfrac{1}{n - 1} \leqslant \dfrac{2}{n}$ for $n \geqslant 2$. Legendre's identity gives

$$n! \leqslant \prod_{p \leqslant n} p^{\frac{n}{p} + \frac{n}{p^2} + \dots} = \prod_{p \leqslant n} p^{\frac{n}{p-1}} \leqslant \left(\prod_{p \leqslant n} p^{\frac{1}{p}} \right)^{2n}$$

and

$$n! \geqslant \prod_{p \leqslant n} p^{\frac{n}{p} - 1} = \left(\prod_{p \leqslant n} p^{-1} \right) \left(\prod_{p \leqslant n} p^{\frac{n}{p}} \right) \geqslant \frac{1}{n!} \left(\prod_{p \leqslant n} p^{\frac{1}{p}} \right)^{n}$$

for integers $n \geqslant 2$. Since $r(n - r + 1) - n = (r - 1)(n - r) \geqslant 0$ when $1 \leqslant r \leqslant n$, it follows that

$$n^n \leqslant [1.n][2(n-1)][3(n-2)] \dots [r(n-r+1)] \dots [(n-1).2][n.1]$$
$$= (n!)^2 \leqslant n^{2n}.$$

Taking logarithms proves the theorem.

Remark: Note that the theorem implies that there always exists a prime p such that $n < p < n^{10}$ for natural numbers $n \geqslant 2$, since

$$\sum_{p \leqslant n^{10}} \frac{\ln p}{p} - \sum_{p \leqslant n} \frac{\ln p}{p} \geqslant \frac{1}{4} (\ln n^{10}) - 2 \ln n = \frac{1}{2} \ln n,$$

which is greater than 0 for $n > 1$. Thus, there exist infinitely many primes.

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HAZAR AYDIN
*Undergraduate Student, Department of Foreign Language Education,
Bogaziçi Üniversitesi, Bebek,
34342 Beşiktaş/Istanbul, Turkey*
e-mail: *hazar.aydin@yandex.com*

108.07 De Moivre's theorem via difference equations

An alternative proof of De Moivre's theorem to the usual methods of *mathematical induction* and *exponential form* (e.g. [1]) is given that is based on the solution of *linear difference equations* [2]. The derivation is surprisingly straightforward and emerges from two trigonometric identities. It should be stressed that this derivation is not claimed to be in any way superior to traditional approaches, only an interesting different approach to achieve the same result. We also see that the difference equation/recurrence formula gives a useful way of expressing $\cos n\theta$ and $\sin n\theta$ as sums of powers of $\cos\theta$ and $\sin\theta$, as well as the reverse process of expressing powers of $\cos\theta$ and $\sin\theta$ as a sum of multiple angles of these functions.

First, we consider the well-known *addition of cosines* identity [3]

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) \equiv 2\cos\alpha\cos\beta$$

with $\alpha = (n+1)\theta$ and $\beta = \theta$, to give

$$\cos(n+2)\theta + \cos n\theta \equiv 2\cos(n+1)\theta\cos\theta$$

$$\text{or} \qquad c_{n+2} + c_n = 2c_1c_{n+1}, \qquad \text{where} \quad c_k = \cos k\theta$$

$$\text{i.e.} \qquad c_{n+2} - 2c_1c_{n+1} + c_n = 0 \tag{1}$$

$n = 0, 1, 2, 3, \ldots$, with initial conditions $c_0 = 1$, $c_1 = \cos\theta$.

Notice that (1) is a linear, constant-coefficient difference equation as c_1 is a fixed value for any particular θ. Following the procedure for solving such equations [2], the *auxiliary equation* for (1) is

$$m^2 - 2c_1m + 1 = 0$$