Note on the Elementary Divisors of Some Related Determinants.

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§ 1. Introductory

The matrices considered in the following note are non-singular, and are related to a given matrix A having elements taken from the field of positive and negative integers and zero. The invariant properties[1] of such a matrix A, under multiplications by matrices of determinant equal to unity, can be formulated, as is well known, in terms of the "elementary divisors" of the determinant $|A|$. Thus if A is of the nth order, and p is a prime occurring in $|A|$ to the power h_n, in the H.C.F. of the first minors of $|A|$ to the power h_{n-1}, in the H.C.F. of the second minors to the power h_{n-2}, and so on, h_0 by convention being zero, then the first differences of the h's,

$$e_r = h_r - h_{r-1}, \qquad (r = 1, 2, \ldots n)$$

are invariant under the transformations considered, and it is known that $e_r \geqslant e_{r-1}$. The numbers

$$E_r = \Pi \, p^{e_r}, \qquad (r = 1, 2, \ldots n)$$

where the product includes all prime factors of $|A|$, are called the elementary divisors of $|A|$, and

$$|A| = E_1 E_2 \ldots E_n.$$

It is further known that for any matrix A in the field considered, others P and Q exist, such that P and Q have determinants equal to 1, and

$$A = PCQ,$$

where the diagonal elements of C are simply the elementary divisors E_r, the non-diagonal elements being zero. C is called the "normal form" of A.

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[1] See Muir, *History of the Theory of Determinants*, vol. IV, chap. XXI, p. 439 (on H. J. S. Smith).

In a previous paper[1] we considered certain matrices related to A. For the sake of uniformity we redesignate these here, the "mth compound" by $A^{(m)}$, the "mth Schläflian" by $A^{[m]}$, the "mth Burnside matrix" by $A^{\{m\}}$, the last being the case, when all the matrices A_j are identical with A, of the "multilinear matrix" of $A_1, A_2, \ldots A_m$, denoted here by $(A_j)^{\{m\}}$, in the former paper by $M(A_j)$. At the same time we shall denote the number of ways of choosing m things from n when repetitions are not allowed by $(n)_m$, when repetitions are allowed by $[n]_m$, while we shall write n^m as $\{n\}_m$.

With these preliminaries we propose to find the elementary divisors of $|A^{(m)}|$, $|A^{[m]}|$, $|(A_j)^{\{m\}}|$, and $|A^{\{m\}}|$, given those of $|A|$, or of $|A_1|, |A_2|, \ldots |A_m|$.

§ 2. General Rules for Finding the Elementary Divisors.

The chief results of the earlier paper can now be summarised as

$$(AB \ldots K)^{(m)} \equiv A^{(m)} B^{(m)} \ldots K^{(m)} \tag{1}$$

$$(AB \ldots K)^{[m]} \equiv A^{[m]} B^{[m]} \ldots K^{[m]} \tag{2}$$

$$(A_j B_j \ldots K_j)^{\{m\}} \equiv (A_j)^{\{m\}} (B_j)^{\{m\}} \ldots (K_j)^{\{m\}} \tag{3}$$

$$(AB \ldots K)^{\{m\}} \equiv A^{\{m\}} B^{\{m\}} \ldots K^{\{m\}}. \tag{3a}$$

Hence we have $\quad A^{(m)} \equiv P^{(m)} C^{(m)} Q^{(m)}$, by (1),

and the three analogous results. $|P^{(m)}|$, $|Q^{(m)}|$, $|P^{[m]}|$, $|Q^{[m]}|$, etc. being, as is known, powers of $|P|$ and $|Q|$ and thus all equal to unity, we have only to consider the form of $C^{(m)}$, $C^{[m]}$, $(C_j)^{\{m\}}$ and $C^{\{m\}}$. These are all diagonal matrices, and slight consideration of the ways in which they are formed from C tells us that the exponents of elementary divisors of $|A^{(m)}|$, $|A^{[m]}|$, $|(A_j)^{\{m\}}|$ and $|A^{\{m\}}|$ for each prime p can be found as follows:—

1. *Compounds.* From the n exponents in $|C|$, $(e_n, e_{n-1}, \ldots e_1)$, we take in all possible ways m without repetition, and sum each of the $(n)_m$ sets of m so obtained. These $(n)_m$ sums, when arranged in descending order of magnitude, say

$$e_r^{(m)}, \qquad (r = 1, 2, \ldots (n)_m)$$

are the exponents of the elementary divisors of $|A^{(m)}|$ for the prime p, and the divisors themselves are

$$E_r^{(m)} = \Pi\, p^{e_r^{(m)}}. \qquad (r = 1, 2, \ldots (n)_m)$$

[1] On the Latent Roots of Certain Matrices. *Proc. Edin. Math. Soc.*, Ser. 2, Vol. I, p. 135.

2. *Schläflians.* We proceed just as in the case of compounds, except that in the choice of sets of m exponents repetitions are now allowed. We thus obtain $[n]_m$ sums $e_r{}^{[m]}$, and the elementary divisors of $|A^{[m]}|$ are

$$E_r{}^{[m]} = \Pi\, p^{e_r^{[m]}}. \qquad (r = 1, 2, \ldots [n]_m)$$

3. *Multilinears.* From the m brackets, each of n exponents, for the prime p in $|A_1|$, $|A_2|$, $\ldots |A_m|$, we choose in all possible ways an exponent from each bracket, arranging in descending order the sums $e_r{}^{\{m\}}$ so obtained. Then the elementary divisors of $|(A_j)^{\{m\}}|$ are

$$E_r{}^{\{m\}} = \Pi\, p^{e_r^{\{m\}}}. \qquad (r = 1, 2, \ldots \{n\}_m)$$

3a. *Burnside Matrices.* The procedure resembles that followed in the case of compounds and of Schläflians, except that now not only are repetitions of exponents chosen from the bracket permitted, but *different orders* of the same exponents, though giving equivalent sums, are counted as distinct.

The foregoing results apply equally well to the case where the elements of the matrices concerned are not necessarily integers, but rational integral functions of a variable; and in general to cases in which the highest common factor of any two elements can be found by the familiar process of division.