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RESEARCH ARTICLE

Semistable degenerations of Calabi–Yau manifolds and mirror P=W conjectures

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Abstract

Mirror symmetry for a semistable degeneration of a Calabi–Yau manifold was first investigated by Doran–Harder–Thompson when the degenerate fiber is a union of two quasi-Fano manifolds. They proposed a topological construction of a mirror Calabi–Yau by gluing of two Landau–Ginzburg models that are mirror to those Fano manifolds. We extend this construction to a general type semistable degeneration where the dual boundary complex of the degenerate fiber is the standard N-simplex. Since each component in the degenerate fiber comes with the simple normal crossing anticanonical divisor, one needs the notion of a hybrid Landau–Ginzburg model – a multipotential analogue of classical Landau–Ginzburg models. We show that these hybrid Landau–Ginzburg models can be glued to be a topological mirror candidate for the nearby Calabi–Yau, which also exhibits the structure of a Calabi–Yau fibration over $\mathbb{P}^N$. Furthermore, it is predicted that the perverse Leray filtration associated to this fibration is mirror to the monodromy weight filtration on the degeneration side [12]. We explain how this can be deduced from the original mirror P=W conjecture [18].

Contents

1. Introduction

Traditionally, mirror symmetry is a conjectural relationship between two compact Kähler n-dimensional Calabi–Yau manifolds X and $X^\vee$: The complex (algebraic) geometry of X (B-side) is equivalent to the symplectic geometry of $X^\vee$ (A-side) and vice versa [1][19]. In [26], Strominger–Yau–Zaslow proposed a geometric construction of such mirror pair as dual special Lagrangian torus fibrations. This idea extends mirror symmetry beyond the Calabi–Yau case, particularly to the case of quasi-Fano manifolds [2]. We say X is quasi-Fano if the anticanonical divisor is effective, base-point free and $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$. In this case, a mirror object is given by a *Landau–Ginzburg (LG) model* $(Y, \omega, \mathsf{w} : Y \to \mathbb{C})$ where (Y, ω) is an n-dimensional Kähler Calabi–Yau manifold and $\mathsf{w} : Y \to \mathbb{C}$ is a locally trivial symplectic fibration near infinity. We refer readers to [2] for more details.

1.1. Generalization of Doran–Harder–Thompson construction

A relationship between two different kinds of mirror symmetries for Calabi–Yau manifolds and quasi-Fano manifolds was first addressed by Doran–Harder–Thompson in the case of Tyurin degenerations [11]. Recall that Tyurin degeneration is a semistable degeneration of a Calabi–Yau manifold X into a union of two quasi-Fano varieties $X_1 \cup X_2$ over the smooth anticanonical hypersurface $X_{12} := X_1 \cap X_2$. This degeneration restricts the behavior of the normal bundles of X_{12} in X_1 and X_2 to be inverse to each other. For $i = 1, 2$, suppose one has a mirror LG model (Y_i, w_i) for each pair (X_i, X_{12}) and generic fibers of each w_i are topologically the same. Note that the anticanonical divisor $-K_{X_i}$ is mirror to the monodromy of a generic fiber $\mathsf{w}_i^{-1}(t)$ near the infinity. Then by the adjunction formula, the relation on the normal bundles of X_{12} corresponds to the condition that the monodomies of $\mathsf{w}_i^{-1}(t)$ are inverse to each other. This allows one to topologically glue two LG models to obtain a mirror candidate of X which is also equipped with the map to $\mathbb{P}^1$.[1] One natural question is how to generalize this construction when X degenerates into a more general simple normal crossing variety.

Question 1.1. How do we extend the construction of Doran–Harder–Thompson for a semistable degeneration of more general types?

The aim of this article is to answer this question for a certain case and study related topics. We use a recently developed language of hybrid LG models [24]. This is a multipotential analogue of classical LG model, whose idea goes back to [3, Section 5.3]. A triple $(Y, \omega, h : Y \to \mathbb{C}^N)$, called a *hybrid LG model of rank N* if (Y, ω) is a Kähler Calabi–Yau manifold of dimension n and $h = (h_1, \ldots, h_N) : Y \to \mathbb{C}^N$ is a Calabi–Yau fibration which is locally trivial around the infinity boundaries of the base (See Definition 3.1). In fact, this turns out to be a suitable model to capture mirror symmetry of the quasi-Fano pair $(X, D = \cup_{i=1}^N D_i)$ in the following way: For any $I \subset \{1, \ldots, N\}$, the induced quasi-Fano pair $(D_I := \cap_{i \in I} D_i, \cup_{j \notin I} D_j \cap D_I)$ is expected to be mirror to the induced hybrid LG model $(Y_I := \cap_{i \in I} \mathsf{w}_i^{-1}(t_i), \omega|_{Y_I}, h|_{Y_I} : Y_I \to \mathbb{C}^{N-|I|})$, where $\mathsf{w}_i^{-1}(t_i)$ is a generic fiber of w_i. We will review the precise notion of the hybrid LG model and the associated mirror symmetry relations in Section 3.

Let's consider a semistable degeneration of a Calabi–Yau manifold X into a simple normal crossing variety $X_c = \cup_{i=0}^N X_i$ whose dual boundary complex is the standard N-simplex. Suppose we have a hybrid LG model $(Y_i, \omega_i, h_i : Y_i \to \mathbb{C}^N)$ mirror to each pair $(X_i, \cup_{j \neq i} X_{ij})$ with additional topological conditions (Hypothesis 4.6). Similar to the Tyurin degeneration case, the semistability corresponds to

[1] Here, we view $\mathbb{P}^1$ as the topological gluing of the base disks of LG models.

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the condition on the monodromies associated to the hybrid LG models (See Ansatz 3.9). By shrinking the base of h_i to a polydisk Δ_{h_i}, this condition allows to topologically glue the hybrid LG models and produces a symplectic fibration $\pi : Y \longrightarrow \mathbb{P}^N$ (Proposition 4.7) where the base Δ_{h_i} is identified with the locus $\{|z_j| \leq |z_i| | j = 0, \dots, N\} \subset \mathbb{P}^N$. We also take a general hyperplane $H \subset \mathbb{P}^N$ and its complement $\mathbb{P}^N \setminus H \cong \mathbb{C}^N$. We write the induced fibration $\tilde{\pi} : \tilde{Y} \longrightarrow \mathbb{C}^N$ for $\tilde{Y} := \pi^{-1}(\mathbb{P}^N \setminus H)$.

Theorem 1.2 (Theorem 4.9). *Suppose that $(X_i, \cup_{j\neq i} X_{ij})$ is topological mirror to $(Y_i, h_i : Y_i \longrightarrow \mathbb{C}^N)$ for all i. Then*

1. *Y is topological mirror to X. In other words, $e(Y) = (-1)^n e(X)$,*
2. *$\tilde{Y}$ is topological mirror to X_c. In other words, $e(\tilde{Y}) = (-1)^n e(X)$,*

where $e(-)$ is the Euler characteristic.

1.2. Mirror P=W conjecture

The topological mirror relation in Theorem 1.2 is the weakest form of the mirror symmetry one would expect. We could ask further about other kinds of mirror symmetry relations for this construction. In the semistable degeneration, there is a geometric autoequivalence on the cohomology of X induced by the monodromy of X around the degenerate fiber X_c. It gives rise to the *monodromy weight filtration* on $H^*(X)$ that constitutes the limiting mixed Hodge structure. On the other hand, on the cohomology of the degenerate fiber X_c, there is Deligne's canonical weight filtration that constitutes the mixed Hodge structure. The natural question is what the corresponding filtrations on the mirror Y and $\tilde{Y}$ are.

Question 1.3. What is the filtration on the cohomology of Y (resp. $\tilde{Y}$) that is mirror to the monodromy weight filtration (resp. Deligne's canonical weight filtration)?

The answer is expected to be the perverse Leray filtration associated to $\pi : Y \longrightarrow \mathbb{P}^N$ (resp. $\tilde{\pi} : \tilde{Y} \longrightarrow \mathbb{C}^N$), as proposed by Doran–Thompson [12, Conjecture 4.3]. The following is a simplified version that we discuss in this article.

Conjecture 1.4.

1. *For X and Y, we have*
$$\dim_{\mathbb{C}} Gr_F^p Gr_{p+q}^{W_{\lim}} H^{p+q+l}(X, \mathbb{C}) = \dim_{\mathbb{C}} Gr_F^{n-q} Gr_{n+p-q}^P H^{n+p-q+l}(Y, \mathbb{C}),$$
where $P_\bullet$ is the perverse Leray filtration associated to π.
2. *For X_c and $\tilde{Y}$, we have*
$$\dim_{\mathbb{C}} Gr_F^p Gr_{p+q}^{W} H^{p+q+l}(X_c, \mathbb{C}) = \dim_{\mathbb{C}} Gr_F^{n-q} Gr_{n+p-q}^P H_c^{n+p-q+l}(\tilde{Y}, \mathbb{C}),$$
where $P_\bullet$ is the perverse Leray filtration associated to $\tilde{\pi}$.

Remark 1.5. We should emphasize that we could not discuss complex geometric properties of the both topological mirror candidates $(Y, \pi : Y \longrightarrow \mathbb{P}^N)$ and $(\tilde{Y}, \tilde{\pi} : \tilde{Y} \longrightarrow \mathbb{C}^N)$ as we do not know how to glue complex structures. This means that the perverse Leray filtrations associated to π and $\tilde{\pi}$ are not the usual ones discussed in the literature [7]. Instead, we consider the potentially equivalent filtrations, called general flag filtrations, whose description is purely topological. See Section 2.3 for more details.

The motivational work for the appearance of the weight and perverse filtration in mirror symmetry, which we shall call *mirror P=W conjecture*, is the proposal of Harder–Katzarkov–Przyjalkowski [18] in the context of mirror symmetry of log Calabi–Yau varieties. For a given log-Calabi–Yau variety U of dimension n, one can consider the mixed Hodge structure on the cohomology $H^*(U)$ that consists of Deligne's canonical weight filtration $W_\bullet$ and Hodge filtration $F^\bullet$. On the other hand, we have a canonical

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affinization map Aff : $U \to \operatorname{Spec} H^0(U, \mathcal{O}_U)$ and it provides the perverse Leray filtration $P_\bullet$. Since those filtrations are compatible with each other, we can define the perverse-mixed Hodge polynomial of U by

$$PW_U(u,t,w,p) := \sum_{a,b,r,s} (\dim_{\mathbb{C}} \mathrm{Gr}_F^a \mathrm{Gr}_{s+b}^W \mathrm{Gr}_{s+r}^P (H^s(U,\mathbb{C})) u^a t^s w^b p^r .$$

Conjecture 1.6 (Mirror P=W conjecture)[18]. *Assume that two n-dimensional log-Calabi–Yau varieties U and $U^\vee$ are mirror to each other. Then we have the following polynomial identity:*

$$PW_U(u^{-1}t^{-2}, t, p, w)u^n t^n = PW_{U^\vee}(u,t,w,p). \tag{1.1}$$

In case that U has compactification (X, D) where X is a smooth (quasi-)Fano and D is simple normal crossing anticanonical divisor, the mirror P=W conjecture can be deduced from mirror symmetry for the pair (X, D). Note that the choice of a pair (X, D) corresponds to the choice of a hybrid LG potential $h : Y \to \mathbb{C}^N$ which plays a role of a proper affinization map. Then mirror symmetry expects that one could match the (part of) E_1-page of the spectral sequence for the weight filtration on $H^*(U)$ with the E_1-page of the spectral sequence for G-flag filtration (=perverse Leray filtration) associated to h on $H^*(Y)$. Explicitly, this is an isomorphism of the E_1-pages

$$\Big(\bigoplus_{p-q=a} \mathrm{Gr}_F^p\, {}^W E^{-l,p+q+l}, d_1\Big) \cong ({}^G E^{-l,n+a+l}, d_1^G) \tag{1.2}$$

where both are known to degenerate at the E_2-page (see Section 2.1,2.3 for the notations). We say the mirror pair $(X, D)|(Y, \omega, h : Y \to \mathbb{C}^N)$ satisfies the mirror P=W conjecture in a strong sense if the relation (1.2) holds.

Theorem 1.7. *Suppose that each mirror pair $(X_i, \cup_{j\neq i} X_{ij})|(Y_i, \omega_i, h_i : Y_i \to \mathbb{C}^N)$ satisfies the mirror P=W conjecture in a strong sense. Then*

1. *for X and Y as above, we have*

$$\bigoplus_{p-q=a} Gr_F^p Gr_{p+q}^{W_{\lim}} H^{p+q+l}(X) \cong Gr_{n+a}^{P^\pi} H^{n+a+l}(Y),$$

2. *for X_c and $\tilde{Y}$ as above, we have*

$$\bigoplus_{p-q=a} Gr_F^p Gr_{p+q}^W H^{p+q+l}(X_c) \cong Gr_{n+a}^{P^{\tilde{\pi}}} H_c^{n+a+l}(\tilde{Y}).$$

The main idea is to apply the gluing property (Proposition 3.2) of each hybrid LG potential $h_i : Y_i \to \mathbb{C}^N$ to describe the E_1-pages of the spectral sequences for $P^\pi_\bullet$ and $P^{\tilde{\pi}}_\bullet$ in a way that they become isomorphic to those for $W_{\lim\bullet}$ and $W_\bullet$, respectively. One of the key lemmas is the Poincaré duality statement for hybrid LG models, which we will prove in Section 7.

Theorem 1.8 (Theorem 7.4)(Poincaré duality). *Let $(Y, h : Y \to \mathbb{C}^N)$ be a rank N hybrid LG model. Then for $a \geq 0$, there is an isomorphism of cohomology groups*

$$H^a(Y, Y_{sm}, \mathbb{C}) \cong H^{2n-a}(Y, Y_{sm}, \mathbb{C})^*,$$

where $n = \dim_{\mathbb{C}} Y$.

1.3. The degeneration-fibration correspondence for Batyrev mirror pairs

One can see that the mirror construction discussed above proposes the conjectural mirror correspondence between semistable degenerations and Calabi–Yau fibrations, which we shall call the *degeneration-*

fibration correspondence. In Section 6, we provide further evidence for such mirror correspondence for Batyrev mirror pairs [4].

We first consider a semistable degeneration of a smooth toric Fano variety X_Δ which is induced by a semistable partition Γ of the polytope Δ (Definition 6.3). The degenerate fiber is the union of toric varieties associated to the maximal subpolytopes $\{\Delta_{(i)}|i = 0, \ldots, N\}$ in Δ for some N. This induces a type $(N+1)$ semistable degeneration of a general Calabi–Yau hypersuface X of X_Δ whose degenerate fiber $X_c = \cup_{i=0}^{N} X_i$ is the simple normal crossing union of general hypersurfaces X_i of $X_{\Delta_{(i)}}$, determined by $\Delta_{(i)}$. We will show that on the mirror side, the partition Γ canonically induces a morphism $\pi : Y \to \mathbb{P}^N$ from a mirror dual Calabi–Yau Y. It follows from the construction that the deepest intersection of the components of the degenerate fiber is mirror to a generic fiber of π. Moreover, as the base of $\pi : Y \to \mathbb{P}^N$ comes with the toric chart, we obtain a natural candidate of a mirror hybrid LG model to $(X_i, \cup_{j\neq i} X_{ij})$. In other words, for each i, we take $\Delta_i := \{|z_j| \leq |z_i||j \neq i\} \subset \mathbb{P}^N$ and set $Y_i := \pi^{-1}(\Delta_i)$ and $h_i := \pi|_{Y_i}$.

Conjecture 1.9. (Conjecture 6.10) *For each i, the hybrid LG model* $(Y_i, h_i : Y_i \to \Delta_i)$ *is mirror to the pair* $(X_i, \cup_{j\neq i} X_{ij})$.

Conjecture 1.9 can be considered as the reverse construction of the topological gluing of hybrid LG models. We leave verifying this conjecture for future work.

2. Backgrounds

In this section, we set up the notations and review basic concepts about mixed Hodge structures and perverse filtrations.

2.1. The weight filtration

We recall basic concepts about the mixed Hodge structures [10][9] following the exposition in [25]. Let U be a smooth quasi-projective variety over $\mathbb{C}$ and (X, D) be a good compactification of U. Recall that a pair (X, D) is called a good compactification of U if X is a smooth and compact variety and D is a simple normal crossing divisor. Let $j : U \to X$ be a natural inclusion. Consider the logarithmic de Rham complex

$$\Omega_X^\bullet(\log D) \subset j_*\Omega_U^\bullet.$$

Locally at $p \in D$ with an open neighborhood $V \subset X$ with coordinates $(z_1, \cdots, z_n)$ in which D is given by $z_1 \cdots z_k = 0$, one can see

$$\Omega_X^1(\log D)_p = \mathcal{O}_{X,p}\frac{dz_1}{z_1} \oplus \cdots \oplus \mathcal{O}_{X,p}\frac{dz_k}{z_k} \oplus \mathcal{O}_{X,p}dz_{k+1} \oplus \cdots \oplus \mathcal{O}_{X,p}dz_n$$
$$\Omega_X^r(\log D)_p = \bigwedge^r \Omega_X^1(\log D)_p.$$

There are two filtrations on the logarithmic de Rham complex $(\Omega_X^\bullet(\log D), d)$:

1. (Hodge filtration) A decreasing filtration $F^\bullet$ on $\Omega_X^\bullet(\log D)$ defined by

$$F^p\Omega_X^\bullet(\log D) := \Omega_X^{\geq p}(\log D).$$

2. (Weight filtration) An increasing filtration $W_\bullet$ on $\Omega_X^\bullet(\log D)$ defined by

$$W_m\Omega_X^r(\log D) := \begin{cases} 0 & m < 0 \\ \Omega_X^r(\log D) & m \geq r \\ \Omega_X^{r-m} \wedge \Omega_X^m(\log D) & 0 \leq m \leq r. \end{cases}$$

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Theorem 2.1 [25, Theorem 4.2].

1. *The logarithmic de Rham complex* $\Omega_X^\bullet(\log D)$ *is quasi-isomorphic to* $j_*\Omega_U^\bullet$*:*

$$H^k(U;\mathbb{C}) = \mathbb{H}^k(X, \Omega_X^\bullet(\log D)).$$

2. *The decreasing filtration* $F^\bullet$ *on* $\Omega_X^\bullet(\log D)$ *induces the filtration in cohomology*

$$F^p H^k(U;\mathbb{C}) = \mathrm{Im}(\mathbb{H}^k(X, F^p\Omega_X^\bullet(\log D)) \to H^k(U;\mathbb{C}))$$

which is called the Hodge filtration on $H^\bullet(U)$*. Similarly, the increasing filtration* $W_\bullet$ *on* $\Omega_X^\bullet(\log D)$ *induces the filtration in cohomology*

$$W_m H^k(U;\mathbb{C}) = \mathrm{Im}(\mathbb{H}^k(X, W_{m-k}\Omega_X^\bullet(\log D)) \to H^k(U;\mathbb{C}))$$

which is called the weight filtration on $H^\bullet(U)$*.*

3. *The package* $(\Omega_X^\bullet(\log D), W_\bullet, F^\bullet)$ *gives a* $\mathbb{C}$*-mixed Hodge structure on* $H^k(U;\mathbb{C})$*.*

Remark 2.2. In general, the weight filtration can be defined over the field of rational numbers $\mathbb{Q}$ so that the $\mathbb{Q}$-mixed Hodge structures are considered. However, as we will mainly focus on filtrations on the cohomology group with complex coefficients, we will not explicitly denote the rational structures in the notation.

The key properties of these two filtrations are the degenerations of the associated spectral sequences. More precisely, we have

Proposition 2.3 [25, Theorem 4.2, Proposition 4.3].

1. *The spectral sequence for* $(\mathbb{H}(X, \Omega_X^\bullet(\log D)), F^\bullet)$ *whose* E_1*-page is given by*

$$E_1^{p,q} = \mathbb{H}^{p+q}(X, Gr_F^p \Omega_X^\bullet(\log D))$$

degenerates at the E_1*-page. Thus, we have*

$$Gr_F^p \mathbb{H}^{p+q}(X, \Omega_X^\bullet(\log D)) = \mathbb{H}^{p+q}(X, Gr_F^p \Omega_X^\bullet(\log D)).$$

2. *The spectral sequence for* $(\mathbb{H}(X, \Omega_X^\bullet(\log D)), W_\bullet)$ *whose* E_1*-page is given by*

$$E_1^{-m,k+m} = \mathbb{H}^k(X, Gr_m^W \Omega_X^\bullet(\log D))$$

degenerates at the E_2*-page and the differential* $d_1 : E_1^{-m,k+m} \to E_1^{-m+1,k+m}$ *is strictly compatible with the filtration* $F_\bullet$*. In other words,*

$$E_2^{-m,k+m} = E_\infty^{-m,k+m} = Gr_{m+k}^W \mathbb{H}^k(X, \Omega_X^\bullet(\log D)).$$

For a given mixed Hodge structure $V = (V_\mathbb{C}, W_\bullet, F^\bullet)$ and $m \in \mathbb{Z}$, we define the m-th *Tate twist* of V by setting $V(m) := (V_\mathbb{C}(m), W(m)_\bullet, F(m)^\bullet)$, where $V_\mathbb{C}(m) := (2\pi i)^m V_\mathbb{C}$ and

$$W(m)_k := W_{k+2m} \quad F(m)^p := F^{m+p}$$

for all k and p.

In order to compute the mixed Hodge structures, we introduce the geometric description of the E_1-page of the spectral sequence. Let D be a simple normal crossing divisor with N irreducible components $D_1, \ldots, D_N$. For any index set $I \subset \{1, \ldots, N\}$, we write $D_I = \cap_{i\in I} D_i$ for the intersection.

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We set $D(k)$ to be the disjoint union of k-tuple intersections of the components of D and $D(0)$ to be X. Also, for $I = (i_1, \cdots, i_m)$ and $J = (i_1, \cdots, \hat{i_j}, \cdots, i_m)$, there are inclusion maps

$$\iota^I_J : D_I \hookrightarrow D_J$$
$$\iota^m_j = \bigoplus_{|I|=m} \iota^I_J : D(m) \hookrightarrow D(m-1)$$

which induce canonical Gysin morphisms on the level of cohomology. Therefore, we have

$$\gamma_m = \bigoplus_{j=1}^{m} (-1)^{j-1} (\iota^m_j)_! : H^{k-m}(D(m))(-m) \to H^{k-m+2}(D(m-1))(-m+1), \tag{2.1}$$

where $(-)_!$ is the Gysin morphism. We call this sign convention the *Mayer-Vietoris sign rule* which is unique up to ± 1. Under the residue map, this gives a geometric description of the differential $d_1 : E_1^{-m,k+m} \to E_1^{-m+1,k+m}$ of the E_1-page of the spectral sequence for the weight filtration as follows:

Proposition 2.4 [25, Proposition 4.7]. *The following diagram is commutative*

$$\begin{array}{ccc} E_1^{-m,k+m} & \xrightarrow{res_m} & H^{k-m}(D(m);\mathbb{C})(-m) \\ \downarrow d_1 & & \downarrow -r_m \\ E_1^{-m+1,k+m} & \xrightarrow{res_{m-1}} & H^{k-m+2}(D(m-1);\mathbb{C})(-m-1), \end{array} \tag{2.2}$$

where res_m is the residue map for all $m \geq 0$.

Note that all the morphisms in the diagram (2.2) are compatible with Hodge filtration $F^\bullet$. This description provides several computational tools as well as functorial properties of the mixed Hodge structures under geometric morphisms. For more details, we refer the reader to [25].

One can extend the above construction to the case when U is singular. This can be done by taking a simplicial or cubical resolution of the singular variety U and associated good compactifications. We will not review this construction but describe one particular case which we will deal with.

Example 2.5. Let D be a simple normal crossing variety. Consider the long exact sequences

$$0 \to \bigoplus_{|I|=1} H^j(D_I) \xrightarrow{d_0} \bigoplus_{|I|=2} H^j(D_I) \xrightarrow{d_1} \bigoplus_{|I|=3} H^j(D_I) \xrightarrow{d_2} \cdots, \tag{2.3}$$

where d_i is the alternating sum of the restriction map. Then we have

$$\mathrm{Gr}^W_j H^{i+j}(D) = \frac{\ker d_i}{\mathrm{Im}\, d_{i+1}}.$$

In fact, the sequence (2.3) is the E_1-page of the spectral sequence for the weight filtration $W_\bullet$. It is also compatible with the Hodge filtrations on each term to yield the Hodge filtration on $H^*(D)$.

2.2. The monodromy weight filtration

Let X be a smooth complex manifold and Δ be the unit disk. We consider a holomorphic map $f : X \to \Delta$ that is smooth over the punctured disk $\Delta^* := \Delta \setminus \{0\}$. We also assume that $E := f^{-1}(0)$ is a simple normal crossing divisor. Let E_i be the components of E and write

$$E_I = \bigcap_{i \in I} E_i, \qquad E(m) = \coprod_{|I|=m} E_I$$

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as before. We present the de Rham theoretic description of the monodromy weight filtration on a generic fiber $X_t := f^{-1}(t)$.

Define the relative de Rham complex on X with logarithmic poles along E:

$$\Omega^\bullet_{X/\Delta}(\log E) := \Omega^\bullet_X(\log E)/f^*(\Omega^1_\Delta(\log 0)) \wedge \Omega^{\bullet-1}_X(\log E).$$

By definition, this fits into the short exact sequence

$$0 \to f^*(\Omega^1_\Delta(\log 0)) \wedge \Omega^{\bullet-1}_X(\log E) \to \Omega^\bullet_X(\log E) \to \Omega^\bullet_{X/\Delta}(\log E) \to 0.$$

By taking $-\otimes \mathcal{O}_E$ on the above sequence, we will have

$$0 \to \Omega^\bullet_{X/\Delta}(\log E) \otimes \mathcal{O}_E[-1] \xrightarrow{\wedge dt/t} \Omega^\bullet_X(\log E) \otimes \mathcal{O}_E \to \Omega^\bullet_{X/\Delta}(\log E) \otimes \mathcal{O}_E \to 0.$$

The connecting homomorphism induces the residue at 0 of the logarithmic extension of the Gauss–Manin connection:

$$res_0(\nabla) : \mathbb{H}^q(E, \Omega^\bullet_{X/\Delta}(\log E) \otimes \mathcal{O}_E) \to \mathbb{H}^q(E, \Omega^\bullet_{X/\Delta}(\log E) \otimes \mathcal{O}_E).$$

Note that the cohomology of the induced complex on E, $\Omega^\bullet_{X/\Delta}(\log E) \otimes \mathcal{O}_E$ becomes isomorphic to the cohomology group $H^*(X_t)$. Also, the morphism $res_0(\nabla)$ recovers the monodromy action on $H^k(X_t)$.

Define the increasing filtration $W_\bullet$ on $\Omega_{X/\Delta}(\log E) \otimes \mathcal{O}_E$ by

$$W_k\Omega^\bullet_{X/\Delta}(\log E) \otimes \mathcal{O}_E := \mathrm{Im}\Big(W_k\Omega^\bullet_X(\log E) \to \Omega^\bullet_{X/\Delta}(\log E) \otimes \mathcal{O}_E\Big)$$

and the decreasing filtration $F^\bullet$ by the simple truncation. To describe the monodromy weight filtration, we consider the resolution of $\Omega^\bullet_{X/\Delta}(\log E) \otimes \mathcal{O}_E$ as follows. Define a tri-filtered double complex

$$(A^{\bullet,\bullet}, d', d'', W_\bullet, W(M)_\bullet, F^\bullet)$$

on E by

$$A^{p,q} = \frac{\Omega^{p+q+1}_X(\log E)}{W_p\Omega^{p+q+1}_X(\log E)}, \quad d' = (-) \wedge dt/t : A^{p,q} \to A^{p+1,q}, \quad d'' = d_{dR} : A^{p,q} \to A^{p,q+1}$$
$$W_r A^{p,q} = \frac{W_{r+p+1}\Omega^{p+q+1}_X(\log E)}{W_p\Omega^{p+q+1}_X(\log E)}, \quad W(M)_r A^{p,q} = \frac{W_{r+2p+1}\Omega^{p+q+1}_X(\log E)}{W_p\Omega^{p+q+1}_X(\log E)}$$
$$F^r A^{p,q} = \frac{F^r\Omega^{p+q+1}(\log E)}{W_p\Omega^{p+q+1}_X(\log E)}.$$

We have the map

$$\begin{aligned} \mu : \Omega^q_{X/\Delta}(\log E) \otimes \mathcal{O}_E &\to A^{0,q} \\ \omega &\mapsto (-1)^q (dt/t) \wedge \omega \mod W_0 \end{aligned}$$

which defines a quasi-isomorphism of bifiltered complexes

$$\mu : (\Omega^\bullet_{X/\Delta}(\log E) \otimes \mathcal{O}_E, W_\bullet, F^\bullet) \to (s(A^{\bullet,\bullet}), W_\bullet, F^\bullet),$$

where $s(A^{\bullet,\bullet})$ is the associated single complex. Consider the natural morphism $\nu : A^{p,q} \to A^{p+1,q-1}$ given by $\omega \mapsto \omega\,(\mathrm{mod} W_{p+1})$. As it commutes with both differentials d' and d'', it induces the

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endomorphism of the associated simple complex $s(A^{\bullet,\bullet})$. Note that this sends $W(M)_r$ to $W(M)_{r-2}$ and F^p to F^{p-1}.

Theorem 2.6 [25, Theorem 11.21]. *The following diagram is commutative:*

$$\begin{array}{ccc}
\mathbb{H}^q(E, \Omega^\bullet_{X/\Delta}(\log E) \otimes \mathcal{O}_E) & \xrightarrow{\mu} & \mathbb{H}^q(E, s(A^{\bullet,\bullet})) \\
\big\downarrow{\scriptstyle res_0 \nabla} & & \big\downarrow{\scriptstyle -\nu} \\
\mathbb{H}^q(E, \Omega^\bullet_{X/\Delta}(\log E) \otimes \mathcal{O}_E) & \xrightarrow{\mu} & \mathbb{H}^q(E, s(A^{\bullet,\bullet})).
\end{array}$$

By taking the residue map, we have

$$\begin{aligned}
\mathrm{Gr}_r^{W(M)} s(A^{\bullet,\bullet}) &\cong \bigoplus_{k \geq 0, -r} \mathrm{Gr}^W_{r+2k+1} \Omega^\bullet_X(\log E)[1] \\
&\cong \bigoplus_{k \geq 0, -r} \Omega^\bullet_{E(r+2k+1)}[-r-2k].
\end{aligned}$$

Therefore, the E_1 page of the spectral sequence for the monodromy weight filtration $W(M)_\bullet$ is given by

$$E_1^{p,q} = \bigoplus_{k \geq 0, p} H^{q+2p-2k}(E(2k-p+1), \mathbb{C})(p-k).$$

More explicitly, the E_1-page is given by the following diagram:

$$\begin{array}{ccccccc}
E_1^{-2,q} & \xrightarrow{d_1} & E_1^{-1,q} & \xrightarrow{d_1} & E_1^{0,q} & \xrightarrow{d_1} & E_1^{2,q} \\
\| & & \| & & \| & & \| \\
\bigoplus_{k \geq 0} H^{q-2k-4}(E(2k+3)) & & \bigoplus_{k \geq 0} H^{q-2k-2}(E(2k+2)) & & \bigoplus_{k \geq 0} H^{q-2k}(E(2k+1)) & & \bigoplus_{k \geq 1} H^{q-2k+2}(E(2k))
\end{array}$$

$$\begin{tikzcd}
H^{q-4}(E(3)) \arrow[r] \arrow[dr] & H^{q-2}(E(2)) \arrow[r] \arrow[dr] & H^q(E(1)) \arrow[dr] & \\
\bigoplus & \bigoplus & \bigoplus & \\
H^q(E(5)) \arrow[r] & H^{q-4}(E(4)) \arrow[r] \arrow[dr] & H^{q-2}(E(3)) \arrow[r] \arrow[dr] & H^q(E(2)) \\
\bigoplus & \bigoplus & \bigoplus & \\
\vdots & \vdots & H^{q-4}(E(5)) \arrow[r] & H^{q-2}(E(4)),
\end{tikzcd}$$

where the horizontal arrows are (the alternating sum of) the Gysin morphisms while the antidiagonal arrows are (the alternating sum of) the pullback morphisms. If we write down two morpshisms by G and d, respectively, the differential $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ is given by $d_1 = G + (-1)^p d$.

Theorem 2.7 [25, Theorem 11.22]. *The spectral sequence for the filtration $W(M)_\bullet$ degenerates at the E_2-page so that we have*

$$E_2^{p,q} = E_\infty^{p,q} = Gr_q^{W(M)} H^{p+q}(X).$$

We will also denote the monodromy weight filtration $W(M)_\bullet$ by $W_{\lim \bullet}$.

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2.3. The perverse filtration

We briefly review the notion of perverse filtration [6] and its geometric description [7].

Definition 2.8. Let Y be an algebraic variety or complex analytic space and $D^b_c(Y)$ be a derived category of constructible sheaves on Y. An object $K^\bullet \in D^b_c(Y)$ is called a *perverse sheaf* if it satisfies following two dual conditions:

1. (Support Condition) $\dim \operatorname{supp}(\mathcal{H}^i(K^\bullet)) \leq -i$
2. (Cosupport Condition) $\dim \operatorname{supp}(\mathcal{H}^i(\mathbb{D}K^\bullet)) \leq i$, where $\mathbb{D} : D^b_c(Y) \to D^b_c(Y)$ is a dualizing functor.

Verdier's dualizing functor on $D^b_c(Y)$ is defined as $\mathbb{D} = \operatorname{Hom}_{\mathcal{O}_Y}(-, p^!(\mathbb{C}_{pt}))$, where $p : Y \to pt$ is a trivial map. We call $p^!(\mathbb{C}_{pt})$ a dualizing complex of Y, and denote it by ω_Y. In particular, if Y is nonsingular of complex dimension n, $\omega_Y = \mathbb{C}_Y[2n]$. Note that the subcategory $\mathcal{P}(Y)$ of perverse sheaves on Y is an abelian category. Also, the support and cosupport condition induces the so-called perverse t-structure $({}^{\mathfrak{p}}D^{b,\geq 0}_c(Y), {}^{\mathfrak{p}}D^{b,\leq 0}_c(Y))$ on $D^b_c(Y)$ whose heart is $\mathcal{P}(Y)$. Explicitly, it is given by

1. $K^\bullet \in {}^{\mathfrak{p}}D^{b,\leq 0}_c(Y)$ if and only if K satisfies the support condition. Also, ${}^{\mathfrak{p}}D^{b,\leq n}_c(Y) := {}^{\mathfrak{p}}D^{b,\leq 0}_c(Y)[-n]$
2. $K^\bullet \in {}^{\mathfrak{p}}D^{b,\geq 0}_c(Y)$ if and only if K satisfies the cosupport condition. Also, ${}^{\mathfrak{p}}D^{b,\geq n}_c(Y) := {}^{\mathfrak{p}}D^{b,\geq 0}_c(Y)[-n]$.

We denote ${}^{\mathfrak{p}}\tau_{\leq n} : D^b_c(Y) \to {}^{\mathfrak{p}}D^{b,\leq n}_c(Y)$ (resp. ${}^{\mathfrak{p}}\tau_{\geq n} : D^b_c(Y) \to {}^{\mathfrak{p}}D^{b,\geq n}_c(Y)$) the natural truncation functor. This induces *perverse cohomology functors* ${}^{\mathfrak{p}}\mathcal{H} : D^b_c(Y) \to \mathcal{P}(Y)$ defined by ${}^{\mathfrak{p}}\mathcal{H}^k := {}^{\mathfrak{p}}\tau_{\leq 0} \circ {}^{\mathfrak{p}}\tau_{\geq 0} \circ [k]$. Applying the perverse truncation, one can define the perverse filtration on the hypercohomology of a constructible sheaf $\mathcal{K}^\bullet$ on Y as follows;

Definition 2.9. For $\mathcal{K}^\bullet \in D^b_c(Y)$, the perverse filtration $P_\bullet$ on $\mathbb{H}^k(Y, \mathcal{K}^\bullet)$ is defined to be

$$P_b\mathbb{H}^k(Y, \mathcal{K}^\bullet) := \operatorname{Im}\Big(\mathbb{H}^k(Y, {}^{\mathfrak{p}}\tau_{\leq b}\mathcal{K}^\bullet) \to \mathbb{H}^k(Y, \mathcal{K}^\bullet)\Big).$$

Let $f : X \to Y$ be a morphism of smooth varieties. Then we can define the perverse (f-)Leray filtration on the cohomology $H^\bullet(X, \mathbb{C})$ by setting

$$P^f_l H^k(X, \mathbb{C}) := P^f_l \mathbb{H}^k(Y, Rf_*\mathbb{C})$$

Theorem 2.10 [25, Corollary 14.41]. *If f is proper, then the spectral sequence for the perverse Leray filtration degenerates at the E_2 page. In other words, we have*

$$Gr^{P^f}_l \mathbb{H}^k(X, \mathbb{C}) = E^{k-l,l}_2 = \mathbb{H}^{k-l}(Y, {}^{\mathfrak{p}}\mathcal{H}^l(Rf_*\mathbb{C})).$$

We will provide a geometric description of the perverse Leray filtration in case that the base space is either affine or quasi-projective. For this, we introduce some notations used in the next subsection. Let $f : X \to Y$ be a locally closed embedding. Then the restriction functor is given by $(-)|_X = Rf_!f^*$ on $D^b_c(Y)$, which is exact. If f is closed, then we also have the right derived functor of sections with support in X, denoted by $R\Gamma_X(-) = Rf_*f^!$.

Next, we provide provide a geometric description of perverse filtrations. We follow the same convention for the indices of filtrations used in [12]. Let's consider the following commutative diagram of varieties

$$\begin{array}{ccc} U & \hookrightarrow & Y \\ \downarrow{\scriptstyle \pi_U} & & \downarrow{\scriptstyle \pi} \\ B_U & \hookrightarrow & B, \end{array}$$

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where

- Y is a smooth complex projective variety of complex dimension n.
- B is a complex projective variety of complex dimension m. We fix an embedding $B \hookrightarrow \mathbb{P}^N$.
- $\pi : Y \longrightarrow B$ is a proper morphism.
- B_U is the affine subvariety of B and $U := \pi^{-1}(B_U)$. We write π_U for the restriction of π onto U.

Recall that there is a smooth projective variety $F(N,m)$ parametrizing m-flags $\mathfrak{F} = \{F_{-m} \subset \cdots \subset F_{-1}\}$ on $\mathbb{P}^N$, where F_{-p} is a codimension p linear subspace. A linear m-flag $\mathfrak{F}$ on $\mathbb{P}^N$ is general if it belongs to a suitable Zariski open subset of $F(N,m)$. Similarly, we say a pair of linear m-flags $(\mathfrak{F}_1, \mathfrak{F}_2)$ is general if it belongs to a suitable Zariski open subset of $F(N,m) \times F(N,m)$.

Fix a general pair of m-flags $(H_\bullet, L_\bullet)$ on $\mathbb{P}^N$. Intersecting with B, it gives a pair of flags of subvarieties $(B_\bullet, C_\bullet)$ of B,

$$\emptyset = B_{-m-1} \subset B_{-m} \subset \cdots \subset B_{-1} \subset B_0 = B,$$
$$\emptyset = C_{-m-1} \subset C_{-m} \subset \cdots \subset C_{-1} \subset C_0 = B,$$

where $B_\bullet := H_\bullet \cap B$ and $C_\bullet := L_\bullet \cap B$. We set $Y_\bullet = \pi^{-1}(B_\bullet)$ and $Z_\bullet = \pi^{-1}(C_\bullet)$. By following [7, 12], we define the following flag filtrations.

Definition 2.11.

1. The flag filtration $G^\bullet$ (of the first kind) on the cohomology of U is a decreasing filtration defined by

$$G^i H^k(U,\mathbb{C}) := \ker\{H^k(U,\mathbb{C}) \to H^k(U, \mathbb{C}|_{U\cap Y_{i-1}})\}.$$

2. The flag filtration $G^\bullet$ (of the second kind) on the compactly supported cohomology of U is a decreasing filtration defined by

$$G^j H^k_c(U,\mathbb{C}) := \mathrm{Im}\{H^k_{Z_{-j}\cap U,c}(U,\mathbb{C}) \to H^k_c(U,\mathbb{C})\}.$$

3. The δ-flag filtration $\delta^\bullet$ on the cohomology of Y is a decreasing filtration defined by

$$\delta^p H^k(U,\mathbb{C}) := \mathrm{Im}\{\bigoplus_{i+j=p} H^k_{Z_{-j}}(Y, \mathbb{C}|_{Y-Y_{i-1}}) \to H^k(Y,\mathbb{C})\}.$$

Note that both two G-filtraions can be defined on other cohomology theories as well. We describe the E_1-page of the spectral sequence for each filtration.

1. The E_1-page of the spectral sequence for the flag filtration $G^\bullet$ (of the first kind) on $H^*(U)$ is given by

$${}^G E_1^{p,q} = H^{p+q}(U\cap Y_p, U\cap Y_{p-1}, \mathbb{C}) \Longrightarrow H^*(U,\mathbb{C}),$$

and the differential $d_1 : {}^G E_1^{p,q} \to {}^G E_1^{p+1,q}$ is the connecting homomorphism of the long exact sequence of cohomology groups of the triple (Y_{p+1}, Y_p, Y_{p-1}). Furthermore, we have

$${}^G E_\infty^{p,q} = \mathrm{Gr}^p_G H^{p+q}(U,\mathbb{C}).$$

2. The E_1-page of the spectral sequence for the flag filtration $G^\bullet$ (of the second kind) on $H^*_c(U,\mathbb{C})$ is given by

$${}^G E_1^{p,q} = H^{p+q}_{Z_{-p}\cap U - Z_{-p-1}\cap U, c}(U,\mathbb{C}) \Longrightarrow H^*_c(U,\mathbb{C}),$$

and the differential $d_1 : {}^G E_1^{p,q} \to {}^G E_1^{p+1,q}$ is the connecting homomorphism of the long exact sequence of cohomology groups with supports $(Z_{-p}, Z_{-p-1}, Z_{-p-2})$. Furthermore, we have

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$$ {}^{G}E_{\infty}^{p,q} = \mathrm{Gr}_G^p H_c^{p+q}(U, \mathbb{C}). $$

3. The E_1-page of the spectral sequence for the δ-flag filtration $\delta^\bullet$ on $H^*(Y)$ is given by

$$ {}^{\delta}E_1^{p,q} = \bigoplus_{i+j=p} H^{p+q}_{Z_{-j}-Z_{-j-1}}(Y, \mathbb{C}|_{Y_i - Y_{i-1}}) \Longrightarrow H^*(Y, \mathbb{C}). $$

More explicitly, the E_1-page is given by the following diagram.

$$
\begin{array}{ccccccc}
E_1^{-2,q} & \xrightarrow{d_1} & E_1^{-1,q} & \xrightarrow{d_1} & E_1^{0,q} & \xrightarrow{d_1} & E_1^{1,q} \\
\\
H^{q-2}_{Z_0-Z_{-1}}(\mathbb{C}|_{Y_{-2}-Y_{-3}}) & \to & H^{q-1}_{Z_0-Z_{-1}}(\mathbb{C}|_{Y_{-1}-Y_{-2}}) & \to & H^{q}_{Z_0-Z_{-1}}(\mathbb{C}|_{Y_0-Y_{-1}}) & & \\
\bigoplus & \searrow & \bigoplus & \searrow & \bigoplus & \searrow & \\
H^{q-2}_{Z_{-1}-Z_{-2}}(\mathbb{C}|_{Y_{-3}-Y_{-4}}) & \to & H^{q-1}_{Z_{-1}-Z_{-2}}(\mathbb{C}|_{Y_{-2}-Y_{-3}}) & \to & H^{q}_{Z_{-1}-Z_{-2}}(\mathbb{C}|_{Y_{-1}-Y_{-2}}) & \to & H^{q+1}_{Z_{-1}-Z_{-2}}(\mathbb{C}|_{Y_0-Y_{-1}}) \\
\bigoplus & & \bigoplus & \searrow & \bigoplus & \searrow & \bigoplus \\
\vdots & & \vdots & & H^{q}_{Z_{-2}-Z_{-3}}(\mathbb{C}|_{Y_{-2}-Y_{-3}}) & \to & H^{q+1}_{Z_{-2}-Z_{-3}}(\mathbb{C}|_{Y_{-1}-Y_{-2}}).
\end{array}
\tag{2.4}
$$

- For fixed j, the antidiagonal sequence is the same with the E_1-page of the spectral sequence for the G-filtration of the first kind on $H^*_{Z_{-j}}(Y)$ with respects to the induced flag $Z_{-j} \cap Y_\bullet$. Let's write d_I for the differential.
- For fixed i, the horizontal sequence is the same with the E_1-page of the spectral sequence for the G-filtration of the second kind on $H^*(Y_i)$ with respects to the induced flag $Z_\bullet \cap Y_i$. Let's write d_{II} for the differential.
- The differential $d_1 : {}^{\delta}E_1^{p,q} \to {}^{\delta}E_1^{p+1,q}$ is given by $d_1 = d_I + (-1)^p d_{II}$.

 Moreover, we have

$$ {}^{\delta}E_{\infty}^{p,q} = \mathrm{Gr}_{\delta}^p H^{p+q}(Y, \mathbb{C}). $$

Theorem 2.12 [7, Theorem 4.1.3 and 4.2.1]. *There are identification of the perverse and flag filtrations:*

1. $P_l^{\pi_U} H^k(U) = G^{k-l} H^k(U)$, *where l starts from k up to $k+m$.*
2. $P_l^{\pi_U} H_c^k(U) = G^{k-l} H_c^k(U)$, *where l starts from $k-m$ to k.*
3. $P_l^{\pi} H^k(Y) = \delta^{k-l} H^k(U)$, *where where l starts from $k-m$ up to $k+m$.*

Corollary 2.13. *The spectral sequences for all the flag filtrations in Definition 2.11 degenerate at the E_2-page.*

Proof. Theorem 2.12 implies that there are natural isomorphisms between two spectral sequences that induces the identity on the abutment. Since the morphisms π and π_U are proper, Theorem 2.10 implies that the spectral sequence for the associated perverse filtration degenerates at the E_1-page. Note that the E_2-term of the Grothendieck spectral sequence used in Theorem 2.10 is the same with E_1-term of the perverse filtration. Also, the spectral sequence for the shifted flag filtrations $G^{k-\bullet}$ and $\delta^{k-\bullet}$ is the shifted spectral sequence for $G^\bullet$ and $\delta^\bullet$, respectively. In other words, $E_1^{p,q}$ for the shifted filtration is the same with $E_2^{2p+q,-p}$ for the original filtration (see [9] [7, Section 3.7].) Therefore, we have the E_2-degeneration results for the flag filtrations $G^\bullet$ and $\delta^\bullet$. □

3. Extended Fano/LG correspondence

3.1. Hybrid LG models

We start to recall the notion of hybrid LG models introduced in [24]. Let's first introduce some notations. Let $h = (h_1, \ldots, h_N) : Y \to \mathbb{C}^N$ be a N-tuple of (holomorphic) functions and $(z_1, \ldots, z_N)$ be the coordinates of the base $\mathbb{C}^N$. For each nonempty subset $I = \{i_1, \ldots, i_l\} \subset \{1, \ldots, N\}$, we write $h_I = (h_{i_1}, \ldots, h_{i_l}) : Y \to \mathbb{C}^{|I|}$ and the coordinate $(z_{i_1}, \ldots, z_{i_l})$ for the base $\mathbb{C}^{|I|}$, which implicitly determines the natural inclusion $\mathbb{C}^{|I|} \subset \mathbb{C}^N$.

Definition 3.1 [24, Definition 3.1, 5.9]. A *hybrid LG model of rank N* is a triple $(Y, \omega, h = (h_1, h_2, \ldots, h_N) : Y \to \mathbb{C}^N)$ where

1. (Y, ω) is n-dimensional complex Kähler Calabi–Yau manifold with a Kähler form $\omega \in \Omega^2(Y)$;
2. $h : Y \to \mathbb{C}^N$ is a proper (surjective) holomorphic map such that
   (a) (Local trivialization) There exists a constant $R > 0$ such that for any nonempty subset $I \subset \{1, \ldots, N\}$, the induced map $h_I : Y \to \mathbb{C}^{|I|}$ is a locally trivial symplectic fibration over the region $B_I := \{|z_i| > R | i \in I\}$ with smooth fibers. Furthermore, over B_I we have $v(h_j) = 0$ for any horizontal vector field $v \in T^{h_I}Y$ associated to h_I and $j \notin I$;
   (a) (Compatibility) For $I \subset J$, such local trivializations are compatible under the natural inclusions $B_J \times \mathbb{C}^{N-|J|} \subset B_I \times \mathbb{C}^{N-|I|} \subset \mathbb{C}^N$.
   We call $h : Y \to \mathbb{C}^N$ a *hybrid LG potential*.

When $N = 1$, this definition recovers the usual notion of LG models $(Y, \omega, h : Y \to \mathbb{C})$, where h becomes a locally trivial symplectic fibration with smooth fibers near infinity. One can see that the second condition in Definition 3.1 controls the geometry of the local fibration h near the infinity boundary of the base. For each nonempty subset $I \subset \{1, \ldots, N\}$, let's write Y_I for a generic fiber of $h_I : Y \to \mathbb{C}^{|I|}$ and $h_{Y_I} : Y_I \to \mathbb{C}^{N-|I|}$ for the restriction of h into Y_I. Then the induced triple $(Y_I, \omega|_{Y_I}, h_{Y_I})$ can be regraded as a hybrid LG model of rank $N - |I|$. From this point of view, the condition $(2) - (a)$ in Definition 3.1 is rephrased as the condition that $h_I : Y \to \mathbb{C}^{|I|}$ is a local trivialization of the induced hybrid LG models of rank $N - |I|$.

Associated to the hybrid LG model $(Y, \omega, h : Y \to \mathbb{C}^N)$, we define the ordinary LG model to be a triple $(Y, \omega, \mathsf{w} := \Sigma \circ h : Y \to \mathbb{C})$, where $\Sigma : \mathbb{C}^N \to \mathbb{C}$ is the summation map. The following proposition justifies this terminology.

Proposition 3.2 [24, Proposition 3.2]**(Gluing property).** *Let $(Y, \omega, h : Y \to \mathbb{C}^N)$ be a hybrid LG model and H be a generic hyperplane in the base $\mathbb{C}^N$, which is not parallel to any coordinate lines. There exists an open cover $\{U_i\}_{i=1}^N$ of H such that for any nonempty subset $I \subset \{1, \ldots, N\}$, the induced map $h^{-1}(U_I) \to U_I$ is isotopic to the induced hybrid LG potential $h_{Y_I} : Y_I \to \mathbb{C}^{N-|I|}$ which is linear along the base.*

Proof. We present the proof for the reader's convenience. Take a hyperplane $H = \{a_1 z_1 + \cdots + a_N z_N = M\}$, where $a_i \neq 0$ for all i. By changing the coordinate $z_i \mapsto z_i / a_i$, we reduce to the case where $a_i = 1$ for all i. We also further reduce to the case when M is real due to the rotational symmetry. By generality, we take $M > NR$. First, note that $H \cap (\cap_{i=1}^N \{|z_i| \leq R\}) = \emptyset$. Let $R_i = \{Re(z_i) > R\}$ and the simply connected region

$$U_i := \{Re(z_i) > R\} \cap H = \{Re(z_1 + \cdots + \hat{z}_i + \cdots + z_n) < M - R\} \cap H$$

for each i. Since $U_i \subset \{|z_i| > R\}$, one can project U_i to the locus $\{z_i = 2R\}$ inside the region $\{|z_i| < R\}$. The image of the projection is $V_i := \{z_i = 2R, Re(z_1 + \cdots + \hat{z}_i + \cdots + z_N) < M - R\}$ which contains $\bigcap_{j \neq i} \{|z_j| \leq R\}$. Therefore, this projection identifies $h : h^{-1}(U_i) \to U_i$ with $h : h^{-1}(V_i) \to V_i$ due to the local triviality of the hybrid LG model. Moreover, the latter map is completed to $h_{Y_i} : Y_i \to \mathbb{C}^{N-1}$ by the inductive argument. In general, for each I, $U_I = \cap_{i \in I} U_i$ is nonempty and simply connected. Since $U_I \subset \{|z_i| > R, i \in I\}$, one can apply the same argument to get the conclusion. □

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Definition 3.3. Let $(Y, \omega, h : Y \to \mathbb{C}^N)$ be a hybrid LG model. We define the induced triple $(Y, \omega, \mathsf{w} : Y \to \mathbb{C})$ to be the ordinary LG model associated to the hybrid LG model (Y, ω, h) and denote a generic fiber of w by Y_{sm}.

Remark 3.4. In general, Proposition 3.2 is expected to hold in the symplectic category (see [24, Section 5] for more details). In this article, we mainly focus on the topological properties of hybrid LG models.

On the cohomology level, Proposition 3.2 implies that the cohomology group of $\pi^{-1}(H)$ is (non-canonically) isomorphic to that of the normal crossing union of Y_i's. We will use this fact to study the perverse Leray filtration associated to $h : Y \to \mathbb{C}^N$ on $H^*(Y)$.

Consider a general flag of hyperplanes in $\mathbb{C}^N$,

$$\mathfrak{H} : 0 = H_{-N-1} \subset H_{-N} \subset \cdots \subset H_{-1} \subset H_0 = \mathbb{C}^N$$

which is transversal to the discriminant locus of h in the sense of [7, Definition 5.2.4] and each H_{-l} is not parallel to any coordinate lines. We write $Y_{sm^{(l)}}$ for $h^{-1}(H_{-l})$ so that we have a general flag of subvarieties

$$0 \subset Y_{sm^{(N)}} \subset \cdots \subset Y_{sm^{(1)}} \subset Y$$

which will be used to compute the flag filtration $G^\bullet$ (equivalently, the perverse Leray filtration $P^h_\bullet$) on $H^*(Y)$ (see Section 2.3). In other words, the E_1-page of the spectral sequence is given by the sequence

$$H^{a-N}(Y_{sm^{(N)}}) \xrightarrow{d_1} H^{a-N+1}(Y_{sm^{(N-1)}}, Y_{sm^{(N)}}) \xrightarrow{d_1} \cdots \xrightarrow{d_1} H^{a-1}(Y_{sm^{(1)}}, Y_{sm^{(2)}}) \xrightarrow{d_1} H^a(Y, Y_{sm^{(1)}}). \tag{3.1}$$

We use the same notation in the proof of Proposition 3.2. Take open (simply connected) regions $\{R_i \subset \mathbb{C}^N | i = 1, \dots, N\}$ which induce an open covering of H_{-1}, $\{U_i := R_i \cap H_{-1} | i = 1, \dots, N\}$ that yields the gluing property. Let $V_i = \{z_i = const\}$ be the region that U_i projects to. Due to the genericity of the flag, we may assume that $H_{-2} \cap U_i \subset U_i$ projects to a hyperplane that is contained in $\cup_{j \neq i} R_j \cap V_i$ for all i. As both H_{-1} and H_{-2} are not parallel to any coordinate lines, this can be done by scaling M sufficiently large to place H_{-2} far enough from each coordinate line. It ensures that $Y_{sm} \cap h^{-1}(U_i)$ is isotopic to $Y_{i,sm}$ for each i. Inductively, for each k, we could assume that the collection of regions $\{R_i \subset \mathbb{C}^N | i = 1, \dots, N\}$ yields the gluing property for H_{-k} in $V_I := \cap_{i \in I} V_i$ for any $|I| = k - 1$. Then the gluing property implies the following:

$$Y_{sm^{(k)}} \cap h^{-1}(U_I) \cong \begin{cases} Y_{I, sm^{(k-|I|)}} & |I| < k \\ Y_I & |I| \geq k. \end{cases}$$

Lemma 3.5. *For any $a \geq 0, k \geq 1$, the relative cohomology $H^a(Y_{sm^{(k)}}, Y_{sm^{(k+1)}})$ is isomorphic to $\bigoplus_{|I|=k} H^a(Y_I, Y_{I,sm})$.*

Proof. Take the (simply connected) open region $\{R_i \subset \mathbb{C}^N | i = 1, \dots, N\}$ and the induced cover $\{U_i = R_i \cap H_{-1}\}$ as above. When $k = 1$, the Mayer–Vietoris argument with respect to the open cover $\{h^{-1}(U_i)\}$ and the gluing property implies that $H^a(Y_{sm^{(1)}}, Y_{sm^{(2)}}) \cong \bigoplus_{i=1}^N H^a(Y_i, Y_{i,sm^{(1)}})$ where $H^a(Y_i, Y_{i,sm^{(1)}})) \cong H^a(Y_i, Y_{i,sm})$. In general, we apply the Mayer–Vietoris sequence to the cohomology group $H^a(Y_{sm^{(k)}}, Y_{sm^{(k+1)}})$ with the induced open cover by R_i's. The E_1-page of the spectral sequence is given by

$$\bigoplus_{|I|=1} H^a(Y_{I,sm^{(k-1)}}, Y_{I,sm^{(k)}}) \xrightarrow{d_1} \dots \xrightarrow{d_1} \bigoplus_{|I|=k-1} H^a(Y_{I,sm^{(1)}}, Y_{I,sm^{(2)}}) \xrightarrow{d_1} \bigoplus_{|I|=k} H^a(Y_I, Y_{I,sm^{(1)}}) \to 0.$$

By induction, each direct summand is the direct sum of $H^a(Y_J, Y_{J,sm})$ for some J with $|J| = k$. Then the differential d_1 becomes the alternating sum of the identity morphisms, where the signs are determined

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by the Mayer–Vietoris sign rule (2.1). Then it follows from a simple combinatorial fact that this sequence is exact except at the first term, and $H^a(Y_{sm^{(k)}}, Y_{sm^{(k+1)}}) = \ker(d_1) \cong \bigoplus_{|I|=k} H^a(Y_I, Y_{I,sm})$. □

For any $I \subset J$ with $|J| = |I| + 1$, we write ρ_I^J for the composition of morphisms

$$\rho_I^J : H^\bullet(Y_J, Y_{J,sm}) \hookrightarrow H^\bullet(Y_{I,sm}, Y_{I,sm^{(2)}}) \to H^{\bullet+1}(Y_I, Y_{I,sm}),$$

where the first one is given by Lemma 3.5 and the second one is the connecting homomorphism of the long exact sequence of cohomology groups of the triple $(Y_I, Y_{I,sm}, Y_{I,sm^{(2)}})$. Since we choose an open cover globally, Lemma 3.5 allows one to rewrite the E_1-page of the spectral sequence (3.1) as follows:

$$H^{a-N}(Y_{sm^{(N)}}) \xrightarrow{d_1} \bigoplus_{|I|=N-1} H^{a-N+1}(Y_I, Y_{I,sm}) \xrightarrow{d_1} \cdots \xrightarrow{d_1} \bigoplus_{|I|=1} H^{a-1}(Y_I, Y_{I,sm}) \xrightarrow{d_1} H^a(Y, Y_{sm}),$$

where the differential d_1 is the signed sum of the induced morphisms ρ_I^J's that follows the Mayer–Vietoris sign rule (2.1).

For later use, we introduce the Poincaré dual of ρ_I^J. For a given hybrid LG model $(Y, h : Y \to \mathbb{C}^N)$ of rank N, we will show that there is a canonical isomorphism

$$PD : H^a(Y, Y_{sm}, \mathbb{C}) \xrightarrow{\cong} H^{2n-a}(Y, Y_{sm}, \mathbb{C})^* \tag{3.2}$$

for all $a \geq 0$ (Theorem 7.4). We define the morphism $(\rho_I^J)^\vee : H^\bullet(Y_I, Y_{I,sm}) \to H^{\bullet-1}(Y_J, Y_{J,sm})$ to be the composition $(\rho_I^J)^\vee = PD_J \circ (\rho_I^J)^* \circ PD_I^{-1}$, where PD_I (resp. PD_J) is the same one in (3.2) for the induced hybrid LG model (Y_I, h_{Y_I}) (resp. (Y_J, h_{Y_J})).

3.2. *Extended Fano/LG correspondence*

Let X be a smooth (quasi-)Fano manifold and D be an effective simple normal crossing anticanonical divisor with N components $D_1, D_2, \ldots, D_N$. For any index set $I = \{i_1, i_2, \cdots, i_m\} \subset \{1, 2, \cdots, N\}$, we define

$$D_I := D_{i_1} \cap \cdots \cap D_{i_m}, \qquad D(I) := \Sigma_{j \notin I} D_I \cap D_j.$$

For example, if $I = \{1\}$, then $D_{\{1\}} = D_1$ and $D(\{1\}) = (D_2 \cup \cdots \cup D_k) \cap D_1$. We also assume that all pairs $(D_I, D(I))$ are (quasi-)Fano. We also write the normal crossing union of l-th intersections by $D\{l\} = \sum_{|I|=l} D_I$ for all $l \geq 0$.

Definition 3.6. A hybrid LG model $(Y, \omega, h : Y \to \mathbb{C}^N)$ is mirror to (X, D) if it satisfies the following mirror relations:

1. the associated ordinary LG model $(Y, \omega, \mathsf{w} : Y \to \mathbb{C})$ is mirror to (X, D);
2. for $i = 1, 2, \ldots, N$, a hybrid LG model $(Y_i, \omega|_{Y_i}, h_{Y_i} : Y_i \to \mathbb{C}^{N-1})$ is mirror to $(D_i, D(\{i\}))$.

Such a mirror pair is called a *(quasi-)Fano mirror pair*, and we write it by $(X, D)|(Y, \omega, h)$.

To elaborate the precise sense of the mirror relations, we introduce some notations. Let $\square$ be a cubical category whose objects are finite subsets of $\mathbb{N}$ and morphisms $Hom(I, J)$ consists of a single element if $I \subset J$ and else is empty. Given a category C, we define a cubical object to be a contravariant functor $F : \square \to \mathsf{C}$, which is also called a cubical diagram of categories. For a cubical object F and $I \subset \mathbb{N}$, we write

$$F_I := F(I)$$
$$d_{IJ} := F(I \to J) : X_J \to X_I, \quad I \subset J.$$

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We also define a morphism of cubical objects in an obvious way and mainly consider the category of finite-dimensional vector spaces over $\mathbb{C}$, denoted by $\mathsf{Vect}_{\mathbb{C}}$.

First, on the B-side, consider the natural inclusions $\iota_I^J : D_J \hookrightarrow D_I$ for $I \subset J$. We consider a cubical object $\mathfrak{H}\mathfrak{H}_a(X, D)$ in $\mathsf{Vect}_{\mathbb{C}}$ defined as

$$\mathfrak{H}\mathfrak{H}_a(X, D)_I = \bigoplus_{p-q=a} H^{p,q}(D_I)$$
$$\mathfrak{H}\mathfrak{H}_a(X, D)_{IJ} = \iota_{I!}^J : \bigoplus_{p-q=a} H^{p,q}(D_J) \to \bigoplus_{p-q=a} H^{p,q}(D_I),$$

where $\iota_!$'s are the Gysin morphisms. We also take the Poincaré dual of $\mathfrak{H}\mathfrak{H}_a(X, D)$, denoted by $\mathfrak{H}\mathfrak{H}_a^c(X, D)$ where

$$\mathfrak{H}\mathfrak{H}_a^c(X, D)_I = \bigoplus_{p-q=a} H^{p,q}(D_I)$$
$$\mathfrak{H}\mathfrak{H}_a^c(X, D)_{IJ} = (\iota_I^J)^* : \bigoplus_{p-q=a} H^{p,q}(D_I) \to \bigoplus_{p-q=a} H^{p,q}(D_J).$$

On the A-side, let $(Y, \omega, h : Y \to \mathbb{C}^N)$ be a hybrid LG model of rank N and $n = \dim_{\mathbb{C}} Y$. We consider a cubical object $\mathfrak{H}\mathfrak{H}_a(Y, h) \in \mathsf{Vect}_{\mathbb{C}}$

$$\mathfrak{H}\mathfrak{H}_a(Y, h)_I = H^{n+a-|I|}(Y_I, Y_{I,sm})$$
$$\mathfrak{H}\mathfrak{H}_a(Y, h)_{IJ} = \rho_I^J : H^{n+a-|J|}(Y_J, Y_{J,sm}) \to H^{n+a-|I|}(Y_I, Y_{I,sm})$$

for $n \leq a \leq n$. We also take the Poincaré dual $\mathfrak{H}\mathfrak{H}_a^c(Y, h)$, where

$$\mathfrak{H}\mathfrak{H}_a^c(Y, h)_I = H^{n+a-|I|}(Y_I, Y_{I,sm})$$
$$\mathfrak{H}\mathfrak{H}_a^c(Y, h)_{IJ} = (\rho_I^J)^\vee : H^{n+a-|I|}(Y_I, Y_{I,sm}) \to H^{n+a-|J|}(Y_I, Y_{J,sm}).$$

Conjecture 3.7. *Let $(X, D)|(Y, \omega, h : Y \to \mathbb{C}^N)$ be a (quasi-)Fano mirror pair. For $-n \leq a \leq n$, there exists isomorphisms of the cubical objects in $\mathsf{Vect}_{\mathbb{C}}$:*

$$\mathfrak{H}\mathfrak{H}_a(X, D) \cong \mathfrak{H}\mathfrak{H}_a(Y, h), \qquad \mathfrak{H}\mathfrak{H}_a^c(X, D) \cong \mathfrak{H}\mathfrak{H}_a^c(Y, h).$$

Remark 3.8. Conjecture 3.7 is motivated from the relative version of homological mirror symmetry conjecture for (quasi-)Fano mirror pairs [24, Section 4.3]. In particular, this is expected to follow from applying Hochschild homology to the categorical statement. Additionally, it is expected that one of the above isomorphisms follows from the other via Poincaré duality.

3.3. Line bundles/Monodromy correspondence

Let (X, D) be a (quasi-)Fano pair, where D is smooth and $(Y, \omega, \mathsf{w} : Y \to \mathbb{C})$ be its mirror LG model. In this case, there is a mirror correspondence between the anticanonical line bundle $-K_X$ and the monodromy T of a generic fiber $\mathsf{w}^{-1}(t)$ around infinity. Such correspondence can be made precise on the categorical level via the homological mirror symmetry conjecture. On the B-side, tensoring with $-K_X$ provides autoequivalences on the derived category of coherent sheaves on X, $D^b\mathrm{Coh}(X)$, as well as on $D^b\mathrm{Coh}(D)$ by restriction. On the A-side, the monodromy operator T induces autoequivalences on the relevant Fukaya categories associated with Y_{sm} and $\mathsf{w} : Y \to \mathbb{C}$.

On the other hand, when D has more than one component, one can ask a more refined version of the above correspondence. On the B-side, we have N line bundles $\mathcal{O}_X(D_i)$ for $i = 1, \ldots, N$, whose sum is the anticanonical line bundle $-K_X$. Each line bundle induces an autoequivalence on $D^b\mathrm{Coh}(X)$ by

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taking the tensor product with itself. On the A-side, there are N monodromy operators, each of which is induced by taking a loop T_i near infinity on the base of $h : Y \to \mathbb{C}^N$,

$$T_i := (t_1, \dots, t_{i-1}, e^{\sqrt{-1}\theta} t_i, t_{i+1}, \dots, t_N) \quad (0 \le \theta \le 2\pi) \tag{3.3}$$

for a generic $(t_1, \dots, t_N) \in \mathbb{C}^N$ and $i = 1, \dots, N$. We denote such operators by ϕ_{T_i}. Note that the monodromy operatorϕ_{T_i} induces not only the automorphism of a generic fiber $Y_i = h_i^{-1}(t)$ but also the automorphism of the induced fibration $h|_{Y_j} : Y_j \to \mathbb{C}^{N-1}$ for any i, j. This will play a key role in Section 4. Moreover, note that the composition of T_i's is the loop T near infinity on the base of $\mathsf{w} : Y \to \mathbb{C}$. Each monodromy operator is expected to induce an autoequivalence, denoted by ϕ_{T_i} as well, on the relevant Fukaya category of $(Y, \omega, h : Y \to \mathbb{C}^N)$.

Ansatz 3.9. *There are correspondences between the line bundle $\mathcal{O}_X(D_i)$ and the monodromy ϕ_{T_i} for all $i = 1, \dots, N$.*

The main source of Ansatz 3.9 can be found in [15][16] where the mirror symmetry of smooth toric Fanos has been discussed. See also [24, Section 4.3] for more details.

4. Mirror construction for a smoothing of a semistable degeneration

4.1. Semistable degeneration

Let $\mathfrak{X}$ be a complex connected analytic space and Δ be the unit disk. A degeneration is a proper flat surjective map $\pi : \mathfrak{X} \to \Delta$ such that $\mathfrak{X} - \pi^{-1}(0)$ is smooth and the fiber $\mathfrak{X}_t$ is a compact Kähler manifold for every $t \neq 0$. The fiber at the zero $\mathfrak{X}_0 := \pi^{-1}(0)$ is called the *degenerate fiber*. Given the degeneration $\pi : \mathfrak{X} \to \Delta$ and for $t \neq 0$, we say that $\mathfrak{X}_t$ degenerates to $\mathfrak{X}_0$ or equivalently $\mathfrak{X}_0$ is smoothable to $\mathfrak{X}_t$. In particular, if the total space $\mathfrak{X}$ is smooth and the degenerate fiber $\mathfrak{X}_0$ is a simple normal crossing divisor of $\mathfrak{X}$, then the degeneration $\pi : \mathfrak{X} \to \Delta$ is called *semistable*. We define a *type* of the semistable degeneration to be the dimension of the dual complex of the degenerate fiber.

Due to Friedman [13], the semistability condition on the degeneration $\pi : \mathfrak{X} \to \Delta$ controls the behavior of the degenerate fiber in a way that the normal bundle of singular locus of $\mathfrak{X}_0$ in $\mathfrak{X}$ is trivial. This property is called *d-semistabilty*.

Definition 4.1 [13, Definition 1.13]. Let $X = \bigcup_{i=0}^N X_i$ be a normal crossing variety of pure dimension n whose irreducible component is smooth. We define X to be *d-semistable* if

$$\bigotimes_{i=0}^{N} I_{X_i}/I_{X_i} I_D \cong \mathcal{O}_D, \tag{4.1}$$

where D is the singular locus of X and I_D (resp. I_{X_i}) is the ideal sheaf of I_D (resp. I_{X_i}).

From now on, we specialize to the case where the degenerate fiber of the semistable degeneration of type $(N+1)$ consists of $N+1$ irreducible components. In this case, we have an equivalent description of d-semistability, which will be used in the mirror construction. Let's write $X_c = \bigcup_{i=0}^N X_i$ for the degenerate fiber $\mathfrak{X}_0$. For any $i, j \in \{0, \dots, N\}$, we write X_{ij} for the intersection of X_i and X_j as a divisor of X_i. Since the degeneration $\pi : \mathfrak{X} \to \Delta$ is semistable, we have $X_c|_{X_i} = \mathfrak{X}_0|_{X_i} \cong \mathfrak{X}_t|_{X_i} = 0$ for $t \neq 0$. It implies that in $\mathrm{Pic}(X_{ij}) \cong \mathrm{Pic}(X_{ji})$, we have the following relation

$$\begin{aligned} 0 &= \mathcal{O}(X_0 + \cdots + X_N)|_{X_i}|_{X_{ij}} \\ &= \mathcal{O}(X_i)|_{X_{ij}} \otimes \mathcal{O}(\sum_{j \neq i} \mathcal{O}(X_j))|_{X_{ji}}. \end{aligned}$$

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The right-hand side, denoted by $N(X_{ij})$, is called *the normal class of* X_{ij}. A collection of all the normal classes is a $\binom{N}{2}$-tuple

$$N_{X_c} := (N(X_{ij})) \in \bigoplus_{i<j} \mathrm{Pic}(X_{ij}).$$

The triviality of the collection of normal classes of X_c implies (4.1). In our case, this is indeed equivalent.

Proposition 4.2. *Suppose that the normal crossing variety* $X_c = \bigcup_{i=0}^N X_i$ *introduced above is smoothable with a semistable degeneration of type* $(N+1)$. *Then the d-semistablity is equivalent to the triviality of the collection of normal classes of* X_c.

Proof. The same argument for the type III case [23, Proposition 4.1] applies to this case. □

In general, the d-semistability condition is not sufficient to imply the smoothability with a smooth total space. In case that X_c is Calabi–Yau, which is of our main interest, this direction has been studied by Kawamata–Namikawa [22].

Theorem 4.3 [22, Theorem 4.2]. *Let* $X_c = \bigcup X_i$ *be a compact Kähler normal crossing variety of dimension n such that*

1. X_c *is d-semistable;*
2. *its dualizing sheaf* ω_{X_c} *is trivial;*
3. $H^{n-2}(X_c, \mathcal{O}_{X_c}) = 0$ *and* $H^{n-1}(X_i, \mathcal{O}_{X_i}) = 0$ *for all i.*

Then X_c *is smoothable to a Calabi–Yau n-fold X with a smooth total space.*

The following definition is motivated by the type III case [23, Definition 2.1].

Definition 4.4. Let X be a Calabi–Yau projective normal crossing variety. X is called *d-semistable of type* $(N+1)$ if there exists a type $(N+1)$ semistable degeneration $\phi : \mathfrak{X} \to \Delta$ whose degenerate fiber $\mathfrak{X}_0$ is X.

Example 4.5. Let $Q_5 \subset \mathbb{P}^4$ be a smooth quintic 3-fold. In the anticanonical linear system, it degenerates to a normal crossing union of two smooth hyperplanes H_1 and H_2 and a smooth cubic 3-fold Q_3. For simplicity, we denote it by $Z_c := Z_1 \cup Z_2 \cup Z_3$, where $Z_1 = H_1$, $Z_2 = H_2$ and $Z_3 = Q_3$. Note that the total space of such degeneration is singular so that one needs to modify X_c to obtain a semistable degeneration. First, consider the intersection between a generic quintic 3-fold and Z_c. It becomes a union of three curves C_1, C_2 and C_3, where C_i lies in Z_{jk} and $C_i \cap Z_{123}$ are all the same for $\{i, j, k\} = \{1, 2, 3\}$. For $\{i, j\} = \{1, 2\}$, we take a blow up of Z_i along C_3, denoted by $\pi_i : \mathrm{Bl}_{C_j} Z_i \to Z_i$. Let E_i be an exceptional divisor and write $(-)'$ for the proper transformation of the subvariety $(-)$. While the proper transform Z'_{i3} is isomorphic to Z_{i3}, Z'_{ij} is the blow up of Z_{ij} along $C_3 \cap Z_{ij}$. By construction C'_3 is disjoint from Z'_{123}. The last step is to blow up $\mathrm{Bl}_{C_3} Z_1$ along C'_3. If we write the resulting normal crossing variety as $X_c = X_1 \cup X_2 \cup X_3$, we have

$$\begin{aligned}(X_1, X_{12} \cup X_{13}) &\cong (\mathrm{Bl}_{C'_3} \mathrm{Bl}_{C_2} Z_1, Z'_{12} \cup Z_{13}),\\ (X_2, X_{21} \cup X_{23}) &\cong (\mathrm{Bl}_{C_1} Z_2, Z'_{12} \cup Z_{13}),\\ (X_3, X_{31} \cup X_{32}) &\cong (Z_3, Z_{12} \cup Z_{13}).\end{aligned} \tag{4.2}$$

In [12], the authors present a mirror construction for this example by considering this degeneration as an iterative Tyurin degeneration.

4.2. Mirror construction

Let $X_c = \bigcup_{i=0}^N X_i$ be a d-semistable Calabi–Yau n-fold of type $(N+1)$ and X be a smoothing of X_c. For each i, the irreducible component X_i is quasi-Fano with the canonically chosen anticanonical divisor $\bigcup_{j\neq i} X_{ij}$. Suppose that the mirror hybrid LG model of the pair $(X_i, \cup_{j\neq i} X_{ij})$ is given by $(Y_i, \omega_i, h_i = (h_{i0}, \ldots, h_{i\hat{i}}, \ldots, h_{iN}) : Y_i \to \Delta^N)$. Here, we shrink the base of the hybrid LG potential to a sufficiently large polydisks Δ^N. We propose a topological construction of a mirror Calabi–Yau manifold of X as a gluing of the hybrid LG models (Y_i, ω_i, h_i).

To perform the gluing, we require more topological conditions on the hybrid LG models. In X_c, two divisors $X_{ij} \subset X_i$ and $X_{ji} \subset X_j$ are topologically identified for $i \neq j$. This should be reflected on the mirror side by requiring that two induced hybrid LG models $(Y_{ij} := h_{ij}^{-1}(t_j), h_i|_{Y_{ij}} : Y_{ij} \to \Delta^{N-1})$ and $(Y_{ji} := h_{ji}^{-1}(t_i), h_j|_{Y_{ji}} : Y_{ji} \to \Delta^{N-1})$ are topologically the same for $t_i, t_j \in \partial\Delta$. Furthermore, one can enhance the topological identification by taking into account complex structures and symplectic structures. For instance, once the preferred choice of the topological identification $X_{ij} = X_{ji}$ has been made, the complex isomorphism between X_{ij} and X_{ji} is given by an element $f_{ij} \in \mathrm{Aut}(X_{ij})$ which is homotopic to the identity. Also, these identifications should be compatible to endow X_c with a well-defined complex structure: For $i \neq j \neq k$, the composition of the restrictions $f_{ki}|_{X_{kij}} \circ f_{jk}|_{X_{jki}} \circ f_{ij}|_{X_{ijk}} \in \mathrm{Aut}(X_{ijk})$ is homotopic to the identity. A mirror counterpart should be the identification given by an element of symplectomorphisms $g_{ij} \in \mathrm{Symp}(Y_{ij}, \omega_i|_{Y_{ij}}, h_i|_{Y_{ij}})$ which preserves the hybrid LG potentials. For $i \neq j \neq k$, the composition of the restrictions $g_{ki}|_{(Y_{kij}, h_k)} \circ g_{jk}|_{(Y_{jki}, h_j)} \circ g_{ij}|_{(Y_{ijk}, h_i)} \in \mathrm{Symp}(Y_{ijk}, \omega_i|_{Y_{ijk}}, h_i|_{Y_{ijk}})$ is required to be homotopic to the identity.

Hypothesis 4.6.

1. *For $i \neq j$, two induced hybrid LG models $(Y_{ij} := h_{ij}^{-1}(t_j), h_i|_{Y_{ij}} : Y_{ij} \to \Delta^{N-1})$ and $(Y_{ji} := h_{ji}^{-1}(t_i), h_j|_{Y_{ji}} : Y_{ji} \to \Delta^{N-1})$ are topologically the same for any $t_i, t_j \in \partial\Delta$. In particular, if symplectic structures are taken into account, this identification is given by a symplectomorphism $g_{ij} \in \mathrm{Symp}(Y_{ij}, \omega_i|_{Y_{ij}}, h_i|_{Y_{ij}})$, which is homotopic to the identity.*
2. *For $i \neq j \neq k$, the composition of the induced symplectomorphism $g_{ki}|_{(Y_{kij}, h_k)} \circ g_{jk}|_{(Y_{jki}, h_j)} \circ g_{ij}|_{(Y_{ijk}, h_i)} \in \mathrm{Symp}(Y_{ijk}, \omega_i|_{Y_{ijk}}, h_i|_{Y_{ijk}})$ is homotopic to the identity.*

Since there is already a global complex structure on X_c, without loss of generality, we may assume that all such gluing automorphisms are indeed the identity. This follows from perturbing the complex and symplectic structures in the beginning.

In fact, the identification on the bases Δ^N along the boundary components is modelled on the normal crossing union. For example, we can consider the base of the i-th hybrid LG model, denoted by $\Delta^N_{h_i}$, sits in $\mathbb{C}^{N+1}$ as

$$\Delta^N_{h_i} \cong \{|z_j| \leq 1, z_i = t_i | j \neq i\}$$

for some t_i with $|t_i| = 1$. Thus, we get a normal crossing union of (Y_i, ω_i) equipped with the induced map to a normal crossing union of the base $\Delta^N_{h_i}$ of each potential h_i. Moreover, topologically, we can further glue these bases $\Delta^N_{h_i}$ along the boundary components until the monodromies associated to h_i come into this procedure. Then the resulting base space becomes topologically the same with $\mathbb{C}^N$, hence we obtain a topological fibration $\tilde{\pi} : \tilde{Y} \to \mathbb{C}^N$. We will give more precise description of $\tilde{\pi} : \tilde{Y} \to \mathbb{C}^N$ after Proposition 4.7.

From now on, we assume that the collection of the hybrid LG models $(Y_i, \omega_i, h_i : Y_i \to \Delta^N)$ satisfies Hypothesis 4.6. Then we can interpret the vanishing of the normal classes of X_c as the relation of the monodromies associated to the hybrid LG models based on Ansatz 3.9. Recall that d-semistability is equivalent to the triviality of normal class in $\mathrm{Pic}(X_{ij})$

$$\begin{aligned} 0 &= \mathcal{O}(X_0 + \cdots + X_N)|_{X_i}|_{X_{ij}} \\ &= \mathcal{O}(X_i)|_{X_{ij}} \otimes \mathcal{O}(\sum_{j\neq i} \mathcal{O}(X_j))|_{X_{ji}} \end{aligned} \tag{4.3}$$

for any $i \neq j$. For each hybrid LG model $(Y_i, \omega_i, h_i : Y \to \Delta^N)$, we write monodromies induced by the loop along the j-th coordinate and the diagonal by $\phi_{T_{ij}}$ and ϕ_{T_i}, respectively. Then the mirror counterpart

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of the relation (4.3) corresponds to

$$\phi_{T_{ij}} \circ \phi_{T_j} = \mathrm{Id} \in \mathrm{Symp}(Y_{ij}, \omega_i|_{Y_{ij}}, h_i|_{Y_{ij}}). \tag{4.4}$$

In other words, we have the following correspondence of monodromies

$$\begin{array}{ccc} Y_i & & Y_j \\ \big\downarrow h_i & & \big\downarrow h_j \\ \Delta^N & & \Delta^N \end{array}$$

$$(t_{i0}, \ldots, t_{i\hat{i}}, \ldots, e^{\sqrt{-1}\theta} t_{ij}, \ldots t_{iN}) \longleftrightarrow e^{-\sqrt{-1}\theta}(t_{j0}, \ldots, t_{j\hat{j}}, \ldots, t_{jN}).$$

Proposition 4.7. *Suppose that the hybrid LG models* $\{(Y_i, \omega_i, h_i : Y_i \to \Delta^N) | i = 0, \ldots, N\}$ *introduced above satisfy Hypothesis 4.6 and the relation (4.4). Then they can be glued to yield a symplectic fibration* $\pi : Y \to \mathbb{P}^N$.

Proof. Let $[z_0 : \cdots : z_N]$ be homogeneous coordinates on $\mathbb{P}^N$. Consider the closed subsets $\Delta_i \subset \mathbb{P}^N$

$$\Delta_i = \left\{ |z_j| \le |z_i| \text{ for } j \neq i \right\}$$

for $i = 0, \ldots, N$. First, note that $\cup_{i=0}^N \Delta_i = \mathbb{P}^N$. Since $\Delta_i \subset U_i = \{z_i \neq 0\}$, we have $\Delta_i \cong \Delta^N$ and any k-th intersection of Δ_i's is homeomorphic to $(S^1)^k \times \Delta^{N-k}$. Due to Hypothesis 4.6, we can identify the base of $h_i : Y \to \Delta^N$ with Δ_i for all i's. If we take the coordinates $(t_{i0}, \cdots, t_{i\hat{i}}, \cdots, t_{iN})$ of Δ_i, where $t_{ik} = \frac{z_k}{z_i}$, then Δ_i becomes the closed unit disk. Then the chart map between Δ_i and Δ_j is exactly the same as the relation (4.4) because this is given by multiplying t_{ij}^{-1}. □

We keep the notation used in the proof of Proposition 4.7. Consider the moment map $\mu : \mathbb{P}^N \to \mathbb{R}^N$ which is given by

$$\begin{aligned} \mu : \mathbb{P}^N &\to \mathbb{R}^N \\ [z_0 : \cdots : z_N] &\mapsto \left(\frac{|z_0|}{\sum_{i=0}^N |z_i|}, \cdots, \frac{|z_{N-1}|}{\sum_{i=0}^N |z_i|} \right). \end{aligned}$$

Note that the image $\mathrm{Im}(\mu) \in \mathbb{R}^N$ is the standard N-simplex Δ. Also, a fiber over a k-dimensional face σ is $(S^1)^k$. We consider the dual spine Π^N in Δ, defined as the subcomplex of the first barycentric subdivision of Δ spanned by the 0-skeleton of the first barycentric subdivision minus the 0-skeleton of Δ. Decomposing Δ along Π^N, we have $N+1$ cubes $\square_0, \ldots, \square_N$, each of which is pulled a product of disks D^N in $\mathbb{P}^N$. These are exactly the polydisks $\Delta_0, \ldots, \Delta_N$ we have introduced. We illustrate the case of $\mathbb{P}^2$ in Figure 1: the dual spine Π^2 is the union of dotted segments that decomposes Δ into three cubes $\square_0, \square_1$ and $\square_2$. Also, observe that each k-th intersection of these cubes pulls backs to $(S^1)^{N-k} \times D^k$, which is the same as the k-th intersection of Δ_i's.

Now, we can see how $\tilde{\pi} : \tilde{Y} \to \mathbb{C}^N$ sits in the fibration $\pi : Y \to \mathbb{P}^N$ more rigorously. Take an open cover $\{V_i' | i = 0, \ldots, N\}$ of the N-simplex Δ such that for each subset $I \subset \{0, \ldots, N\}$, V_I' only contains $\square_I$ among $\square_J$'s for $|J| = |I|$. Here, we follow our convention to denote the intersections. Let V_i denote the preimage of V_i' under μ. Suppose we remove the image of a generic $\mathbb{P}^{N-1} \subset \mathbb{P}^N$ near the dual spine Π^N. The overlap V_{ij} is now diffeomorphic to $\Delta^{N-1} \times (S^1 \times [0,1] - \{pt\})$ hence not contracts to $\Delta^{N-1} \times S^1$. However, if one instead removes a small closed neighborhood $N_\epsilon(\mathbb{P}^{N-1})$ of $\mathbb{P}^{N-1}$ in $\mathbb{P}^N$ and shrink V_i's if necessary, then V_{ij} becomes diffeomorphic to Δ^{N-1} (See Figure 2 for $N = 2$). Since V_i contracts to the base of the hybrid LG model (Y_i, ω_i, h_i), the induced symplectic

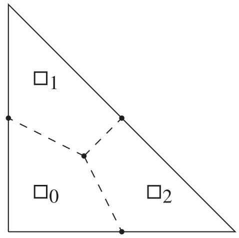

Figure 1. *Description of the dual spine* Π^2.

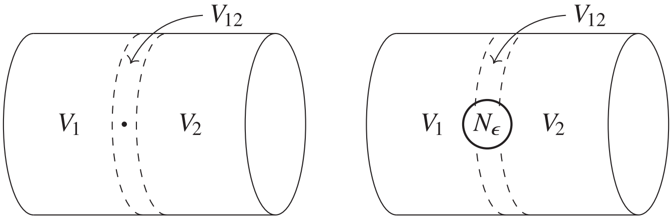

Figure 2. *Description of the intersection of* V_1 *and* V_2 *in* $\mathbb{P}^2$.

fibration $\pi : \pi^{-1}(\mathbb{P}^N \setminus N_\epsilon(\mathbb{P}^{N-1})) \to \mathbb{P}^N \setminus N_\epsilon(\mathbb{P}^{N-1})) \cong \mathbb{C}^N$ can be seen as $\tilde{\pi} : \tilde{Y} \to \mathbb{C}^N$. In other words, Proposition 4.7 is equivalent to saying that $\tilde{\pi} : \tilde{Y} \to \mathbb{C}^N$ is compactifiable (to $\pi : Y \to \mathbb{P}^N$) if the condition (4.4) holds.

Remark 4.8. In general, there is a significant difference between gluing polydisks and the standard open charts of $\mathbb{P}^N$. This is because the former procedure encodes information about singularities of each hybrid LG model while the latter is too rigid to do so.

Theorem 4.9. *Let* $X_c = \bigcup_{i=0}^N X_i$ *be a d-semistable Calabi–Yau n-fold of type* $(N+1)$ *and* X *be its smoothing. Suppose that we have hybrid LG models* $(Y_i, \omega_i, h_i : Y_i \to \Delta^N)$ *mirror to* $(X_i, \cup_{j\neq i} X_{ij})$ *that satisfies Hypothesis 4.6 and the relation (4.4). Let* $\tilde{Y}$ *and* Y *be glued symplectic manifolds constructed above. Then*

1. *Y is topological mirror to X. In other words, $e(Y) = (-1)^n e(X)$,*
2. *$\tilde{Y}$ is topological mirror to X_c. In other words, $e(\tilde{Y}) = (-1)^n e(X)$,*

where $e(-)$ is the Euler characteristic.

Lemma 4.10. *Let* $h : Y \to \mathbb{C}^N$ *be a hybrid LG model. Then* $e(Y_{sm}) = \sum_{|I|=1}^N (-1)^{|I|-1} e(Y_I)$.

Proof. The gluing property (Proposition 3.2) of the hybrid LG model $h : Y \to \mathbb{C}^N$ implies that there exists an open cover $\{U_i | i = 1, \dots, N\}$ of Y_{sm} such that the induced fibration $h|_{U_I} : U_I \to \mathbb{C}^{N-|I|}$ is isotopic to $h|_{Y_I} : Y_I \to \mathbb{C}^{N-|I|}$. The conclusion follows from the Mayer–Vietoris argument. □

Proof of Theorem 4.9. Both items (1) and (2) are proven by the Mayer–Vietoris argument. We prove the item (2) first. By the Mayer–Vietoris sequence, we have

$$\begin{aligned} e(X_c) &= \sum_{|I|=1}^{N+1} (-1)^{|I|-1} e(X_I) \\ &= \sum_{|I|=1}^{N+1} (-1)^{|I|-1} (-1)^{n-|I|+1} e(Y_I, Y_{I,sm}) \\ &= (-1)^n \sum_{|I|=1}^{N+1} e(Y_I, Y_{I,sm}). \end{aligned}$$

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Now, it is enough to show that $e(\tilde{Y}) = \sum_{|I|=1}^{N+1} e(Y_I, Y_{I,sm})$. Take an open cover $\{U_i | i = 0, \ldots, N\}$ of the base of $\tilde{\pi} : \tilde{Y} \longrightarrow \mathbb{C}^N$ such that $\tilde{\pi}^{-1}(V_I)$ contracts to Y_I for all I. By applying the Mayer–Vietoris argument, we have

$$e(\tilde{Y}) = \sum_{|I|=1}^{N+1} (-1)^{|I|-1} e(Y_I).$$

Since $e(Y_I) = e(Y_I, Y_{I,sm}) + e(Y_{I,sm})$, Lemma 4.10 implies that

$$\begin{aligned} e(\tilde{Y}) &= \sum_{|I|=1}^{N+1} (-1)^{|I|-1} (e(Y_I, Y_{I,sm}) + e(Y_{I,sm})) \\ &= \sum_{|I|=1}^{N+1} (-1)^{|I|-1} \left(e(Y_I, Y_{I,sm}) + \sum_{|J|=1, J\cap I=\emptyset}^{N+1-|I|} (-1)^{|J|-1} e(Y_{I,J}) \right). \end{aligned}$$

Here, $Y_{I,J}$ is the same with $Y_{I\cup J}$ by Hypothesis 4.6 (1), but we use different notation to emphasize the Mayer–Vietoris procedure. By rewriting $e(Y_{I,J}) = e(Y_{I,J}, Y_{I,J,sm}) + e(Y_{I,J,sm})$, we can iteratively apply Lemma 4.10. Then we get $e(\tilde{Y}) = \sum_{|I|=1}^{N+1} e(Y_I, Y_{I,sm})$ by taking the resummation.

We apply similar argument to prove an item (1). It follows from the same method in [23, Proposition 3.2] that the Euler characteristic of the smoothing manifold X is given by

$$e(X) = \sum_{|I|=1}^{N+1} (-1)^{|I|} |I| e(X_I).$$

By assumption, we have $e(X) = (-1)^n \sum_{|I|=1}^{N+1} |I| e(Y_I, Y_{I,sm})$. Take an open cover $\{U_i | i = 0, \ldots, N\}$ of the base of $h : Y \longrightarrow \mathbb{P}^N$ as before. Note that for any I, the intersection U_I contracts to $(S^1)^{|I|-1} \times \Delta^{N+1-|I|}$. Applying the Mayer–Vietoris argument with respect to the induced open cover $\{h^{-1}(U_i)\}$, one can see that the Euler characteristic $e(h^{-1}(U_I))$ vanishes for $|I| > 1$ because $h^{-1}(U_I)$ contracts to a fiber bundle over $(S^1)^{|I|-1}$ with a fiber Y_I. Therefore,

$$e(Y) = \sum_{|I|=1} e(Y_I) = \sum_{|I|=1} e(Y_I, Y_{I,sm}) + e(Y_{I,sm}).$$

By iteratively applying Lemma 4.10, we get the conclusion. □

5. Mirror P=W conjectures

We maintain the notation used in the previous section. Let X be a smoothing of $X_c = \bigcup_{i=0}^N X_i$, a d-semistable Kähler Calabi–Yau n-fold of type $(N+1)$. We have introduced the topological construction of their mirror objects that comes with additional symplectic fibration structure $(Y, \pi : Y \longrightarrow \mathbb{P}^N)$ and $(\tilde{Y}, \tilde{\pi} : \tilde{Y} \longrightarrow \mathbb{C}^N)$, respectively. In this section, we discuss a refined version of Theorem 4.9. In the degeneration picture, we have two filtrations on the cohomology groups: the monodromy weight filtration $W_{\lim \bullet}$ on $H^*(X)$ and the Deligne's canonical weight filtration $W_\bullet$ on $H^*(X_c)$. The corresponding mirror filtrations are conjectured to be the perverse Leray filtration (equivalently, the δ-flag filtration) on $H^*(Y)$ associated to h and the perverse Leray filtration (equivalently, the G-flag filtration) on $H^*(\tilde{Y})$ associated to $\tilde{h}$, respectively.

Degeneration (B-side)	Fibration (A-side)
Monodromy weight filtration $W_{\lim \bullet}$	Perverse Leray filtration $P^{\pi}_\bullet$ associated to π
Deligne's canonical weight filtration $W_\bullet$	Perverse Leray filtration $P^{\tilde{\pi}}_\bullet$ associated to $\tilde{\pi}$

As noted in Remark 1, since the gluiing construction (Proposition 4.7) is not performed in the complex category, we mainly consider the flag filtrations introduced in Section 2.3 which are potentially equivalent to the perverse Leray filtrations.

Theorem 5.1. *Suppose that each mirror pair* $(X_i, \cup_{j\neq i} X_{ij})|(Y_i, \omega_i, h_i : Y_i \to \mathbb{C}^N)$ *satisfies Conjecture 3.7. Then*

1. *for X and Y, we have*

$$\bigoplus_{p-q=a} Gr_F^p Gr_{p+q}^{W_{lim}} H^{p+q+l}(X) \cong Gr_{n+a}^{P^\pi} H^{n+a+l}(Y),$$

   where $-N \leq l \leq N$.
2. *for* X_c *and* $\tilde{Y}$, *we have*

$$\bigoplus_{p-q=a} Gr_F^p Gr_{p+q}^{W} H^{p+q+l}(X_c) \cong Gr_{n+a}^{P^{\tilde{\pi}}} H_c^{n+a+l}(\tilde{Y}),$$

   where $0 \leq l \leq N$.

Proof of Theorem 5.1-(1). Let $\pi : Y \to \mathbb{P}^N$ be a gluing of $N+1$ hybrid LG models $(Y_i, h_i : Y_i \to \mathbb{C}^N)$. Take an open cover $\{V_s\}$ of $\mathbb{P}^N$ as explained in the discussion after Proposition 4.7 such that the induced fibration $\pi^{-1}(V_s) \to V_s$ contracts to $h_s : Y_s \to \Delta^N$. We consider a general linear flag in $\mathbb{P}^N$

$$\mathfrak{H} : 0 = H_{-N-1} \subset H_{-N} \subset \cdots \subset H_{-1} \subset H_0 = \mathbb{P}^N$$

which satisfies several properties:

1. $\mathfrak{H}$ intersects transversally with the discriminant locus of π in the sense of [7, Definition 5.2.4];
2. the induced flag $\mathfrak{H} \cap V_s$ is not parallel to any coordinate lines of the base of $h_s : Y_s \to \Delta^N$.

We also consider a pair of such general linear flags $(H_\bullet, L_\bullet)$ of $\mathbb{P}^N$,

$$\mathfrak{H} : H_{-n} \subset \cdots \subset H_{-1} \subset H_0 = \mathbb{P}^N$$
$$\mathfrak{L} : L_{-n} \subset \cdots \subset L_{-1} \subset H_0 = \mathbb{P}^N.$$

Due to the genericity, for $i = -N, \ldots, 0$ and $j = 0, \ldots, N$, we may assume there exists a collection of sufficiently small $\epsilon_j > 0$ which yields isomorphisms of pairs

$$((H_{-j} - H_{-j-1}) \cap L_i, (H_{-j} - H_{-j-1}) \cap L_{i-1}) \cong ((H_{-j} - N_{\epsilon_j}(H_{-j-1})) \cap L_i, (H_{-j} - N_{\epsilon_j}(H_{-j-1})) \cap L_{i-1}). \tag{5.1}$$

Not only that, we further assume that a flag $\mathfrak{L}$ satisfies the following: if necessary, one can modify the cover $\{V_s\}$ in a way that the induced cover of $\mathbb{P}^N - N_{\epsilon_0}(H_{-1})$, denoted by $\{V_s^{(0)}\}$, satisfies the following properties:

1. for nonempty index set I, $V_I^{(0)} \cong \Delta^{N+1-|I|}$;
2. the induced open regions $\{V_{st}^{(0)} | t \neq s\}$ yield the gluing property of the induced flag $V_s^{(0)} \cap \mathfrak{L}$ in the sense of the discussion before Lemma 3.5.

The first condition is the one explained in the discussion after Proposition 4.7. The second condition can be obtained by rescaling each hybrid LG potential $h_s : Y_s \to \Delta^N$ before the gluing. Again, one can further modify the open cover $\{V_s\}$ in a way that the induced cover of $H_{-1} \setminus N_{\epsilon_1}(H_{-2})$, denoted by $\{V_s^{(1)}\}$ satisfies the following properties

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$$V_I^{(1)} \cong \begin{cases} \Delta_I \cap H_{-1} & |I| < 2 \\ D^{N-|I|} & |I| \geq 2, \end{cases}$$

where $D^{N-|I|}$ is some polydisk in the intersection $\Delta_I \cong S^{|I|-1} \times D^{N-|I|}$ for $|I| > 1$. Similarly, we may assume that the induced open regions $\{V_{st} | t \neq s\}$ restricted to $\mathbb{P}^N \setminus N_{\epsilon_1}(H_{-2})$ yields the gluing property of the induced flag $\mathfrak{L} \cap H_{-1} \cap V_s$. Inductively, we obtain an open cover $\{V_s\}$ such that the induced open cover of $H_{-j} \setminus N_{\epsilon_j}(H_{-j-1})$, denoted by $\{V_s^{(j)}\}$, satisfies the following properties

$$V_I^{(j)} \cong \begin{cases} \Delta_I \cap H_{-j} & |I| < j+1 \\ D^{N-|I|} & |I| \geq j+1 \end{cases}$$

for $j = 0, \ldots, N$. Also, the induced open regions $\{V_{st} | t \neq s\}$ restricted to $\mathbb{P}^N \setminus N_{\epsilon_j}(H_{-j-1})$ yields the gluing property of the induced flag $\mathfrak{L} \cap H_{-j} \cap V_s$. Let's write $Z_{-j} := \pi^{-1}(H_{-j})$ and $W_i := \pi^{-1}(L_i)$.

Lemma 5.2. *For each $q \geq 0$, we have*

$$H^q_{Z_{-j}-Z_{-j-1}}(Y, \mathbb{C}|_{W_i - W_{i-1}}) \cong \bigoplus_{|I|=j-i+1} H^{q-2r}(Y_I, Y_{I,sm}; \mathbb{C}), \tag{5.2}$$

where r is the codimension of $Z_{-j} \cap W_i$ in W_i.

Proof of Lemma 5.2. We first rewrite the cohomology $H^q_{Z_{-j}-Z_{-j-1}}(Y, \mathbb{C}|_{W_i - W_{i-1}})$ by considering the excision principle for local cohomology groups. Then we have

$$\begin{aligned} H^q_{Z_{-j}-Z_{-j-1}}(Y, \mathbb{C}|_{W_i - W_{i-1}}) &\cong H^q_{Z_{-j} \cap (Y - Z_{-j-1})}(Y - Z_{-j-1}, \mathbb{C}|_{W_i - W_{i-1} \cap (W - Z_{-j-1})}) \\ &\cong \left[H^q_{Z_{-j} \cap W_i^\circ}(W_i^\circ) \longrightarrow H^q_{Z_{-j} \cap W_{i-1}^\circ}(W_{i-1}^\circ) \right] \\ &\cong H^{q-2r}(Z_{-j} \cap W_i^\circ, Z_{-j} \cap W_{i-1}^\circ), \end{aligned}$$

where $W_\bullet^\circ := W_\bullet - W_\bullet \cap Z_{-j-1}$. The last isomorphism comes from the tubular neighborhood theorem. By the condition (5.1) on the flag $\mathfrak{H}$, we have

$$H^{q-2r}(Z_{-j} \cap W_i^\circ, Z_{-j} \cap W_{i-1}^\circ) \cong H^{q-2r}((Z_{-j} - \pi^{-1}(N_{\epsilon_j}(H_{-j-1})) \cap W_i, (Z_{-j} - \pi^{-1}(N_{\epsilon_j}(H_{-j-1}))) \cap W_{i-1}).$$

Now, we take Mayer–Vietoris sequence with respects to $\{U_s := \pi^{-1}(V_s) | s = 0, \cdots, N\}$. Note that over U_I, the gluing property yields

$$U_I \cap (Z_{-j} - \pi^{-1}(N_{\epsilon_j}(H_{-j-1}))) \cong \begin{cases} Y_{I,sm^{(-i+j-|I|+1)}} & |I| < j-i+1 \\ Y_I & |I| \geq j-i+1. \end{cases}$$

Therefore, the Mayer–Vietoris sequence is given by

$$\begin{aligned} &\bigoplus_{|I|=1} H^{q-2r}(Y_{I,sm^{(-i+j)}}, Y_{I,sm^{(-i+j+1)}}) \xrightarrow{d_1} \bigoplus_{|I|=2} H^{q-2r}(Y_{I,sm^{(-i+j-1)}}, Y_{I,sm^{(-i+j)}}) \\ &\qquad \xrightarrow{d_1} \cdots \xrightarrow{d_1} \bigoplus_{|I|=j-i+1} H^{q-2r}(Y_I, Y_{I,sm}) \to 0, \end{aligned} \tag{5.3}$$

where the differential satisfies the Mayer–Vietoris sign rule (2.1). Also, for $I = \{i_1, \ldots, i_k\}$, the direct summand $H^{q-2r}(Y_{I,sm^{(-i+j+1-k)}}, Y_{I,sm^{(-i+j-k)}})$ of the k-th term can be computed by regarding the pair $(Y_{I,sm^{(-i+j+1-k)}}, Y_{I,sm^{(-i+j-k)}})$ as subspaces of Y_{i_1}. The choice of the index i_1 does not matter because of the topological restriction we've made (Hypothesis 4.6). Then it becomes $\bigoplus_J H^{q-2r}(Y_{I\cup J}, Y_{I\cup J,sm})$ for $J \subset \{0,\dots,N\}\setminus I$ with $|J| = -i+j+1-k$. In other words, the k-th term is given by

$$S_k := \bigoplus_{|I|=k}\ \bigoplus_{J\subset\{0,\dots,N\}\setminus I,\,|J|=-i+j+1-k} H^{q-2r}(Y_{I\cup J}, Y_{I\cup J,sm})$$

so that the sequence (5.3) becomes

$$S_1 \xrightarrow{d} S_2 \xrightarrow{d} \cdots \xrightarrow{d} S_{j-i+1} \to 0,$$

where each map d is the signed sum of the isomorphisms where the signs are determined by the Mayer–Vietoris rule (2.1). In fact, it fits into the simple combinatorial sequence

$$0 \to S_0 \xrightarrow{d} S_1 \xrightarrow{d} S_2 \xrightarrow{d} \cdots \xrightarrow{d} S_{j-i+1} \to 0$$

hence the conclusion follows. □

Next, we rewrite the E_1-page of the spectral sequence for the δ-filtration. Recall that ${}^{\delta}E_1^{l,n+a} = \bigoplus_{i+j=l} H^{n+a+l}_{Z_{-j}-Z_{-j-1}}(Y, \mathbb{C}|_{Y_i-Y_{i-1}})$ and the differential d_1 is the signed sum of the connecting homomorphisms (see 2.4 for the precise description). Since we choose the open cover $\{U_s | s = 0,\cdots,N\}$ in the proof of Lemma 5.2 that is independent of i and j, the isomorphisms in Lemma 5.2 respects the functoriality. Then we have

$${}^{\delta}E_1^{l,n+a} \cong \bigoplus_{i+j=l}\ \bigoplus_{|I|=j-i+1} H^{n+a+l-2r}(Y_I, Y_{I,sm}; \mathbb{C}),$$

where r is the codimension of $Z_{-j}\cap W_i$ in W_i. More explicitly, we have

$$\begin{array}{ccccc}
E_1^{-1,n+a} & \xrightarrow{\quad d_1\quad} & E_1^{0,n+a} & \xrightarrow{\quad d_1\quad} & E_1^{1,n+a}\\[2ex]
\bigoplus_{|I|=2} H^{n+a-1}(Y_I, Y_{I,sm}) & \longrightarrow & \bigoplus_{|I|=1} H^{n+a}(Y_I, Y_{I,sm}) & & \\
\bigoplus & \searrow & \bigoplus & \searrow & \\
\bigoplus_{|I|=4} H^{n+a-3}(Y_I, Y_{I,sm}) & \longrightarrow & \bigoplus_{|I|=3} H^{n+a-2}(Y_I, Y_{I,sm}) & \longrightarrow & \bigoplus_{|I|=2} H^{n+a-1}(Y_I, Y_{I,sm})\\
\bigoplus & \searrow & \bigoplus & \searrow & \bigoplus\\
\vdots & & \bigoplus_{|I|=5} H^{n+a-4}(Y_I, Y_{I,sm}) & \longrightarrow & \bigoplus_{|I|=4} H^{n+a-3}(Y_I, Y_{I,sm}).
\end{array}$$

The horizontal (resp. antidiagonal) differential d_I (resp. d_{II}) is the signed sum of the relevant connecting homomorphisms ρ_I^J (resp. $(\rho_I^J)^\vee$) following the Mayer–Vietoris sign rule (2.1). The differential $d_1 : {}^{\delta}E_1^{l,n+a} \to {}^{\delta}E_1^{l+1,n+a}$ is given by $d_1 = d_I + (-1)^l d_{II}$. Therefore, we have the following equivalence of the E_1-page of the spectral sequences

$$\Big(\bigoplus_{p-q=a} \mathrm{Gr}_F^p\, {}^{W(M)}E_1^{l,p+q}, d_1\Big) \cong \big({}^{\delta}E_1^{l,n+a}, d_1\big).$$

Since both spectral sequences degenerate at the E_2-page, we have

$$\bigoplus_{p-q=a} \mathrm{Gr}_F^p \mathrm{Gr}^{W_{lim}}_{p+q} H^{p+q+l}(X) = \bigoplus_{p-q=a} \mathrm{Gr}_F^p\big({}^{W(M)}E_2^{l,p+q}\big) \cong {}^{\delta}E_2^{l,n+a} = \mathrm{Gr}_{\delta}^l H^{n+a+l}(Y).$$

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The conclusion follows from Theorem 2.12 (3):

$$^{\delta}E_2^{l,n+a} = \mathrm{Gr}^l_{\delta} H^{n+a+l}(Y) = \mathrm{Gr}^P_{n+a} H^{n+a+l}(Y).$$ □

Proof of Theorem 5.1-(2). The proof of item (2) is almost the same. We use the Mayer–Vietoris argument to describe the E_1-page of the spectral sequence for the flag (=perverse) filtration. To do so, we should work with the regular cohomology groups, not compactly supported ones. The idea is to apply the well-known Poincaré duality statement for the perverse filtration on $H^*_c(\tilde{Y})$:

$$\mathrm{Gr}^P_{n+a} H_c^{n+a+l}(\tilde{Y}) \cong (\mathrm{Gr}^P_{n-a} H^{n-a-l}(\tilde{Y}))^*.$$

In terms of the G-flag filtration, this is isomorphic to $(\mathrm{Gr}_G^{-l} H^{n-a-l}(\tilde{Y}))^*$. Therefore, it is enough to show that

$$\bigoplus_{p-q=a} \mathrm{Gr}^p_F \mathrm{Gr}^W_{p+q} H^{p+q+l}(X_c) \cong (\mathrm{Gr}_G^{-l} H^{n-a-l}(\tilde{Y}))^*.$$

By applying the Mayer–Vietoris argument, the E_1-page of the spectral sequence for the G-flag filtration is given by

$$^{G}E_1^{-l,n-a} \cong \bigoplus_{|I|=l} H^{n-a-l}(Y_I, Y_{I,sm}),$$

and the differential $d_1^G : {}^{G}E_1^{-l,n-a} \to {}^{G}E_1^{-l+1,n-a}$ is the signed sum of the relevant connecting homomorphisms (ρ_I^J) following the Mayer–Vietoris sign rule (2.1). Since the spectral sequence for the G-filtration degenerates at E_2-page, we have ${}^{G}E_2^{-l,n-a} \cong \mathrm{Gr}_G^{-l} H^{n-a-l}(\tilde{Y})$. To compute the Poincaré dual $(\mathrm{Gr}_G^{-l} H^{n-a-l}(\tilde{Y}))^*$, we take the dual of the E_1-page $({}^{G}E_1^{-l,n-a}, d_1^G)$, denoted by $({}^{G}E_1^{-l,n-a^*}, (d_1^G)^*)$. By Poincaré duality (3.2), this becomes

$$({}^{G}E_1^{-l,n-a})^* \cong \bigoplus_{|I|=l} H^{n+a-l}(Y_I, Y_{I,sm})$$

with the induced differential, the signed sum of the relevant connecting homomorphisms $(\rho_I^J)^\vee$ following the Mayer–Vietoris sign rule (2.1). By the assumption (Conjecture 1.6), we have the mirror equivalence of the E_1-page of the spectral sequences

$$(\bigoplus_{p-q=a} \mathrm{Gr}^p_F {}^{W}E_1^{l,p+q}, d_1) \cong (({}^{G}E_1^{l,n+a})^*, (d_1^G)^*).$$

As both spectral sequences degenerate at the E_2-page, we get the conclusion. □

6. Toric degeneration

In this section, we provide a combinatorial way of achieving the degeneration-fibration correspondence for Batyrev mirror paris. After that, we propose how one can see the previous gluing construction.

6.1. Backgrounds on toric varieties

We recollect some backgrounds about toric varieties. We refer for more details to [8]. Let N and M be dual lattices of rank n with the natural bilinear pairing $\langle -, - \rangle : N \times M \to \mathbb{Z}$. We write $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. A rational convex polyhedra cone (simply called *cone*) α in $N_{\mathbb{R}}$ is a convex cone generated by finitely many vectors in N. Associated to a cone α, one can construct an affine toric variety $X_\alpha := \mathrm{Spec}(\mathbb{C}[\alpha^\vee \cap M])$ where $\alpha^\vee \subset M_\mathbb{R}$ is a dual cone of α defined by

$$\alpha^\vee := \{v \in M_\mathbb{R} | \langle v, u\rangle \geq 0 \text{ for all } u \in \alpha\}.$$

Such affine toric varieties can be glued to produce more general toric varieties. This gluing data is combinatorially encoded in a *fan* $\Sigma \subset N_\mathbb{R}$ which is a collection of cones such that

1. each face of a cone in Σ is also a cone in Σ,
2. the intersection of two cones in Σ is a face of each cone.

Given a fan Σ, we define a toric variety $X = X_\Sigma$ by gluing the affine toric varieties $X_\alpha := \mathrm{Spec}(\mathbb{C}[\alpha^\vee \cap M])$: Two affine toric varieties X_α and X_β are glued over $X_{\alpha\beta} := \mathrm{Spec}(\mathbb{C}[(\alpha \cap \beta)^\vee \cap M])$. We call $\{X_\alpha\}$ a toric chart of X_Σ. If $|\Sigma| = N_\mathbb{R}$, it is called *complete*, and the corresponding toric variety X_Σ is compact.

Let $\Sigma[1]$ be a collection of integral primitive ray generators of Σ. Consider the lattice morphism $g : N \longrightarrow \mathbb{Z}^{\Sigma[1]}$ given by $g(v) = (\langle v, \rho\rangle)_{\rho\in\Sigma[1]}$. This induces a short exact sequence

$$0 \longrightarrow N \xrightarrow{g} \mathbb{Z}^{\Sigma[1]} \longrightarrow A_{n-1}(X_\Sigma) \longrightarrow 0,$$

where $\mathbb{Z}^{\Sigma[1]}$ is the set of torus invariant Weil divisors and $A_{n-1}(X_\Sigma)$ is the Chow group of X_Σ. Applying the functor $\mathrm{Hom}(-, \mathbb{C}^*)$ to the above sequence, we get a short exact sequence

$$1 \longrightarrow G \longrightarrow (\mathbb{C}^*)^{\Sigma[1]} \longrightarrow M \otimes_\mathbb{Z} \mathbb{C}^* \longrightarrow 1. \tag{6.1}$$

Let $\{x_\rho\}_{\rho\in\Sigma[1]}$ be a standard basis of rational functions on $\mathbb{C}^{\Sigma[1]}$ and V be the vanishing locus of $\{\prod_{\rho\notin\sigma} x_\rho | \sigma \subset \Sigma\}$. The sequence (6.1) shows that G acts naturally on $\mathbb{C}[(x_\rho)_{\rho\in\Sigma[1]}]$ and leaves V invariant. Then the toric variety X_Σ is the quotient $(\mathbb{C}[(x_\rho)_{\rho\in\Sigma[1]}] \setminus V)//G$ and the homogeneous coordinate ring of X_Σ is equipped with the grading given by the action of G. The sublocus of X_Σ corresponding to $D_\rho = \{x_\rho = 0\}$ is exactly the torus invariant divisors associated to the ray generator ρ. A torus invariant divisor $D = \sum_{\rho\in\Sigma[1]} a_\rho D_\rho$ is Cartier if and only if there is some piecewise linear function ρ on $M_\mathbb{R}$ which linear on the cones of Σ, which takes the integral values on M.

A rational convex polytope Δ in $M_\mathbb{R}$ is the convex hull of finite number of points. We say Δ is a *lattice polytope* if every vertex of Δ is in M. For example, a lattice polytope is given by the intersection of some half spaces cut out by affine hyperplanes

$$\Delta = \{v \in M_\mathbb{R} | \langle v, n_i\rangle \geq -a_i, n_i \in N, a_i \in \mathbb{Z} \text{ for } i = 1, \ldots, s\}.$$

A *l-face* σ is the intersection of Δ with $n - l$ supporting hyperplanes, and we will denote it by $\sigma \prec \Delta$. We also write $\Delta[l]$ for the collection of l-faces of Δ. In particular, a 0-face, a 1-face and a $(n-1)$-face are called a vertex, an edge and a facet of Δ, respectively. For each face $\sigma \prec \Delta$, the cone α_σ dual to σ is defined by

$$\alpha_\sigma = \{u \in N_\mathbb{R} | \langle v, u\rangle \leq \langle v', u\rangle \text{ for all } v \in \sigma \text{ and } v' \in \Delta\}.$$

A collection of dual cones α_σ forms a fan Σ^Δ, called a normal fan of Δ, and we write X_Δ for the associated toric variety.

To a lattice polytope Δ, the associated toric variety X_Δ comes with the divisor $D_\Delta = -\sum_\rho a_\rho D_\rho$ (or simply denoted by D) where the sum is taken over all facets $\rho \prec \Delta$. Equivalently, we get a support function of D, a piecewise linear function ϕ_D such that $\phi_D(v_\rho) = -a_\rho$ for the privimite vector v_ρ to the dual cone of the face $\rho \prec \Delta$. Let $\Delta_D = \{u \in M_\mathbb{R} | u \geq \phi_D \text{ on } N_\mathbb{R}\}$. Geometrically, $\Delta_D \cap M$ generates the space of sections of the line bundle $\mathcal{O}_X(D)$. Note that D is trivial, generated by sections and ample if and only if ϕ_D is affine, convex and strictly convex, respectively.

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A polytope $\Delta \subset M_{\mathbb{R}}$ is called *simplicial*, if there are exactly n edges at each vertex and the primary vectors at each vertex span $M_{\mathbb{R}}$ as a vector space. A fan Σ in $N_{\mathbb{R}}$ is simplicial if all the maximal cones in Σ is simplicial. In particular, if the primary vectors span the lattice, then it is called *nonsingular*.

Proposition 6.1. *If Δ is simplicial (resp. nonsingular), then X_Δ is an orbifold (resp. manifold).*

6.2. Batyrev mirror pairs

We introduce Batyrev mirror pairs [5]. Let Δ be a simplicial lattice polytope in $M_{\mathbb{R}}$. A polar dual of the polytope Δ is defined to be $\Delta^\circ := \{u \in N_{\mathbb{R}} | \langle u, v\rangle \geq -1 \text{ for } v \in \Delta\}$. A lattice polytope is called *reflexive* if its polar dual Δ° is also a lattice polytope. This is equivalent to the condition that the zero 0_M is the only one interior lattice point of Δ. Geometrically, the associated toric variety X_Δ is a Gorenstein Fano variety. From now on, we fix a reflexive polytope $\Delta \subset M_{\mathbb{R}}$ and write Σ_Δ for the fan over the facets of Δ and $\mathbb{P}_\Delta := X_{\Sigma_\Delta}$ for the associated toric variety. Note that Σ_Δ is also the normal fan of the polar dual Δ°, so we have $\mathbb{P}_\Delta = X_{\Sigma_\Delta} = X_{\Sigma^{\Delta^\circ}} = X_{\Delta^\circ}$.

Consider a general Calabi–Yau hypersurface V_Δ of $\mathbb{P}_\Delta$. Since $\mathbb{P}_\Delta$ is an orbifold in general, the hypersurface V_Δ may have singularities. We assume that the hypersurface V_Δ is Δ-regular, meaning that the singular locus of V_Δ is induced from the singular locus of ambient space $\mathbb{P}_\Delta$. Then one may desingularize V_Δ by taking a partial resolution of $\mathbb{P}_\Delta$. Such resolution is given by a refinement $\widetilde{\Sigma}_\Delta$ of the fan Σ_Δ whose cone is contained in a cone of Σ_Δ. In this case, to Σ_Δ, one can add all rays pointing to the elements in $\partial\Delta \cap M$ to obtain $\widetilde{\Sigma}_\Delta$. Batyrev shows that the induced resolution $f : X_{\widetilde{\Sigma}_\Delta} \to \mathbb{P}_\Delta$ is crepant and this is called a maximal projective crepant partial (MPCP) resolution of $\mathbb{P}_\Delta$ [5, Section 2.2]. In particular, if $X := f^*(V_\Delta)$ is smooth, then we say Δ satisfies maximal projective crepant smooth (MPCS) resolution condition. Note that this condition always holds for $n \leq 4$ [5, Section 2.2]. Similarly, consider the dual construction for Δ° and write $X^\vee$ for a MPCP resolution of V_{Δ°.

Theorem 6.2 [5]. *The pair of Calabi–Yau hypersurfaces $(V_\Delta, V_{\Delta^\circ})$ (or $(X, X^\vee)$) satisfies (stringy) Hodge number mirror relation. We call this pair a Batyrev mirror pair.*

6.3. The degeneration-fibration correspondence for Batyrev mirror pairs

We start with reviewing a semistable toric degeneration introduced in [20]. Fix an n-dimensional simplicial polytope Δ. Let Γ be a partition of the polytope Δ into smaller polytopes $\{\Delta_{(i)}\}$. We say the partition Γ is *simplicial* if all subpolytopes $\{\Delta_{(i)}\}$ are simplicial polytopes. We define σ to be l-face of Γ, denoted by $\sigma \prec \Gamma$, if σ is a l-face of $\Delta_{(i)}$ for some i.

Definition 6.3. A simplicial partition Γ is *semistable* if the following conditions hold:

1. each vertex of Δ belongs to only one of $\Delta_{(i)}$'s;
2. for any l-face $\sigma \prec \Gamma$ and k-face $\tau \prec \Delta$ with $\sigma \subset \tau$, then there are exactly $k - l + 1 \Delta_{(i)}$'s such that $\sigma \prec \Delta_{(i)}$.

From now on, we distinguish vertices in Γ from those of Δ: When we say p is a vertex of Γ, it means that p is a vertex of $\Delta_{(i)}$ for some i that is not a vertex of Δ. The restriction of Γ to some face $\sigma \prec \Delta$ is defined to be the partition induced by $\{\Delta_{(i)} \cap \sigma\}$, and denote it by $\Gamma \cap \sigma$.

By definition, if $\cap_{k=1}^{l} \Delta_{(i_k)} \neq \emptyset$, then it has dimension $n - l + 1$. Also, for any vertex p of Γ, there are exactly $n+1$-edges $\sigma_0, \ldots, \sigma_n$ of Γ such that $p \prec \sigma_i$. This allows us to define a dual simplicial complex K_Γ whose vertex set is the set of polytopes $\Delta_{(j)}$'s in the partition Γ. For example, K_Γ is l-simplex if and only if there is a l-face of Δ that contains all vertices of Γ.

Definition 6.4. A vertex $p \prec \Gamma$ is nonsingular if p is nonsingular in one (thus all) subpolytope containing p. A semistable partition Γ is nonsingular if all of vertices are nonsingular.

Definition 6.5. A lifting of Δ by a semistable partition Γ is a triple $(\tilde{\Delta}, \tilde{M}, \pi)$, where $\tilde{\Delta}$ is a lattice polytope of $\tilde{M}$ and $\pi : \tilde{M} \to M$ is a surjective morphism satisfying the following condition: for $\tilde{\sigma} \prec \tilde{\Delta}$,

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either $\pi_*(\tilde{\sigma}) \prec \Delta$ or $\pi_*(\tilde{\sigma}) \prec \Gamma$ where $\pi_* : \tilde{M}_{\mathbb{R}} \to M_{\mathbb{R}}$ is the induced map from π. If $\pi_*(\tilde{\sigma}) \prec \Gamma$, $\pi_*(\tilde{\sigma})$ is said to be a lift of $\pi_*(\tilde{\sigma})$. Also, the lifting is called nonsingular if all polytopes involved are nonsingular.

Proposition 6.6 [20, Proposition 3.12]. *For a nonsingular semistable partition* Γ*, there exists a concave integral piecewise linear function* F_Γ *on* Δ *that is linear on each face of* Γ.

One can take a minimal integral lifting of F_Γ, and we denote it by F.

Theorem 6.7 [20, Theorem 3.13]. *Let* π *be the projection* $\mathbb{Z} \oplus M \to M$ *and* $\tilde{\Delta} = \{(y,x) | y \geq F(x)\} \subset \mathbb{R} \times \Delta \subset \mathbb{R} \times M_{\mathbb{R}}$. *Then* $(\tilde{\Delta}, \mathbb{Z} \oplus M, \pi)$ *is a lifting of* Δ *by* Γ. *If* γ *is a nonsingular partition of a nonsingular polytope, then the lifting is nonsingular.*

Theorem 6.8 [20, Theorem 4.1]. *Suppose* Δ *and* Γ *are both nonsingular. Then there exists a semistable degeneration* $p : X_{\tilde{\Delta}} \to \mathbb{C}$ *of* X_Δ *to* $p^{-1}(0)$. *The dual complex* G *of the degenerate fiber* $p^{-1}(0)$ *is isomorphic to* K_Γ*, and each component in* $p^{-1}(0)$ *is the toric variety defined by the corresponding subpolytope in* Δ.

The semistable degeneration $p : X_{\tilde{\Delta}} \to \mathbb{C}$ induces a semistable degeneration of a Calabi–Yau hypersurface X in X_Δ whose degenerate fiber consists of a generic hypersurface X_i of $X_{\Delta_{(i)}}$ in the linear system $|D_{\Delta_{(i)}}|$. In other words, the normal crossing variety $X_c = \cup_{i=0}^{l} X_i$ is d-semistable of type $l+1$ and X is its smoothing.

Next, we assume that the nonsingular polytope $\Delta \subset M_{\mathbb{R}}$ is reflexive and the origin 0_M is a unique lattice point.

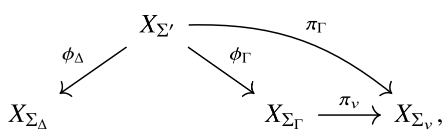
Definition 6.9. A semistable partition Γ is *central* if $0_M \prec \Delta_{(i)}$ for all i.

Let Γ be a nonsingular, central, semistable partition of the polytope Δ. If K_Γ is l-dimensional, there is a unique codimension l linear subspace $L \subset M_{\mathbb{R}}$ passing through the origin such that $\Delta \cap L \prec \Delta_{(i)}$ for all i as a $(n-l)$-face. Consider l primitive vectors given by the intersection of each $\Delta_{(ik)} := \Delta_{(i)} \cap \Delta_{(k)}$ with the orthogonal complement $L^\perp$ of L. Note that the restriction $\Delta \cap L^\perp$ is also reflexive and simplicial while the induced partition is not necessarily semistable. Thus, there are $l+1$ such primitive vectors $v_0, \ldots, v_l$ such that for each i, $\Delta_{(i)}$ contains all v_j except $j = i$.

Next, consider the dual reflexive polytope $\Delta^\circ \subset N_{\mathbb{R}}$ whose dual fan is $\Sigma_\Delta \subset M_{\mathbb{R}}$. For simplicity, we assume that Δ° satisfies the MPCS resolution condition. We explain what the central semistable partition corresponds to in the dual picture. Let $v_0, \ldots, v_l$ be primitive vectors introduced above. We define new fans $\Sigma'_\Gamma \subset \Sigma_\Gamma \subset M_{\mathbb{R}}$ where

1. Σ'_Γ is generated by primitive vectors in all $\Delta_{(ik)}$'s and all v_i's;
2. Σ_Γ is a subfan of Σ'_Γ that is generated by all primitive vectors lying in $\Delta \cap L$ and all $v_i's$.

Furthermore, we consider the fan $\Sigma' := \Sigma_\Delta \cup \Sigma'_\Gamma$. Geometrically, this refinement amounts to taking a (maximal) projective crepant partial resolution of X_{Σ_Δ}, denoted by $\phi_\Delta : X_{\Sigma'} \to X_{\Sigma_\Delta}$. We also have a blow-down map $\phi_\Gamma : X_{\Sigma'} \to X_{\Sigma_\Gamma}$ which is not necessarily crepant. Consider the projection of the lattices $\Pi_v : M \to M_v$, where M_v is the sublattice of M generated by v_i's. This provides a surjective toric morphism $\pi_v : X_{\Sigma_\Gamma} \to X_{\Sigma_v}$, where $\Sigma_v \subset M_{v,\mathbb{R}}$ is the fan generated by all v_i's. In fact, π_v is a trivial fibration whose fiber is a toric variety associated to a fan generated by all primitive vectors in $\Delta \cap L$, denoted by X_L. In summary, we have the following diagrams of toric morphisms



where $\pi_\Gamma := \phi_\Gamma \circ \pi_v$. Since $\pi_\Gamma : X_{\Sigma'} \to X_{\Sigma_v}$ is the composition of toric morphisms, a generic fiber is the toric variety X_L. In fact, over the open dense torus $(\mathbb{C}^*)^l \subset X_{\Sigma_v}$, the morphism π_Γ is trivial fibration

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with fiber X_L. We also present the coordinate description. Consider the homogeneous coordinates of $X_{\Sigma'}$ (resp. X_{Σ_v}), $(z_\sigma | \sigma \in \Sigma')$ (resp. $(z_{\sigma_0}, \dots, z_{\sigma_l})$). In terms of these coordinates, π_Γ is given by

$$\pi_\Gamma = \left(\prod_{\Pi_v(\sigma)=\sigma_0} z_\sigma, \dots, \prod_{\Pi_v(\sigma)=\sigma_l} z_\sigma \right).$$

Let's take a generic anticanonical divisor Y in $X_{\Sigma'}$ and the induced fibration $\pi := \pi_\Gamma|_Y$. Recall that we assume that Δ° satisfies the MPCS resolution condition so that Y is nonsingular. Also, it is clear that a generic fiber of π is Calabi–Yau as this is a nonsingular general member of the anticanonical divisor of X_L.

Since Γ is nonsingular, the base X_{Σ_v} is isomorphic to the projective space $\mathbb{P}^l$ and we write $\{z_i := z_{v_i} | i = 0, \dots, l\}$ for the homogeneous coordinates. For each i, we take a polydisk $\Delta_i = \{|z_j| \leq |z_i| | j \neq i\} \subset \mathbb{P}^l$ and set $Y_i := \pi^{-1}(\Delta_i)$ and $h_i := \pi|_{Y_i} : Y_i \longrightarrow \Delta_i$. It follows from the genericity condition on Y that the restriction of $\pi : Y \longrightarrow \mathbb{P}^l$ over each boundary component of Δ_i is locally trivial with smooth fibers and intersects transversally to each other. This implies that $(Y_i, h_i : Y \longrightarrow \Delta_i)$ is a hybrid LG model.

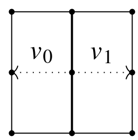
Recall that on the degeneration side, the semistable partition Γ provides a semistable degeneration of a nonsingular Calabi–Yau hypersurface of X_Δ. Each irreducible component X_i of the degenerate fiber $X_c = \cup_{i=0}^l X_i$ is a general hypersurface of the toric variety $X_{\Delta_{(i)}}$ determined by all facets in the facets of Δ.

Conjecture 6.10. *For each i, the hybrid LG model $(Y_i, h_i : Y_i \longrightarrow \Delta_i)$ is mirror to the pair $(X_i, \cup_{j\neq i} X_{ij})$.*

Remark 6.11. One may apply the same construction without imposing MPCS resolution condition on Δ°. In this case, Y becomes an orbifold so that one needs more general notion of hybrid LG models.

One of the major difficulties in proving Conjecture 6.10 and verifying the gluing condition (Ansatz 3.9) is the lack of mirror symmetry results for irreducible components of the degenerate fiber as quasi-Fano varieties. For instance, unlike the Fano case, a toric mirror construction (e.g., Givental's construction [14]) may not be sufficient and needs further modification, especially in the non-nef case. See [11] for the Tyurin degeneration case. In this regard, this conjecture can be viewed as the reverse construction of the one introduced in Section 4. In other words, this could be one way to obtain a mirror hybrid LG model for a quasi-Fano pair. We further explore this direction in subsequent work. We conclude this section by providing two simple pieces of evidence for Conjecture 6.10.

Example 6.12. Consider the reflexive square Δ with the semistable partition Γ given by the vertical line



where dotted arrows are primitive vectors v_0 and v_1. This describes a semistable degeneration of a nonsingular Calabi–Yau hypersurface of $X_\Delta = \mathbb{P}^1 \times \mathbb{P}^1$, which is elliptic curve, into the union of two rational curves X_0 and X_1 intersecting over two points X_{01}. Consider the fans Σ_Δ, Σ' and Σ_v described below (Figure 3):

Geometrically, $X_{\Sigma'}$ is a blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ along the four corners and $X_{\Sigma_v} \cong \mathbb{P}^1$. The morphism $\pi_\Gamma : X_{\Sigma'} \longrightarrow X_{\Sigma_v}$ is the one given by the projection to the first factor. In terms of the homogeneous

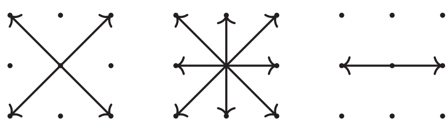
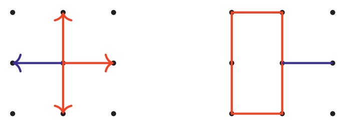


Figure 3. *Fans Σ_Δ, Σ' and Σ_v from the left.*

coordinate $\{z_\sigma | \sigma \in \Sigma'[1]\}$, this is given by

$$\pi_\Gamma = \left[z_{\sigma(-1,1)} z_{\sigma(-1,0)} z_{\sigma(-1,-1)} : z_{\sigma(1,1)} z_{\sigma(1,0)} z_{\sigma(1,1)}\right]. \tag{6.2}$$

Take a generic section Y of $|-K_{X_{\Sigma'}}|$ that is not singular over the locus $\{|z_0| = |z_1|\} \in \mathbb{P}^1 \cong X_v$. Then the induced fibration $\pi : Y \longrightarrow \mathbb{P}^1$ becomes a double cover with four ramification points. Note that two of them lie near $0 \in \mathbb{P}^1$ while the other two points lie near $\infty \in \mathbb{P}^1$. For $i = 0, 1$, we take $Y_i := \pi^{-1}(\Delta_i)$ and $\mathsf{w}_i := \pi|_{Y_i} : Y_i \longrightarrow \Delta_i$. We show that the pair (Y_i, w_i) is mirror to the pair (X_i, X_{01}). To describe mirror of X_0 (the parallel argument works for X_1), we first make the rectangle $\Delta(0)$ reflexive by shifting the middle vertical line (a facet of $\Delta_{(0)}$) to the right by length 1. We still denote it by $\Delta_{(0)}$. Then we apply Givental's construction to get a mirror of X_0 [14, 17]. We consider the polar dual of $\Delta_{(0)}$, denoted by $\Delta^{(0)}$, and regard it belongs to M to match the notation we have used. Note that we have a nef partition of $\Delta^{(0)} = \Delta_1^{(0)} + \Delta_2^{(0)}$ where $\Delta_1^{(0)}[0]$ are the red vertices and $\Delta_2^{(0)}[0]$ is blue one in the following picture (left).



The right picture describes the dual partition $\nabla_1^{(0)}$ and $\nabla_2^{(0)}$. Then the LG model for X_0 is given by

$$Y_0 = \left\{ \sum_{\rho \in \Delta_1^{(1)} \cap M} a_\rho z_\rho = 0 \right\} \subset (\mathbb{C}^*)^2, \qquad \mathsf{w}_0 = \sum_{\rho \in \Delta_2^{(1)} \cap M} a_\rho z_\rho,$$

where a_ρ's are general complex coefficients and $z_\rho = \prod_{i=1}^n z_i^{\langle e_i, \rho \rangle}$, where $\{e_1, \ldots, e_n\}$ is the basis of N. A fiber is of w_0 is compactified to

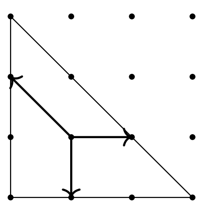
$$\begin{aligned} &a_{\rho(0,0)} z_{\sigma(0,1)} z_{\sigma(-1,1)} z_{\sigma(-1,0)} z_{\sigma(-1,-1)} z_{\sigma(0,-1)} \\ &+ a_{\rho(1,0)} z_{\sigma(0,1)} z_{\sigma(0,-1)} z_{\sigma(1,0)} \\ &+ a_{\rho(0,1)} z^2_{\sigma(0,1)} z^2_{\sigma(-1,1)} z_{\sigma(-1,0)} \\ &+ a_{\rho(0,-1)} z^2_{\sigma(0,-1)} z^2_{\sigma(-1,-1)} z_{\sigma(-1,0)} = 0 \end{aligned}$$

and

$$\lambda z_{\sigma(1,0)} - a_{\rho(-1,0)} z_{\sigma(-1,1)} z_{\sigma(-1,0)} z_{\sigma(-1,-1)} = 0,$$

where z_σ is the homogeneous coordinate of $X_{\nabla^{(0)}}$. For generically chosen a_ρ's, we see that near $\lambda = 0$, this is a double cover of $\mathbb{C}$ with two ramification points. In fact, locally, this morphism is the same with one described in (6.2). As we have the parallel argument for $\Delta_{(1)}$, we get the conclusion.

Example 6.13. Consider the reflexive polytope Δ with $X_\Delta = \mathbb{P}^2$ and the semistable partition Γ described as follows:



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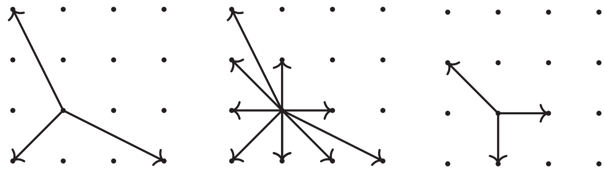

Figure 4. *Fans Σ_Δ, Σ' and Σ_v from the left.*

This describes a semistable degeneration of an elliptic curve into the wheel of three rational curves X_0, X_1 and X_2. Consider the fans Σ_Δ, Σ' and Σ_v described below (Figure 4):

There is a canonical fan morphism $\Sigma' \to \Sigma_v$ which induces the birational map $\pi_\Gamma : X_{\Sigma'} \to X_{\Sigma_v}$. Similar to Example 6.12, one can write down the coordinates and see whether each hybrid LG model is mirror to one of the degeneration components. Instead, we take another approach. Note that $X_{\Sigma'}$ is obtained by taking iterated blow ups of $X_{\Sigma_v} \cong \mathbb{P}^2$. More explicitly, take a coordinate $[z_0 : z_1 : z_2]$ and blow up $X_{\Sigma_v} \cong \mathbb{P}^2$ over three torus invariant points $[1:0:0]$, $[0:1:0]$ and $[0:0:1]$ and denote the exceptional divisors E_0, E_1 and E_2, respectively. We also denote D_i the proper transform of the torus invariant divisor $(z_i = 0)$ for $i \equiv 0, 1, 2 (\mathrm{mod} 3)$. We blow up further along the intersection locus $E_i \cap D_{i+1}$. The resulting variety is $X_{\Sigma'}$. By the adjunction formula, a section of the anticanonical line bundle $-K_{X_{\Sigma'}}$ is given by a section of $-K_{X_{\Sigma_v}}$ that vanishes at $(z_i = z_{i+1} = 0)$ with multiplicity at least 2 along $(z_i = 0)$. Therefore, a generic section is given by $s(z_0, z_1, z_2) = a_1 z_1^2 z_2 + a_2 z_2^2 z_0 + a_3 z_0^2 z_1 + a_4 z_0 z_1 z_2$for some coefficients a_i's. Since the irreducible components of the degenerate fiber are all $\mathbb{P}^1$, we may choose $a_1 = a_2 = a_3$. Then for sufficiently small $|a_4|$ (e.g. $|a_4| \le \frac{1}{2}|a_1|$), over $\Delta_i \subset X_{\Sigma_v}$ one can see that $h_i : Y_i \to \Delta_i$ is a topologically hybrid LG model for $\mathbb{P}^1$. This is due to the fact that Δ_i is much smaller than the standard chart $(z_i \neq 0)$, and there is only one solution of $s = 0$ for fixed $z_{i+1} \in \Delta_i$.

Remark 6.14. One can generalize the notion of a semistable partition by relaxing condition (2) in Definition 6.3 so that the dual complex K_Γ is not just the standard simplex, but a moment polytope of a nonsingular toric variety TV. The same construction provide a Calabi–Yau fibration structure $\pi : Y \to TV$, although one needs to introduce a higher-dimensional base on the degeneration side to achieve semistability. This will be also a topic for the subsequent work.

7. Poincaré duality of hybrid LG models

Let $(Y, \omega, h = (h_1, \ldots, h_N) : Y \to \mathbb{C}^N)$ be a hybrid LG model of rank N. As the Kähler form ω does not play a crucial role in the following discussion, we drop it from the notation. We recall the definition of the compactified hybrid LG model of $(Y, h : Y \to \mathbb{C}^N)$ [24, Section 4.2].

Definition 7.1. A compactified hybrid LG model of $(Y, h : Y \to \mathbb{C}^N)$ is a datum $((Z, D_Z), f : Z \to (\mathbb{P}^1)^N)$ where:

1. Z is a smooth projective variety and $f = (f_1, \ldots, f_N) : Z \to (\mathbb{P}^1)^N$ is a projective morphism where each morphism $\hat{f}_i = (f_1, \ldots, \hat{f}_i, \ldots, f_N) : Z \to (\mathbb{P}^1)^{N-1}$ is the compactification of the potential $\hat{h}_i = (h_1, \ldots, \hat{h}_i, \ldots, h_N) : Y \to \mathbb{C}^{N-1}$ for all $i = 1, \ldots, N$;
2. the complement of Y in Z is a simple normal crossing anticanonical divisor $D_Z := D_{f_1} \sqcup \cdots \sqcup D_{f_N}$ where $D_{f_i} := (f_i^{-1}(\infty))_{red}$ is the reduced pole divisor of f_i for all $i = 1, \ldots, N$;
3. the morphism $f : Z \to (\mathbb{P}^1)^N$ is semistable at $(\infty, \ldots, \infty)$.

In particular, we call the compactified LG model $((Z, D_Z), f)$ *tame* if the pole divisor $f_i^{-1}(\infty)$ is reduced for all i.

Consider the logarithmic de Rham complex $(\Omega_Z^\bullet(\log D_Z), d)$. We define$f$-adapted de Rham complex of Z, denoted by $(\Omega_Z^\bullet(\log D_Z, f), d)$, to be subcomplex which is preserved by the wedge product of all df_i's.

$$\Omega_Z^a(\log D_Z, f) := \{\eta \in \Omega_Z^a(\log D_Z) | \eta \wedge df_i \in \Omega_Z^{a+1}(\log D_Z) \text{ for all } i = 1, \ldots, n.\}$$

First, note that it is a locally free $\mathcal{O}_Z$-module of rank $\binom{n}{a}$. Here is the local description. Denote D_∞ the corner that is the intersection of D_{f_i}'s. For $p \in D_\infty$, we can find local analytic coordinates $z_{i1}, \cdots, z_{ik_i}$ centered at p with $k_1 + \cdots k_N \leq n$ such that in a small neighborhood of p, the divisor D_{f_i} is given by $\prod_{i=1}^{k_1} z_i = 0$ and the potential f_i is given by

$$f_i(z_1, \cdots, z_n) = \frac{1}{z_{i1}^{a_{i1}} \cdots z_{1k_1}^{a_{ik_i}}}$$

for some $a_{ik_i} \geq 1$.

Lemma 7.2. *The f_i-adapted de Rham complex $\Omega_Z^a(\log D_Z, f_i)$ is locally free of rank $\binom{n}{a}$ for all $a \geq 0$. Explicitly,*

$$\Omega_Z^a(\log D_Z, f_i) = \bigoplus_{p=0}^{a} \left[\frac{1}{f_i} \wedge^p W_i \oplus d\log f_i \wedge \left(\wedge^{p-1} W_i \right) \right] \bigotimes \wedge^{a-p} R_i,$$

where W_i is spanned by logarithmic 1-forms associated to the vertical part of $f_i : Z \to \mathbb{P}^1$ and R_i is spanned by holomorphic 1-forms on Y and logarithmic 1-forms associated to the horizontal part of $f_i : Z \to \mathbb{P}^1$.

Proof. See [21, Lemma 2.12] □

The above local description allows one to describe $\Omega_Z^a(\log D_Z, f)$ for all $a \geq 0$. Explicitly, we have

$$\Omega_Z^a(\log D_Z, f) = \bigoplus_{p_1+\cdots+p_N=0}^{a} \bigotimes_{i=1}^{N} \left[\frac{1}{f_i} \wedge^{p_i} W_i \oplus d\log f_i \wedge \left(\wedge^{p_i-1} W_1 \right) \right] \wedge^{a-(p_1+\cdots+p_N)} R, \tag{7.1}$$

where R is spanned by holomorphic 1-forms on Y. Consider the cup(=wedge) product

$$\cup : \Omega_Z^a(\log D_Z, f) \otimes \Omega_Z^{n-a}(\log D_Z, f) \to \Omega_Z^n(\log D_Z, f) = \Omega_Z^n(\log D_Z).$$

Note that this is nondegenerate. From the description of the local form, one can see that it factors through Ω_Z^n, the sheaf of holomorphic n-forms on Z. In other words, we have the following commutative diagram

$$\begin{array}{ccc} \Omega_Z^a(\log D_Z, f) \otimes \Omega_Z^{n-a}(\log D_Z, f) & \xrightarrow{\cup} & \Omega_Z^n(\log D_Z) \\ & \searrow & \uparrow \\ & & \Omega_Z^n. \end{array}$$

This provides the natural isomorphism of locally free $\mathcal{O}_Z$-modules

$$\Omega_Z^a(\log D_Z, f) \cong Hom_{\mathcal{O}_Z}(\Omega_Z^{n-a}(\log D_Z, f), \Omega_Z^n) \cong (\Omega_Z^{n-a}(\log D_Z, f))^* \otimes \Omega_Z^n.$$

Therefore, we have a perfect pairing which is given by the composition of cup product with the natural trace map:

$$\mathbb{H}^q(Z, \Omega_Z^p(\log D_Z, f)) \otimes \mathbb{H}^{n-q}(Z, \Omega_Z^{n-p}(\log D_Z, f)) \to H^n(X, \Omega_Z^n) \xrightarrow{Tr} \mathbb{C},$$

hence we have the Serre duality for the f-adapted de Rham forms.

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Proposition 7.3 [24, Proposition 4.16]. *The f-adapted de Rham complex* $\Omega_Z^{\bullet}(\log D_Z, f)$ *satisfies the Hodge to de Rham degeneration property. In particular, we have*

$$\mathbb{H}^a(Z, \Omega_Z^{\bullet}(\log D_Z, f)) \cong \bigoplus_{p+q=a} \mathbb{H}^q(Z, \Omega_Z^p(\log D_Z, f)).$$

It is known that the cohomology $\mathbb{H}^a(Z, \Omega_Z^{\bullet}(\log D_Z, f))$ is isomorphic to the relative cohomology $H^a(Y, \sqcup_{i=1}^N Y_i, \mathbb{C})$, where Y_i is a smooth generic fiber of $h_i : Y \to \mathbb{C}$ near infinity and $\sqcup_{i=1}^N Y_i$ is the normal crossing union of Y_i's. The gluing property of the hybrid LG model (Proposition 3.2) provides an isomorphism $H^a(Y, \sqcup_{i=1}^N Y_i, \mathbb{C}) \cong H^a(Y, Y_{sm}, \mathbb{C})$, where Y_{sm} is a smooth generic fiber of the associated ordinary LG potential $\mathsf{w} : Y \to \mathbb{C}$. Therefore, we have

Theorem 7.4 (Poincaré duality). *Let* $(Y, h : Y \to \mathbb{C}^N)$ *be a rank N hybrid LG model. Then for* $a \geq 0$, *there is an isomorphism of cohomology groups*

$$H^a(Y, Y_{sm}, \mathbb{C}) \cong H^{2n-a}(Y, Y_{sm}, \mathbb{C})^*,$$

where $n = \dim_{\mathbb{C}} Y$.

We note that the similar result for proper ordinary LG models is done in [17, Section 2.2].

Remark 7.5. For the Poincaré duality statement, we take the topological trace map instead of the algebraic trace map. This amounts to multiplying the sign $(-1)^n$.

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