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GRAPHICAL REGULAR REPRESENTATIONS OF NON-ABELIAN GROUPS, II

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The present paper is a sequel to the previous paper bearing the same title by the same authors [**3**] and which will be hereafter designated by the bold-face Roman numeral **I**. Further results are obtained in determining whether a given finite non-abelian group G has a graphical regular representation. In particular, an affirmative answer will be given if $(|G|, 6) = 1$.

Inasmuch as much of the machinery of **I** will be required in the proofs to be presented and a perusal of **I** is strongly recommended to set the stage and provide motivation for this paper, an independent and redundant introduction will be omitted in the interest of economy. Section 1 of **I** introduces much of the terminology, symbols, and conventions to be employed below. Results from **I** will be indicated by number preceded by "**I**" (e.g., Proposition 2.5 of **I** will be referred to as Proposition **I**.2.5). In particular, the following letters and symbols will retain the meanings assigned to them in Section 1 of **I**:

$$G, H, X, V(X), A(X), X_{G,H}, X_1, \mathrm{Aut}(G), Z.$$

The bibliographical references in this paper, of course, are numbered independently, as are the new results.

1. Classification of non-abelian groups G **with** $(|G|, 6) = 1$**.** The proofs of the two theorems of this section will require the powerful result of W. Feit and J. G. Thompson [**1**].

FEIT-THOMPSON THEOREM. *All finite groups of odd order are solvable.*

THEOREM 1. *Let G be a non-abelian cyclic extension of an abelian group L, and suppose* $(|G|, 6) = 1$. *Then G is in Class* I.

Proof. By hypothesis there is an element $b \in G$ such that each element of G has a unique representation in the form

$$b^j x$$

where $x \in L$ and $j \in \{0, 1, \ldots, s-1\}$; here s is the least positive integer for which $b^s \in L$. (By hypothesis, $s \geqq 5$.) Since L is abelian while G is not, we may select an element $a_0 \in L$ such that $a_0 \notin Z(b)$.

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We now select $a_1, a_2, \ldots \in L\backslash Z(b)$ inductively as follows: Letting the subgroup $G_i = \langle a_0, a_1, \ldots, a_i, b\rangle$ and assuming G_i is not normal in G for $i = 0, 1, \ldots, m-1$, we select $a_m \in L\backslash G_{m-1}$ so that $a_m \notin N(G_{m-1})$, the normalizer of G_{m-1} in G. The process clearly terminates with some subgroup $G_n = \langle a_0, a_1, \ldots, a_n, b\rangle$ such that $G_n \lhd G$. (Conceivably $G_n = G$.)

We assert that it suffices to prove that G_n is in *Class* I. For suppose this were so. Since G is of odd order, it is solvable by the Feit-Thompson Theorem. Since $G_n \lhd G$, there exists a composition series:

$$G_n = N_r \lhd N_{r-1} \lhd \ldots \lhd N_0 = G$$

such that N_{i-1}/N_i is cyclic of prime order (cf. [**2**, p. 139]). With $(|G|, 6) = 1$, we are assured that the index $[N_{i-1}, N_i] \geqq 5$, and so by Theorem **I**.1, N_{i-1} is in *Class* I whenever N_i is $(i = 1, 2, \ldots, r)$, proving inductively that G is in *Class* I. We may therefore assume without loss of generality that $G = G_n$ and that G_i is never normal in G for $0 \leqq i < n$.

Recall that $L \lhd G$ and, trivially, $\langle a_0\rangle \lhd L$.

Case 1: $\langle a_0\rangle \lhd G$. A relation of the form

$$b^{-1}a_0b = a_0^{\,k} \tag{1}$$

must hold. Let $r = |a_0|$. Since $a_0 \notin Z(b)$, and s is odd, b^2 and a_0 do not commute.

We first show that it may be assumed without loss of generality that the two congruences

$$k \equiv -2 \pmod r \tag{2}$$

and

$$2k \equiv -1 \pmod r \tag{3}$$

do not hold in (1)

If (2) were to hold, set $d = b^2$. The group G can as well be considered as a cyclic extension of L by d, with $d^{-1}a_0d = b^{-2}a_0b^2 = a_0^{\,4}$. Since $r \nmid 6$, $k = 4$ cannot satisfy (2), and since $r \nmid 9$, $k = 4$ cannot satisfy (3).

On the other hand, (3) implies

$$b^{-1}a_0^{\,2}b = a_0^{-1}. \tag{4}$$

Substitution in (4) of $f = b^{-1}$ gives $fa_0^{\,2}f^{-1} = a_0^{-1}$, or $f^{-1}a_0^{-1}f = a_0^{\,2}$, which is equivalent to (2). Henceforth we shall assume that (2) and (3) are false.

Define the generating set $H = H' \cup H''$ of G with $H' \cap H'' = \emptyset$ and $H' = (H')^{-1}$, $H'' = (H'')^{-1}$ as follows:

$$H' = \{a_0, a_0^{-1}, b, b^{-1}, ba_0, b^{-1}a_0^{-ks-1}, ba_0^{-k}, b^{-1}a_0\}$$

and

$$H'' = \{a_i, a_i^{-1}, ba_1 \ldots a_i, a_i^{-1} \ldots a_1^{-1}b^{-1} | i = 1, \ldots, n\}. \tag{5}$$

If $n = 0$, then $H = H'$ and X_1 has the form of Figure 1.

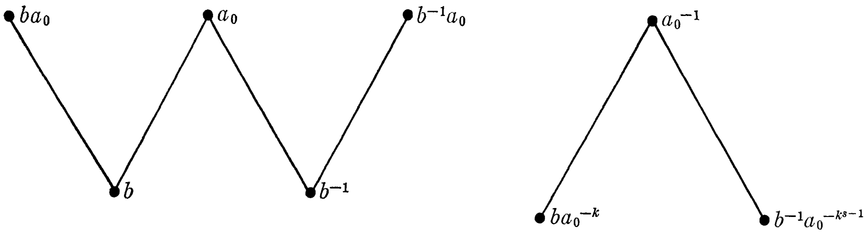

FIGURE 1

A straightforward verification of the 8×8 multiplication table of H' with itself is all that is needed to construct X_1 (cf. the second paragraph following the proof of Proposition I.3.1.)

If $n \geqq 1$, X_1 has the form of Figure 2.

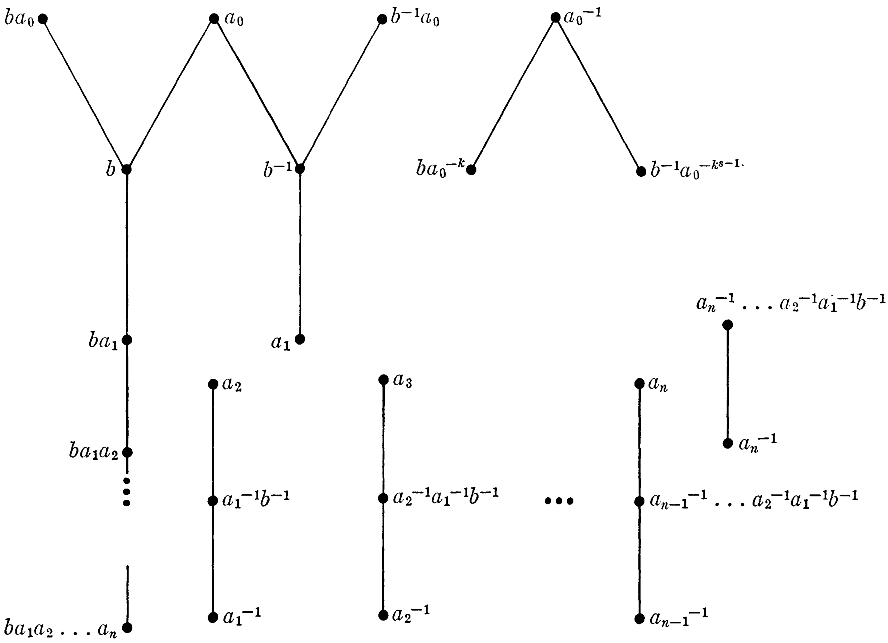

FIGURE 2

Observe that the graph of Figure 1 is a subgraph of this graph attached only at b and b^{-1}. That no other edges (i.e., not shown in Figure 2) can occur in X_1 is justified by the following arguments:

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Suppose

(6) $$xy = z; \qquad x, y, z \in H$$

were to hold. Then

(i) The sums of the exponents of the symbol b in the two members of (6) must be equal.

(ii) If m is the largest subscript i of a factor a_i^j appearing in (6) where $a_i^j \in G_i$, then a_m to some power must appear in both members of (6). (Otherwise a_m to some power could be set equal to an element of G_i for some $i < m$.)

Thus, if any additional edge (i.e., not shown in Figure 2) represented an identity (6) when not all of x, y, z are in H', then that identity would have to be equivalent to one either of the form

(7) $$a_m(bc) = ba_1 \ldots a_m$$

or of the form

(8) $$a_m^{-1}(bc) = ba_1 \ldots a_m,$$

where m is as in (ii) above and $c \in L \cap G_i$ for some $i < m$. However, (7) can be rewritten

$$a_m^{-1}ba_m = bca_{m-1}^{-1} \ldots a_1^{-1},$$

which implies that a_m normalizes G_{m-1}, contrary to our construction. Likewise (8) leads to a contradiction; we first write

(9) $$b^{-1}a_m^{-1}b = a_1 \ldots a_m c^{-1} = c_1 a_m$$

where $c_1 \in L \cap G_{m-1}$. Under inner automorphism by b we obtain $b^{-2}a_m^{-1}b^2 = (b^{-1}c_1b)(b^{-1}a_mb)$, or

(10) $$b^{-2}a_m^{-1}b^2 = (b^{-1}c_1bc_1^{-1})a_m^{-1}$$

by substitution from (9). However, since $b^{-1}c_1bc_1^{-1} \in G_{m-1}$, (10) would imply that the inner automorphism $x \mapsto b^{-1}xb$ has even order, giving a contradiction.

Let $\varphi \in A(X_1)$. If $n = 0$ then b and b^{-1} are the only vertices adjacent to both a vertex of valence 1 and a vertex of valence 2. If $n \geqq 1$, then b and b^{-1} are the only vertices of valence 3. Hence φ either fixes b and b^{-1} or interchanges them. Since a_0 is the only vertex adjacent to both b and b^{-1}, a_0 is a fixed point of $A(X_1)$. Since $b^2 \notin Z(a_0)$ (since $b \notin Z(a_0)$ and b has odd order) Proposition **I**.2.5 implies that b is also a fixed point of $A(X_1)$. Hence each vertex identified with an element of the subgroup $\langle a_0, b\rangle$ is fixed point-wise by Proposition **I**.2.3. If $n = 0$, we conclude that G is in *Class* I. If $n \geqq 1$, it is then obvious that ba_1 is a fixed-point of $A(X_1)$. Inductively, since $ba_1 \ldots a_i$ is a fixed-point of $A(X_1)$, then so is $ba_1 \ldots a_i a_{i+1}$ for $i = 1, \ldots, n - 1$. Since the set of these fixed points generates G, Corollary **I**.2.4 implies that G is in *Class* I.

Case 2: $\langle a_0 \rangle$ *is not normal in* G. Choose the generating set $H = H' \cup H''$ where this time

$$H' = \{a_0, a_0^{-1}, b, b^{-1}, ba_0, a_0^{-1}b^{-1}, a_0^{-1}b, b^{-1}a_0, b^{-1}a_0b, b^{-1}a_0^{-1}b\}$$

but H'' is the same as in (5).

First suppose that $n = 0$; i.e., that $G = \langle a_0, b\rangle$. It is asserted that X_1 assumes the form of Figure 3.

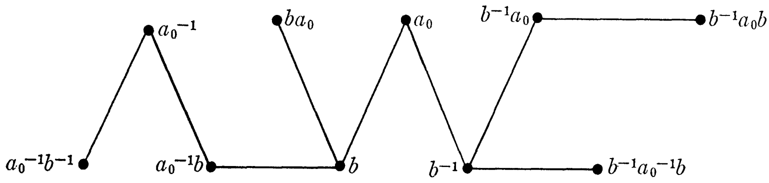

FIGURE 3

As in Case 1, this is verified by a careful term-by-term consideration of the entries in the 10×10 multiplication table of H' and recognition of elements of H' as entries in the table. To facilitate recognition of elements of H' we mention that not only does paragraph (i) of Case 1 above also apply here, but we have

(iii) $b^{\mp j}a_0b^{\pm j} \notin \langle a_0\rangle$ for $j = 1$ and for $(j, 6) > 1$ where $j = 1, \ldots s - 1$.

For example, consider the product $(a_0^{-1}b)(a_0^{-1}b^{-1})$. Were it to lie in H', the only possibilities by (i) would be a_0, a_0^{-1}, $b^{-1}a_0b$, and $b^{-1}a_0^{-1}b$. One immediately rules out a_0 and a_0^{-1}, since $ba_0^{-1}b^{-1} \notin \langle a_0\rangle$ by (iii). Now suppose

$$(a_0^{-1}b)(a_0^{-1}b^{-1}) = b^{-1}a_0b. \tag{11}$$

Then $1 = b^{-1}a_0b^2a_0b^{-1}a_0$, whence an inner automorphism by b yields

$$1 = (b^{-2}a_0b^2)a_0(b^{-1}a_0b). \tag{12}$$

Substitution from (11) for the last factor of (12) yields

$$1 = (b^{-2}a_0b^2)(ba_0^{-1}b^{-1}).$$

Another application of the inner automorphism gives

$$1 = (b^{-3}a_0b^3)a_0^{-1}$$

whence $b^{-3}a_0b^3 \in \langle a_0\rangle$, contrary to (iii).

Finally, suppose

$$(a_0^{-1}b)(a_0^{-1}b^{-1}) = b^{-1}a_0^{-1}b. \tag{13}$$

Here a substitution of $d = b^2$ for b is used. (The reader must verify that no other possible identities arising from the aforementioned multiplication table require a substitution of generators except for the five identities equivalent

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to (13).) It suffices to show that a contradiction arises if (13) still holds after b has been replaced by d. Thus, substituting into (13), we obtain

$$a_0^{-1}b^2a_0^{-1}b^{-2} = b^{-2}a_0^{-1}b^2. \tag{14}$$

However an inner automorphism by b applied to (13) yields $b^{-1}a_0^{-1}ba_0^{-1} = b^{-2}a_0^{-1}b^2$, which with (14) implies

$$(b^{-1}a_0^{-1}b)a_0^{-1} = a_0^{-1}(b^2a_0^{-1}b^{-2}). \tag{15}$$

Since a_0^{-1} commutes with the other factors in (15), one obtains $a_0^{-1} = b^3a_0^{-1}b^{-3}$, contrary to (iii).

Such verification for the other entries in the multiplication table is similar in principle although uniformly more elementary and straightforward than in the foregoing example. This process, nonetheless, is tedious, repetitious, and perhaps best left to the reader. Let it be accepted that the graph X_1 when $n = 0$ is correctly represented in Figure 3. Clearly any $\varphi \in A(X_1)$ either fixes b and b^{-1} or interchanges them. However the vertex b^{-1} lies at distance precisely 2 from a vertex of valence 1 while b does not have this property. Hence b and b^{-1} and, therefore, a_0 are fixed points and G is in *Class* I by Corollary **I**.2.4.

When $n \geqq 1$, X_1 has the form of Figure 4.

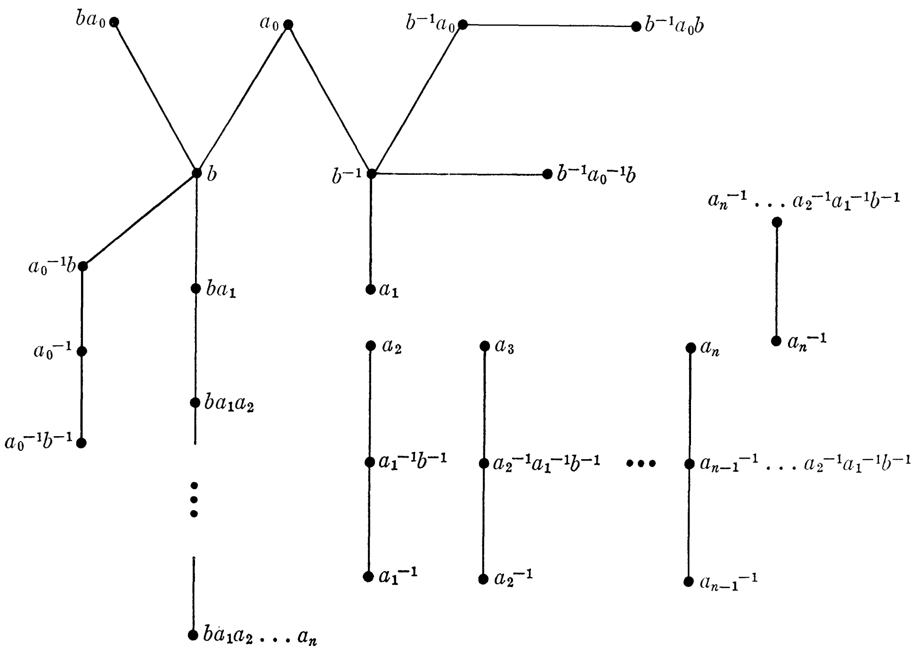

FIGURE 4

The verification is identical to the procedure in Case 1 above. One then argues similarly that b and b^{-1} are fixed-points, whence so is a_0. Thus $\langle a_0, b\rangle$ is fixed point-wise by Proposition **I**.2.3. It follows that ba_1 is a fixed-point, too, and the procedure concludes as in Case 1. The proof of the Theorem is complete.

THEOREM 2. *Every non-abelian finite group whose order is relatively prime to* 6 *is in Class* I.

Proof. Let G_0 be a non-abelian finite group such that $(|G_0|, 6) = 1$. By the Feit-Thompson Theorem, G_0 is solvable.

Let

(16) $$1 = G_n \lhd G_{n-1} \lhd \ldots \lhd G_1 \lhd G_0$$

be a composition series for G_0 where G_{i-1}/G_i is cyclic of prime order p_i $(i = 1, \ldots, n)$, and by hypothesis, $p_i \geqq 5$.

Since G_{n-1} is abelian while G_0 is not, one can select an abelian group G_m in the series (16) such that G_{m-1} is non-abelian $(1 \leqq m \leqq n - 1)$. By Theorem 1, G_{m-1} is in *Class* I. For each $i = 1, \ldots, m - 1$, it follows from Theorem **I**.1 that if G_i is in *Class* I, then so is G_{i-1}. By induction we conclude that G_0 is in *Class* I.

2. An odd non-abelian group with no graphical regular representation. At this point it would be reasonable to ask whether in Theorem 2 the integer 6 could not be replaced by the integer 2. After all, the Feit-Thompson Theorem requires of a group only that its order be odd. Moreover, all the non-abelian groups shown in **I** and by Watkins [**4**] to belong to *Class* II have been of even order. There is, however, one non-abelian group of order 27 which belongs to *Class* II, all other non-abelian groups of order p^3 for odd prime p having been proved in [**4**] to be in *Class* I.

THEOREM 3. *The group* G *of order* 27 *given by* $G = \langle a, b, c | a^3 = b^3 = c^3 = 1, ac = ca; bc = cb, ab = bac\rangle$ *is in Class* II.

Proof. Observe that the identity

(17) $$a^i b^j = b^j a^i c^{ij}$$

for all i, j follows from the defining relations. The group G has precisely four normal subgroups of order 9:

$$N_1 = \langle a, c\rangle, \quad N_2 = \langle b, c\rangle, \quad N_3 = \langle ba, c\rangle, \quad N_4 = \langle ba^{-1}, c\rangle.$$

The pairwise intersection of any two of these is the center $Z = \{1, c, c^{-1}\}$, which is fixed set-wise under $\mathrm{Aut}(G)$.

For $i = 1, 2, 3, 4$, let $M_i = N_i \backslash Z$. Thus $\mathbf{M} = \{M_1, M_2, M_3, M_4\}$ forms a complete block system for the imprimitive group $\mathrm{Aut}(G)$ restricted to $G\backslash Z$.

For later reference let us display these blocks. Observe that each M_i consists of six elements: three *pairs*, each consisting of an element with its inverse:

$$M_1 = \{a, a^{-1}, ac, a^{-1}c^{-1}, ac^{-1}, a^{-1}c\},$$
$$M_2 = \{b, b^{-1}, bc, b^{-1}c^{-1}, bc^{-1}, b^{-1}c\},$$
$$M_3 = \{ba, b^{-1}a^{-1}c, bac, b^{-1}a^{-1}, bac^{-1}, b^{-1}a^{-1}c^{-1}\},$$
$$M_4 = \{ba^{-1}, b^{-1}ac^{-1}, ba^{-1}c^{-1}, b^{-1}a, ba^{-1}c, b^{-1}ac\}.$$

We assert that given $x_1, x_2, y_1, y_2 \in G\backslash Z$ where $x_1 \in M_i \Rightarrow x_2 \notin M_i$ and $y_1 \in M_j \Rightarrow y_2 \notin M_j$ for all $i, j = 1, 2, 3, 4$, then there exists a unique automorphism $\varphi \in \mathrm{Aut}(G)$ such that $\varphi(x_k) = y_k$ $(k = 1, 2)$. It will suffice to prove that for arbitrary $i, j, k, r, s, t \in \{-1, 0, 1\}$ there exists a unique $\varphi \in \mathrm{Aut}(G)$ such that $\varphi(a) = b^i a^j c^k$ and $\varphi(b) = b^r a^s c^t$ unless $b^i a^j c^k$ and $b^r a^s c^t$ are in the same set M_m. For applying φ to the defining relation $ab = bac$, we obtain

$$\varphi(a)\varphi(b) = \varphi(b)\varphi(a)\varphi(c).$$

After substitution and application of (17) this becomes

$$b^{i+r}a^{j+s}c^{jr} = b^{i+r}a^{j+s}c^{is}\varphi(c),$$

whence

$$\varphi(c) = c^{jr-is}. \tag{18}$$

Now $\varphi(c) = 1$ if and only if the determinant

$$\begin{vmatrix} j & i \\ s & r \end{vmatrix} = 0.$$

But that happens only when at least 3 entries are 0 (which is impossible) or $\varphi(a)$ and $\varphi(b)$ lie in the same M_m. Hence $\varphi(c)$ is uniquely determined to be c or c^{-1} by (18) and φ preserves all the defining relations of G.

We next show that $\mathrm{Aut}(G)$ restricted to $G\backslash Z$ acts 4-transitively on the set $\mathbf{M}$. The automorphism determined by $a \mapsto b$ and $b \mapsto ba$ acts on the blocks with cyclic decomposition (M_1, M_2, M_3, M_4) while the automorphism determined by $a \mapsto b$ and $b \mapsto ba^{-1}$ acts with cyclic decomposition $(M_1, M_2, M_4)(M_3)$. These two permutations on $\mathbf{M}$ generate the symmetric group S_4 on $\mathbf{M}$.

Next observe that for $s \in M_i$, the inner automorphism $x \mapsto s^{-1}xs$ permutes all three pairs in each M_j for $j \neq i$ and fixes each element of M_i.

Let $H \subset G$ have the properties: $1 \notin H = H^{-1}$ and $\langle H \rangle = G$. It must be shown that for each such set H, there exists a non-identity automorphism $\varphi_0 \in \mathrm{Aut}(G)$ such that $\varphi_0(H) = H$. Since $\varphi(c) = c$ or c^{-1} for all $\varphi \in \mathrm{Aut}(G)$, it may be assumed that $H \subset G\backslash Z$. Observe that $\langle H \rangle = G$ if and only if H contains at least one pair from each of at least two different sets M_i. Let $H_i = H \cap M_i$. We shall say H_i is *improper* if $H_i = \emptyset$ or M_i. Otherwise H_i

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will be termed *proper*. When H_i is proper, the unique pair in M_i to be included or excluded by H_i is the *distinguished pair* of M_i. If all sets H_i are improper then many $\varphi \in \mathrm{Aut}(G)$ will do for φ_0 while if H_j is the only one which is proper, then choose φ_0 to be an inner automorphism by an element of M_j.

We may assume that at least two of the H_i are proper, and in the light of the foregoing discussion, there is no loss of generality in assuming that H_1 and H_2 are proper with $\{a, a^{-1}\}$ and $\{b, b^{-1}\}$ as distinguished pairs.

Case 1: The distinguished pair of M_3 is $\{bac^{-1}, b^{-1}a^{-1}c^{-1}\}$ or H_3 is improper, and the distinguished pair of M_4 is $\{ba^{-1}c, b^{-1}ac\}$ or H_4 is improper. Then let φ_1 be determined by $a \mapsto a^{-1}$ and $b \mapsto b^{-1}$. Thus $\varphi_1(c) = c$ and φ_1 maps each distinguished pair onto itself, thereby fixing H set-wise.

We have also shown hereby that it may be assumed that not both H_3 and H_4 are improper. Since $\mathrm{Aut}(G)$ is 4-transitive on $\mathbf{M}$, we may assume that H_3 is proper.

Case 2: The distinguished pair of M_3 is $\{ba, b^{-1}a^{-1}c\}$ or $\{bac, b^{-1}a^{-1}\}$, and the distinguished pair of M_4 is $\{ba^{-1}c, b^{-1}ac\}$ or H_4 is improper.

The involution φ_1 of Case 1 interchanges ba with $b^{-1}a^{-1}$ while fixing each of the other three distinguished pairs. Therefore, there is no loss of generality in assuming both that $\{ba, b^{-1}a^{-1}c\}$ is the distinguished pair of M_3 and that H_4 is also proper. Thus some two of the three sets H_1, H_2, H_4 have the same cardinality. It is always possible to construct an automorphism φ_0 interchanging the distinguished pairs of these two sets while fixing the other two distinguished pairs by defining $\varphi_0(ba) = (ba)^{-1}$ and $\varphi_0(d) = d$ where d belongs to the distinguished pair of that set H_1, H_2, or H_4 being fixed by φ_0.

Case 3: The distinguished pair in M_3 is $\{ba, b^{-1}a^{-1}c\}$ and the distinguished pair in M_4 is $\{ba^{-1}, b^{-1}ac^{-1}\}$ or $\{ba^{-1}c^{-1}, b^{-1}a\}$.

It may be assumed that the distinguished pair of M_4 is $\{ba^{-1}, b^{-1}ac^{-1}\}$ since the involution determined by $a \mapsto b^{-1}$ and $b \mapsto a^{-1}$ maps the set of pairs

$$\{\{a, a^{-1}\}, \{b, b^{-1}\}, \{ba, b^{-1}a^{-1}c\}, \{ba^{-1}, b^{-1}ac^{-1}\}\} \tag{19}$$

onto

$$\{\{a, a^{-1}\}, \{b, b^{-1}\}, \{ba, b^{-1}a^{-1}c\}, \{ba^{-1}c^{-1}, b^{-1}a\}\}.$$

Now consider the automorphism φ_2 given by $a \mapsto ba$ and $b \mapsto b^{-1}$. Then $ba \mapsto b^{-1}(ba) = a$ and $ba^{-1} \mapsto b^{-1}(b^{-1}a^{-1}c) = ba^{-1}c$. Thus φ_2 transforms the given set (19) of distinguished pairs into the set considered in Case 2, and the proof is complete.

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