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A REDUCTION THEOREM FOR PERFECT LOCALLY FINITE MINIMAL NON-FC GROUPS

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A group G is said to be a *minimal non-FC group*, if G contains an infinite conjugacy class, while every proper subgroup of G merely has finite conjugacy classes. The structure of imperfect minimal non-*FC* groups is quite well-understood [**3**] (see also [**14**], Section 8). These groups are in particular locally finite. At the other end of the spectrum, a perfect locally finite minimal non-*FC* group must be a p-group [**2**], [**9**]. And it has been an open question for quite a while now, whether such groups exist or not. In [**10**], Theorem 2.4, it was shown that such p-groups have a non-trivial representation as subgroups of the McLain group $\mathrm{M}(\mathrm{Q},\mathrm{F}_p)$, that is, as groups of infinite upper unitriangular matrices of order type Q over the field F_p with p elements, in which all but finitely many non-diagonal entries are zero. The purpose of this note is to obtain the following considerable improvement, which should provide a major step in the discussion of existence of perfect minimal non-*FC* p-groups.

THEOREM. *Every perfect locally finite minimal non-FC group has a quotient, which acts as a barely transitive p-group of finitary permutations on some infinite set.*

Recall, that *finitary permutations* of the set Ω fix all but finitely many elements in Ω. The structure of groups of finitary permutations has been studied intensely in the seventies and again during the last ten years (see [**13**] for references). A subgroup of the symmetric group $\mathrm{Sym}(\Omega)$ on an infinite set Ω is said to be *barely transitive*, if it acts transitively on Ω, while each of its proper subgroups has finite orbits. Barely transitive groups were brought up by B. Hartley [**4**], [**5**] in connection with groups of Heineken-Mohamed type, and have been investigated during the last years mainly by M. Kuzucuoğlu [**7**], [**8**]. Obviously every barely transitive group without proper finite quotients is a minimal non-*FC* group. In particular, the question about existence of perfect locally finite minimal non-*FC* p-groups turns out now to be equivalent to the question about existence of perfect barely transitive p-groups, which in addition act finitarily on the underlying set.

Proof of the Theorem. Let G be a perfect locally finite minimal non-*FC* group. Recall that G is a p-group. Since G is perfect, the centre $\zeta_1(G)$ is the highest term of the upper central series in G. From passing to $G/\zeta_1(G)$ we may assume that G has trivial centre. Consider a non-trivial normal subgroup N of G. The socle S of the *FC*- and p-group N is an elementary-abelian normal subgroup in G ([**14**], p. 10). Consider a fixed non-trivial element $x\in S$, and let $\Omega=\{x^g|g\in G\}$ and $V=\langle\Omega\rangle\leq N$. Since G has no proper finite image and trivial centre, the set Ω must be infinite. Since G is a minimal non-*FC* group without maximal subgroups, it acts barely transitively on Ω via conjugation. Moreover, G acts finitary linearly on the F_p-vector space V: For every $g\in G$, the proper subgroup $V\langle g\rangle$ of G is an *FC*-group, whence

$|V : C_V(g)| \leq |V\langle g\rangle : C_{V\langle g\rangle}(g)| < \infty$. It remains to show, that G acts as a finitary permutation group on Ω.

To this end, we assume that some $g \in G$ has infinite support on Ω. Let $M = \langle g^G \rangle$. Since G is a locally finite p-group, $g \notin M'$, and so $M/M' \neq 1$. Since G is perfect, M is a proper normal subgroup of G. Since G acts barely transitively on Ω, the M-orbits Ω_i ($i \in \omega$) are finite and form a system of imprimitivity. Let $V_i = \langle \Omega_i \rangle \leq N$. Since g has infinite support on Ω, we have $[V_i, g] \neq 1$ for infinitely many $i \in \omega$. However, $[V, g]$ is a finite-dimensional F_p-vector space, hence finite. Thus there is a one-dimensional subspace U in $[V,g]$ such that $U \leq [V_i,g]$ for infinitely many $i \in \omega$. Let I be the set of all such $i \in \omega$. Fix $i_0 \in I$, and choose $g_i \in G$ ($i \in I$) satisfying $\Omega_i^{g_i} = \Omega_{i_0}$. Since V_{i_0} is finite, there is an infinite set $I_0 \subseteq I$ such that $U^{g_i} = U^{g_j}$ for all i, $j \in I_0$. Consider the normalizer $H = N_G(U)$. Fix $\omega_0 \in \Omega_{i_0}$. Since $g_i g_j^{-1} \in H$ for all $i, j \in I_0$, the elements $\omega_0 g_i^{-1} (i \in I_0)$ are contained in an infinite H-orbit on Ω. Hence $H = G$, and U is a normal subgroup of order p in G. But then $1 \neq U \leq \zeta_1(G)$, a contradiction. The proof of the Theorem is complete.

A group G is said to be a *minimal non-CC group*, if $U/C_U(x^U)$ is a Černikov group for all $x \in U < G$, while this property fails for G in place of U. Obviously, every perfect locally finite minimal non-FC group is a minimal non-CC group. Many results about minimal non-FC groups have been transferred to minimal non-CC groups [**12**], [**6**]. The following is an immediate consequence of [**6**], [**1**], and of our Theorem above.

Corollary 1. *Every locally graded minimal non-CC group has a quotient, which acts as a barely transitive p-group of finitary permutations on some infinite set.*

We also obtain a generalization of [**12**].

Corollary 2. *No non-trivial quotient of a locally graded minimal non-CC group lies in a proper variety.*

Proof. Let G be a locally graded minimal non-CC group. Every quotient of G is also such a group [**12**]. Consider $N \triangleleft G$ and assume, that G/N lies in a proper variety. From Corollary 1 we may assume that G/N is a transitive group of finitary permutations of an infinite set Ω. But this contradicts [**11**, Theorem 1].

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