INVITED LECTURE

RECENT DEVELOPMENTS IN ECONOMIC THEORY AND THEIR APPLICATION TO INSURANCE

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1. INTRODUCTION

1.1 The very title of this paper may cause some surprise, since economic theory so far has found virtually no application in insurance. Insurance is obviously an economic activity, and it is indeed strange that general economic theory should seem inapplicable to insurance.

This apparent paradox may to some extent be explained by the historic development. Actuarial mathematics and the essential scientific basis of insurance were developed into a self-contained and fairly complete theory long before economists could claim the name of science for their subject. Actuaries and other insurance people may from time to time have turned to economic theory for help on their problems. In most cases they must have turned away in disappointment, being convinced that actuarial mathematics was well ahead of general economic theory.

1.2. The last point is brought out clearly in a paper on the safety loading of insurance premiums which Tauber (19) presented at the Sixth International Congress of Actuaries in Vienna in 1909. In the introduction to this paper Tauber seems to consider risk bearing as a service and appears to assume that it like all other goods and services must have a *price*, determined by supply and demand in the market. Tauber did not develop this idea, apparently because the economic theory of his time was utterly unable to analyse the problem. That his approach was sound, and that the problem can be formulated and solved by modern economic theory, has been indicated in a previous paper (6).

1.3. During the last thirty years there has been an extremely rapid development in economic theory. The "General Theory" of Keynes (12) which appeared in 1936, is usually considered to have caused a revolution in economic theory. However, this "revolution" has not lead to any developments which seem to have an immediate application in insurance. Post- and pre-Keynesian economics are equally powerless when confronted with the problem which Tauber tried to formulate. It is therefore premature to conclude from the title of this paper that economic theory has caught up with actuarial science, and that the theory of insurance has now become a part of a general theory comprising all economic activities.

1.4. The developments which we shall discuss in this paper all have their origin in one single book: *Theory of Games and Economic Behavior,* published in 1944 by Von Neumann and Morgenstern (16). When this book appeared, it was predicted that it would lead to a revolution in economic thought, even more fundamental than the one caused by Keynes' "General Theory". This revolution has not materialized, and it seems that the basic ideas of game theory have been rather slow in gaining acceptance among economists. The reason may be simply that the theory is *too* revolutionary, and that the game theoretical approach plays havoc with the traditional methods of economic analysis.

Even if game theory to some extent has been ignored, the subject has been developed rapidly since 1944. A bibliography published in 1959 (20) lists more than 1000 books and papers, most of which have appeared during the less than 15 years which had elapsed since the theory was first presented. However, most of the papers listed are written by mathematicians, and the bibliography as a whole confirms the impression that economists in general have not yet grasped the real significance of game theory. It may be appropriate that actuaries should take the lead in putting this extremely versatile theory to practical use, since the theory probably is beyond the mathematical capacity of the rank and file economists.

1.5. Game theory has been a fashionable topic in mathematical circles for more than a decade, and it is natural that we find a number of attempts to apply the theory to insurance problems. However, most of these applications, i.a. (2), (4), (17) and (18) are based on the equivalence between the simplest non-trivial game and the linear programming model, and they deal with insurance problems which are fairly trivial. In this paper we shall try to show that there are other parts of game theory which can be applied to the really important problems of insurance.

2. THE THEORY OF GAMES

2.1. In this section we shall, as briefly as possible, introduce the essential concepts of game theory.

An n-person game in the so-called *normal* form consists of the following three elements.

(i) A set of n players, 1, 2 ... n.
(ii) n sets of *strategies* $X_1 \ldots X_n$.
(iii) n real-valued *payoff* functions $M_1 \ldots M_n$ defined over the members of the sets $X_1 \ldots X_n$.

The game is played as follows: Each player chooses a strategy from the set available to him, i.e. player i chooses a strategy x_i which is a member of X_i. If the strategies chosen by the n players are $x_1, x_2 \ldots x_n$, player i will receive the "payoff" $M_i(x_1 \ldots x_n)$. Hence the payoff function $M_i(x_1 \ldots x_n)$ is interpreted as the *gain* to player i when the strategies $x_1 \ldots x_n$ are used. This gain can be the monetary value of the profit or loss which the player makes by participating in the game. More generally one can interpret M_i as the *utility* which player i attaches to this gain.

2.2. If player i is *rational*, he will try to choose his strategy x_i, so that his payoff M_i becomes as great as possible. However, M_i depends not only on x_i which player i can choose freely. The payoff M_i will depend also on the strategies x_j $(j \neq i)$ chosen by the n-1 other players. In general player i will have no influence on the choices made by the other players. Hence his problem is not just to maximize a given function over the set X_i. To determine the best strategy, he must either anticipate the choices of the other players, or he must seek an agreement with them as to what strategies should be chosen.

2.3. It is evident that the model we have described can be used to analyse a great number of situations in economic life. It also seems that by formulating the problems in the terms of game theory, we come to grips with the real essentials in such situations.

These essentials were usually "assumed away" in classical economic theory. It is generally assumed that each economic agent, producers, consumers, savers, investors etc. in a sense takes the world as given, and chooses the strategy which will maximize his particular "payoff". The assumption implies that the choices made by all other agents are known, at least in a stochastical sense, and that they are independent of the choice made by the particular agent we consider. In most economic situations such assumptions are obviously unrealistic, but they are still made, also in modern theory. The whole "programming" approach to economic problems is based on assumptions of this kind.

The *homo economicus* of classical economics has in the modern theory of "decision making under uncertainty" been replaced by *homo stochasticus*. However, the agent which we really should study is *homo politicus*, the man who makes decisions and acts, fighting or co-operating, with other men pursuing objectives similar to his own.

Haavelmo (11) has recently, in a presidential address to The Econometric Society, admitted that modern mathematical economics often has given poor results when applied to problems in real life. One explanation may be that economists usually have studied a non-existent variety of *homo sapiens*.

2.4. The simplest non-trivial game is usually referred to as the *two-person zero-sum game*. Here $n = 2$ and

$$M_2(x_1,x_2) = - M_1(x_1, x_2).$$

For this case one can prove that there exist two probability distributions $F_1(x_1)$ and $F_2(x_2)$ defined over the sets X_1 and X_2, so that

$$\underset{X_2}{\text{Min}} \int_{X_1} M_1(x_1,x_2)dF_1(x_1) = \underset{X_1}{\text{Max}} \int_{X_2} M_1(x_1,x_2)dF_2(x_2)$$

This is the famous *minimax theorem* which was first proved by Von Neumann (15). A discussion between Frechet (9), (10) and Von Neumann (14) gives some interesting information on the history of this theorem.

The essential idea behind the theorem is that a player must choose his strategy by some random device. If he does not do this, the opponent can guess his strategy and adjust his own strategy accordingly. By selecting the proper random device, i.e. the proper probability distribution, a player can secure for himself a certain expected gain, regardless of what strategy the opponent chooses. The theorem states that these expected gains which both players can secure for themselves, have the same absolute value. Hence none of the players can obtain a greater expected gain if the opponent behaves rationally, i.e. if he chooses the proper random device.

2.5. The minimax theorem is a really deep mathematical theorem, and it has had a profound influence on statistical thinking during the last decade. The economic problems which can be formulated as two-person zero-sum games appear to be rather trivial. However, the solution of a two-person zero-sum game, i.e. the problem of determining the probability distribution $F_1(x_1)$ and $F_2(x_2)$ is as Von Neumann himself has pointed out, equivalent to solving a linear programming problem and its dual. Linear programming has proved to be a very useful technique for solving a number of problems in economic analysis, and it is through this backdoor that game theory has found most of its applications in economics. The papers referred to in para 1.5 are, as we pointed out, all based on this equivalence between linear programming and the simplest of all games.

2.6. Game theory as a whole will have rather restricted applications in economics if M_i is interpreted only as the monetary value of the gain which player i makes in the game. However, by interpreting M_i as the *utility* which player i attaches to this gain, Von Neumann and Morgenstern open vast new fields of application for game theory.

This idea is due to Bernoulli (3) who in 1738, proposed that rational people act so that they maximize, not expected gain, but expected utility of the gain. This principle has played a certain although modest part in statistical theory, but was ignored by most economists until Von Neumann and Morgenstern (16) revived it. To Bernoulli the principle was merely a plausible hypothesis.

Von Neumann and Morgenstern proved that the principle could be derived as a *theorem* from a few simple and apparently very acceptable axioms.

2.7. From this short discussion it appears that the two aspects of game theory which should be most promising for immediate application to problems in insurance, are the following:

(i) The utility concept derived from the Bernoulli principle.

(ii) The analysis of n-person conflict situations, i.e. situations where parties whose interests are opposed can gain by co-operating.

Both aspects have been discussed in considerable detail in a number of recent publications. In the present paper it should therefore be sufficient to give only a brief summary of the essential ideas, illustrated by some simple applications.

3. THE APPROACH OF CLASSICAL ECONOMIC THEORY

3.1. Before discussing in more detail the application of game theory, it may be useful to survey briefly the traditional economic approach to some of the problems in insurance.

The reason why classical economic theory is unable to deal with insurance, is clearly that uncertainty has no place in this theory. The theory assumes that an entrepreneur has full and certain knowledge of future prices and cost when he decides how much to invest in a new factory, where it shall be located and how much it shall produce. This assumption is obviously unrealistic, and most economists replace it by an assumption that the entrepreneur's *expectation* of prices and cost determine his decisions. Practically the whole economic theory is based on this assumption. The theory is internally consistent, and it cannot be refuted when confronted by reality—as long as one does not venture into making statements as to how entrepreneurs form their expectations from the information available to them.

3.2. At a fairly early stage in the development of mathematical economics, authors began more or less tacitly to assume that the expectations of a "rational" entrepreneur would be equal to the

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mathematical expectation in the precise sense given to this term in the theory of probability.

It is obviously not self-evident that the vague concept "entrepreneurial expectations" is identical to, or even connected with the well defined statistical concept "expected value". Suppose for instance that an entrepreneur is considering the marketing of a new product, which may prove either to be worthless, or to sell for $ 20 per unit, with equal probability. The assumption implies that this entrepreneur, if he is rational, should act as if he was certain the product would sell for $ 10 a unit.

Marshall ((13) page 332) discusses this question briefly, and in very cautious terms. He concludes that "the evils of uncertainty must count for something", and seems to assume that the entrepreneur will act as if he was certain to get a lower amount, say $ 9 per unit of his product.

3.3. Many economists do not seem to share the prudence of Marshall, and they confidently assume that entrepreneurs try to maximize *expected profits*, and hence that they keep stocks so that expected storage costs are minimized, locate factories so that expected transportation costs are minimized etc.. This assumption may be justified as a first approximation in some cases, but it cannot be generally valid. If it were, i.e. if businessmen invariably made the decisions which maximize expected profits, they would never take any insurance. As an insurance premium necessarily is greater than the expected loss, a businessman will inevitably reduce his expected profits if he takes insurance to cover some of the risks which he is running.

Hence it appears that a large part of economic theory is based on assumptions which make insurance outright irrational, which of course means that the theory is unable to recognize and analyse insurance as an economic activity.

3.4. The cause of these difficulties is evidently the assumption that a rational man should seek to maximize expected profits, an assumption which can be traced back at least to Pascal. However, the well known "St. Petersburg Paradox" shows that the hypothesis can lead to absurd results when pushed to extremes. The solution which Bernoulli (3) proposed to the paradox, has been strangely ignored by actuaries and economists alike.

In the following we shall see that Bernoulli's assumption that a rational man seeks to maximize expected *utility* will make it possible to bring insurance into a general theory of economic activities.

4. THE UTILITY CONCEPT

4.1. In a paper (5) presented to the ASTIN Colloquium in 1961, it was demonstrated that a utility concept appears to be an essential element in a complete theory of insurance. In that paper we considered an insurance company holding a portfolio of fully paid insurance contracts. If the duration of all contracts is so short that we can ignore interest, the *risk situation* of the company will be completely determined by the following two elements:

(i) $F(x)$ = the probability that claims payable under the contracts shall not exceed x.

(ii) S = the funds which the company holds and can draw upon to pay claims.

If a manager of an insurance company shall be able to make intelligent decisions, he must have some *preference ordering* over the set of all risk situations, i.e. he must have some criterion which enables him to decide whether one risk situation is better than another.

4.2. The preference ordering over the set of risk situations $(S,F(x))$ can be represented by a *utility index*, or a functional $U(S,F(x))$. Von Neumann and Morgenstern(16) have shown that if the preference ordering is consistent in a particular sense, the Bernoulli principle can be proved as a theorem. Hence we must have

$$U(S,F(x)) = \int_{\infty}^{0} u(S-x)dF(x)$$

where

$$u(S) = U(S,\varepsilon(x))$$

Thus $u(S)$ is the utility attached to the degenerate risk situation $(S,\varepsilon(x))$ where the company holds funds amounting to S, and where the probability is one that claims shall be zero. The function $u(x)$

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can be interpreted as the *utility of money* to the insurance company.

The Bernoulli principle states that the utility atached to any risk situation can be expressed as a linear combination of the utility attached to degenerate risk situations. The theorem is obviously related to the familiar theorem that any n linearly independent vectors can be taken as a basis which spans the whole n-dimensional Euclidian space. In many cases the most convenient basis consists of "degenerate" vectors of the type (1,0,0...0), (0,1,0...0).... The Bernoulli principle is nothing but a generalization of this theorem to a space of infinitely many dimensions.

4.3. In the paper (5) referred to we showed that a utility concept was necessary in order to determine the optimal reinsurance arrangement. In the present paper we shall study a different aspect of insurance, which also will illustrate how a utility concept appears essential for a rational formulation of the problem.

We shall consider an insurance company which underwrites only one kind of insurance contracts, and we shall assume that this particular kind of contract has the claim distribution $F(x)$. The net premium of this insurance will then be

$$P = \int_0^\infty x dF(x) = m$$

We assume further that the company accepts such insurance contracts against a gross premium $(1 + \lambda)P$.

If the company underwrites n contracts, its total funds will amount to $S + n(1 + \lambda)P$ and the claim distribution of its portfolio will be

$F^{(n)}(x)$ = the n-th convolution of $F(x)$ with itself.

4.4. In the simplest non-trivial case the utility of money can be represented by a function of the form

$$u(x) = -ax^2 + x$$

When the utility function has this form, it is easy to show that the utility attached to the risk situation $(S,F(x))$ is given by

$$U(S,F(x)) = S - m - a(S - m)^2 - aV$$

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where m and V are the mean and the variance of $F(x)$. With a portfolio of n contracts, the utility of the company will be

$$U(S + n(1 + \lambda) P, F^{(n)}(x)) = S + n\lambda P - a(S + n\lambda P)^2 - anV$$

With this expression we can formulate and solve a number of problems. We can for instance assume that λ is given by market conditions, and determine the value of n, i.e. the number of contracts which will maximize the utility of the company.

4.5. We shall not solve the problems outlined in the preceding paragraph when the utility function has the form $u(x) = -ax^2 + x$. This utility function has been studied in considerable detail in previous papers (6) and (7), so that a complete solution will be a mere repetition of calculations already published.

Instead we shall assume

$$u(x) = -e^{-ax}$$

It is of course of no significance that utility in this case is negative. We can always add a positive constant to the utility function without altering the underlying preference order.

For this utility function we find

$$U(S, F(x)) = -\int_0^\infty e^{-a(S-x)}\, dF(x) = -e^{-aS}\, \varphi(a)$$

If the integral

$$\varphi(a) = \int_0^\infty e^{ax}\, dF(x)$$

exists, $\varphi(a)$ will be the characteristic function of $F(x)$ (for the argument $-ai$).

Hence a portfolio of n stochastically independent insurance contracts each with a claim distribution $F(x)$, will give the company a utility

$$U(S + n(1 + \lambda) P,\ F^{(n)}(x)) = -e^{-a(S+n(1+\lambda)P)}\, \{\varphi(a)\}^n$$

4.6. In the last formula of the preceding paragraph it is convenient to write

$$V(n, \lambda) = \log(-U) = n \log \varphi(a) - aS - an(1 + \lambda) P.$$

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As $V(n, \lambda)$ will decrease with increasing U, the company will seek to maximize $V(n, \lambda)$.

We see that $V(n, \lambda)$ is linear in n. Hence the problem mentioned in para 4.4 is trivial when the utility of money has this particular shape. For a given λ, the company will seek to underwrite as many contracts as possible if

$$a(1 + \lambda) P > \log \varphi(a)$$

If the inequality is reversed, the company will not underwrite any contract of this kind.

4.7. We arrive at more interesting problems if we assume that n depends on λ, i.e. that we have

$$n = n(\lambda)$$

where $n(\lambda)$ is a function which increases with decreasing λ. Here we can interpret $n(\lambda)$ as the market demand for this particular insurance contract. The lower the "price" λ, the greater the demand n. In practice it appears that insurance has a very low price elasticity. A more realistic interpretation may be to assume that n can be increased by using a part of the loading for sales promotion, such as advertising.

If the function $n(\lambda)$ is given, we can determine the value of λ which maximizes the company's utility. The first order condition for a maximum is:

$$\frac{\partial V}{\partial n} \frac{dn}{d\lambda} + \frac{\partial V}{\partial \lambda} = 0$$

If we assume that $n(\lambda)$ is linear, i.e. that

$$n(\lambda) = n_0 - \alpha\lambda .$$

the condition is reduced to

$$\alpha \log \varphi(a) = a n_0 - \alpha a P(1 + 2\lambda)$$

from which we can determine the desired value of λ.

4.8. The problem which we have discussed in the preceding paragraphs is of obvious practical importance, and it has frequently

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been discussed in insurance literature. However, it appears difficult. if not impossible to come to grips with the problem unless the objectives of the insurance company are formulated in an operational manner. Such formulation of the problem requires a utility concept—or something equivalent.

4.9. It is interesting to note that the Bernoulli principle was used as early as 1834 by Barrois (1) in his study of fire insurance. Barrois considered the following problem:

A man who has total assets amounting to S, owns a house worth R and there is a probability p that the house may be destroyed by fire. He can insure his house against fire by paying a premium Q. Shall he, or shall he not take this insurance?

Barrois introduced a utility function $u(x)$ which measures the utility attached to an amount x of money, and shows that a rational person will take insurance if, and only if

$$u(S - Q) > pu(S - R) + (1-p)\, u(S)$$

This inequality says that the person prefers to have his assets reduced from S to S-Q with certainty, rather than risk a reduction to S-R with probability p.

By this simple device Barrois has solved Marshall's problem of determining the amount a rational person should pay to avoid the "evils of uncertainty". He has also provided the analytical tools which Tauber 75 years later sought in vain in economic theory.

Following Bernoulli, Barrois assumes that $u(x) = \log x$. However, like Bernoulli himself, he seems aware that other non-decreasing functions may serve equally well. Barrois should certainly be recognized as one of the pioneers in the theory of insurance. It seems incidentally that the place in the history of insurance which is rightfully his, is about to be occupied by Alice Morrison, who according to a recent paper by Williams (23) was the first to apply Bernoulli's principle to insurance problems. Miss Morrison's work has apparently not been published, but it was rewarded with a Ph. D. by Iowa State University in 1949.

4.10. Barrois was a contemporary of Cournot, whose work also was forgotten by the following generation of economists. Referring to Cournot, Walras ((22) page XX) wrote in 1900:

> „Si la France du XIXme siècle, qui a vu naître la science nouvelle, s'en est complètement désintéressée, cela tient à cette conception d'une étroitesse bourgeoise de la culture intellectuelle qui l'a partagée en deux zones distinctes: l'une produisant des calculateurs dépourvus de connaissances philosophiques, morales, historiques, économiques, et l'autre oû fleurissent des lettrés sans aucunes notions mathématiques".

It may be unfair to make this comment on the France of today, but the remark may have some address to modern actuaries. Barrois has been quoted fairly regularly in the insurance literature—also in *The ASTIN Bulletin*. However, none of the conscientious bibliographers who have referred to Barrois' paper seem to have realised that the paper contains the key to the whole theory of the economics of uncertainty.

Actuaries are certainly not "calculateurs" without any culture, but they do occasionally take a too narrow view of their function. Insurance is an economic activity, and the actuary should not ignore the economic environment in which his company works.

5. CONFLICT SITUATIONS

5.1. Another paper (8) presented to this colloquium discusses the rating of different risk groups in an insurance collective. This is a typical conflict situation. Each group wants to pay the lowest possible premium, and the total amount of premium to be paid will be lowest if all groups can agree to join and form one single company. Hence the groups as a whole will gain if they co-operate. However, the interests of the various groups are opposed, since each group will want to secure for itself the greatest possible share of the collective gain which results from the co-operation. If a group is dissatisfied with its share of this gain, it can threaten to leave the company, and it can try to bribe other groups to join it in setting up a rival company. The remaining groups will then have to consider whether they shall offer the dissident group a higher share, or accept the break up of the big company.

5.2. Economic theory has nothing to offer when it comes to analysing a problem like the one we have outlined. Most economists will probably dismiss the problem all together as not belonging to economics but to some underdeveloped branch of psychology.

The paper (8) referred to offers a solution, which may, or may not, be considered as the final answer. However, whether this solution is accepted or not, it is clear that game theory has something to offer, and that it is through this theory we may hope to gain complete mastery of the problem.

5.3. We get a better basis for comparing game theory and classical economic theory if we study the reinsurance market.

As a starting point we consider n insurance companies, which as a result of their direct underwriting find themselves in the risk situations determined by the elements $F_i\,(x_i)$ and S_i $(i = 1, 2 \ldots n)$.

Using the notation of para 4.2, we find that company i attaches the following utility to its initial risk situation

$$U_i = \int_0^\infty u_i\,(S_i - x_i)\,dF_i\,(x_i)$$

If $x_1 \ldots x_n$ are stochastically independent, and if for all i we have $u_i'\,(x_i) > 0$ and $u_i''(x_i) < 0$ over the whole range which enters into consideration, it is fairly easy to show that there are reinsurance arrangements which will increase the utility of all the companies. Hence the situation is similar to the one described in para 5.1. All companies stand to gain if they co-operate, i.e. if they conclude reinsurance treaties with each other. However, there is a conflict of interest as to how this gain should be distributed among the companies. It is clear that the problem can be analysed in terms of game theory, and that we can arrive at a solution similar to the one given for the problem discussed in the paper (8) already referred to.

5.4. Assume now that the companies make some reinsurance arrangements so that the utility of company i changes from U_i^0 to U_i^1.

If no company will participate in an arrangement which reduces its own utility, we must have

$$U_i^0 \leq U_i^1 \quad \text{for all } i.$$

If there exist no arrangement which will give company i the utility U_i^2 such that

$$U_i^1 \leq U_1^2 \quad \text{for all } i$$

the arrangement corresponding to U_i^1 is said to be *Pareto optimal*.

If the n companies act rationally, we must assume that they somehow reach a Pareto optimal reinsurance arrangement. It is obviously irrational to settle for some other arrangement, since it will then be possible to increase the utility of all companies by switching to a Pareto optimal arrangement.

5.5. In the paper (6) already referred to it has been shown that the Pareto optimal arrangements in a reinsurance market are determined by n functions $y_1(x) \ldots y_n(x)$ which satisfy the conditions

$$k_i u_i'(S_i - y_i(x)) = k_j u_j'(S_j - y_j(x))$$

where $k_1 \ldots k_n$ are positive constants, and $x = x_1 \ldots + x_n$ and where $y_i(x)$ is the amount which company i has to pay if the total amount of claims is x.

The constants $k_1 \ldots k_n$ can be chosen arbitrarily, subject to some rather trivial restrictions which we shall not discuss here. Hence the solution we have arrived at is indeterminate. Our assumption that the companies behave rationally, implies that they will reach some Pareto optimal arrangement, but the assumption is not sufficient to determine which particular arrangement they will settle for. In order to obtain a determinate solution. i.e. to determine the values of $k_1 \ldots k_n$, we must make *additional assumptions* about how the companies behave in the reinsurance market.

5.6. The indeterminate character of the solution is brought out quite clearly by a simple example studied in the paper (7) already referred to. In this paper we assumed that the utility of money to company i was given by

$$u(x) = x - a_i x^2 \qquad i = 1, 2 \ldots n$$

The constant a_i can be interpreted as a measure of the company's risk aversion, and may differ from one company to another.

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It can then be proved that a Pareto optimal reinsurance arrangement between the n companies will give company i the utility

$$U_i = \frac{1}{4a_i} - q_i^2\, a_i\, A \qquad i = 1, 2 \ldots n$$

Here A is a positive "universal constant" for this particular market. The parameters $q_1 \ldots q_n$ can be chosen arbitrarily, subject to the restrictions:

(i) $q_1 + q_2 + \ldots q_n = 1$
(ii) $q_i \geq 0$ for all i

Company i will obviously want q_i to be as small as possible. However, this leads to a conflict with the desires of the other companies, which also want their particular q to be as small as possible. Hence the n companies have to bargain their way to a compromise.

The companies have to reach agreement on a set of numbers $q_1 \ldots q_n$ which satisfy the conditions above. This is the problem, reduced to its bare essentials. It is quite clear that in order to obtain a determinate solution, we must make some assumptions as to how the companies negotiate their way to an agreement.

5.7. In some previous papers (6) and (7) we have approached the problem in the manner of classical economic theory. We have assumed with Tauber (19) that reinsurance cover is a service, and that as any other service it must have its price. We have further assumed that there exists an equilibrium price which will make supply and demand for this service equal.

In the classical theory of commodity markets, it is assumed that all traders in the market take the equilibrium prices for each commodity as given and unalterable, and buy and sell at these prices until their utility is maximized. One can then show that the market will reach a Pareto optimal situation. In this Pareto optimal situation, the commodities are distributed among the traders in such a way that any further exchanges will reduce the utility of some of the participants.

In the following we shall seek to transfer this behavioral assumption to the reinsurance market, and see if the price mechanism

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which leads the classical commodity market to a Pareto optimum can be generalized so that it can fulfil the same function in a reinsurance market.

5.8. In a reinsurance market a company accepts liability for a portfolio of insurance contracts with claim distribution $F(x)$ against payment of an amount P. Since there is no natural "unit of insurance cover", there is no obvious way in which P and $F(x)$ can be connected. Hence our first problem is to define a price concept which can be meaningfully applied to the transactions in a reinsurance market.

In the paper (6) already referred to, it has been shown that a transformation of the form

$$P(F(x)) = \sum_{j=1}^{\infty} p_j \varkappa_j$$

is the only one which satisfies the essential requirements of a price concept. Here $p_1 \ldots p_j \ldots$ are constants, and $\varkappa_j$ is the j-th cumulant of the probability distribution $F(x)$.

The transformation $P(F(x))$ must be interpreted as the amount one has to pay for reinsurance cover for a portfolio with claim distribution $F(x)$.

5.9. Under certain conditions a price of this form can be an equilibrium price, i.e. can lead to a balance between supply and demand for reinsurance cover in the market. It has, however, been shown in another paper (7) that if the companies take an equilibrium price of this kind as given, and set out to do the transactions which will maximize their utility, they will in general end up with a situation which is not Pareto optimal. In this situation there will be transactions—at other prices—which will increase the utility of all the participating companies. If the companies act rationally, they will carry out these transactions and reach a Pareto optimal situation. However, these latter transactions can obviously not be governed by the price mechanism of classical economic theory.

5.10. The reinsurance problem appears at first sight to be a problem which can be analysed in terms of classical economic theory, once the objectives of the companies have been formulated in an operational manner by the help of Bernoulli's utility concept. However, closer investigations show that economic theory can only take us part of the way. The problem is in its very essence a problem of co-operation between parties who have conflicting interests, and who are free to form and break any coalitions which may serve their particular interests. Classical economic theory is powerless when it comes to analyse such problems. The only theory which at present seems to hold some promise of being able to sort out and explain this apparently chaotic situation, is the Theory of Games.

6. A GENERAL THEORY OF INSURANCE

6.1. In the preceding sections we have assumed that all players act rationally. We shall now assume that one player, say player n, does not act rationally. We shall refer to this player as *Nature*, and assume that he burns down houses, sinks ships, and in general plays havoc with the carefully laid plans of the other players. However, Nature is not completely erratic in its behaviour. We shall assume that there exists a probability distribution $F(x_n)$ defined over the set X_n, and that this distribution can be determined, at least to a useful approximation by observing the behaviour of Nature.

The n-1 other players who act rationally must take the erratic behaviour of Nature into account when chosing their strategies. A single player, acting on his own, can do little to protect himself against Nature. However, groups of players can co-operate, and for instance form insurance companies—or in game theory—coalitions to spread the harmful effects of Nature's behaviour. These coalitions can again make reinsurance arrangements to spread the effects even further.

6.2. It is evident that the whole insurance activity can be seen as a game played by a large number of reasonably rational players against an erratic, and generally hostile Nature. The function of the actuary in this game is to study the probability distribution $F(x_n)$ and provide the information which form the basis of the decisions made by the players individually, and by those responsible for managing their coalitions, i.e. the directors of insurance companies.

The concept "Game against Nature" was introduced by Wald (21) in 1950. The concept has proved extremely useful by providing a link between modern statistical theory and game theory and studies of economic behaviour. It also appears that by adopting this concept, it should be possible to construct a unified theory for all aspects of insurance. This will of course be a formidable task, which it may take decades to complete. In the present paper we have only touched upon a few points which illustrates the power of game theory when it comes to solving problems which it is difficult even to formulate in a meaningful manner in the terms of earlier theories.

7. CONCLUSION

7.1. In this paper we have tried to show that game theory can be applied to a number of problems in insurance, problems to which earlier theories have been unable to offer satisfactory solutions. The literature we have quoted seems to indicate that an increasing number of actuaries are becoming aware of the possibilities of game theory, and that they will find far more ingenious applications for the theory than we have been able to suggest.

7.2. We have, however, tried to show something more than this. We have sought to demonstrate that game theory is not just another new mathematical device which may be handy in an insurance company. The real purpose of this paper has been to show that game theory is indeed a necessary—maybe even a sufficient—basis on which a unified general theory of insurance can be constructed.

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