ANZIAM J. 58(2017), 428–435
doi:10.1017/S144618111600033X

THE INVARIANT REGION FOR THE EQUATIONS OF NONISENTROPIC GAS DYNAMICS

WEI-FENG JIANG[✉1] and ZHEN WANG[2]

(Received 11 June, 2016; accepted 15 October, 2016; first published online 6 March 2017)

Abstract

We study the existence of the invariant region for the equations of nonisentropic gas dynamics. We obtain the mean-integral of the conserved quantity after making an intensive study of the Riemann problem. Using the extremum principle and the Lagrangian multiplier method, we prove that the one-dimensional equations of nonisentropic gas dynamics for an ideal gas possess a unique invariant region. However, the invariant region is not bounded.

2010 *Mathematics subject classification*: 35L65.

Keywords and phrases: gas dynamics, nonisentropic, existence of invariant region.

1. Introduction

In this paper, we analyse the existence of the invariant region for the one-dimensional equations of nonisentropic gas dynamics. These are characterized by

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (\rho u^2 + p)_x = 0 \\ (\rho E)_t + (\rho u E + pu)_x = 0 \end{cases} \tag{1.1}$$

with the gas state

$$p = R\rho T, \quad e = c_v T, \quad p(s,\rho) = e^{s/c_v}\rho^\gamma, \tag{1.2}$$

where $\rho > 0$; s and u are the density, entropy and velocity, respectively; $E = u^2/2 + e$ is the energy; and e is the internal energy. Also, R, k, c_v and γ are positive constants with $\gamma > 1$.

The initial data for (1.1) is

$$(\rho, m, q)(x,t)|_{t=0} = \begin{cases} (\rho_l, m_l, q_l) & x < 0 \\ (\rho_r, m_r, q_r) & x > 0, \end{cases} \tag{1.3}$$

where $m = \rho u$, $q = \rho E$ and $\rho_{l,r}$, $m_{l,r}$, $q_{l,r}$ are all given constants.

[1]China Jiliang University, Hangzhou, China; e-mail: casujiang89@cjlu.edu.cn.
[2]Wuhan Institute of Physics and Mathematics of the Chinese Academy of Sciences, Wuhan, China; e-mail: zhenwang@wipm.ac.cn.

The gas dynamics equation is one of the core subjects for a conservation law, for which the most important problem is the existence of global weak solutions with large initial data.

Compensated compactness [10] is one of the most effective methods for investigating this problem. Using this method, the framework for the existence of solutions to the equations of isentropic gas dynamics with Cauchy data is almost complete. Diperna [2, 3] established the existence of the weak entropy solution for the isentropic Euler equations with general L^∞ initial data for $\gamma = 1 + 2/(2n+1)$, where $n \geq 2$ is an integer. Ding et al. [4, 5] also obtained the existence of the isentropic solution by vanishing numerical viscosity for $\gamma \in [1, 5/3]$. Lions, Perthame, Tadmor and Souganidis [1, 7, 8] obtained existence results for $\gamma > 3$, while Huang and Wang [6] obtained existence results for $\gamma = 1$.

However, the compensated compactness theory encountered bottlenecks when proving the existence of a solution for (1.1) and (1.2) with Cauchy data because of the difficulty in obtaining a uniform bounded estimation.

The aim of the present paper is to find a method for obtaining an invariant region for the conservation laws. Hence, we calculate the mean-integral of the Riemann solution

$$\begin{cases} \overline{\rho} = \dfrac{\rho_l + \rho_r}{2} + \dfrac{k}{2}(m_l - m_r) \\ \overline{m} = \dfrac{m_l + m_r}{2} + \dfrac{k}{2}\left\{\dfrac{3-\gamma}{2}\left(\dfrac{{m_l}^2}{\rho_l} - \dfrac{{m_r}^2}{\rho_r}\right) - (\gamma - 1)(q_l - q_r)\right\} \\ \overline{q} = \dfrac{q_l + q_r}{2} + \dfrac{k}{2}\left\{\gamma\dfrac{m_l q_l}{\rho_l} - \dfrac{\gamma-1}{2}\dfrac{{m_l}^3}{{\rho_l}^2} - \gamma\dfrac{m_r q_r}{\rho_r} + \dfrac{\gamma-1}{2}\dfrac{{m_r}^3}{{\rho_r}^2}\right\} \end{cases}$$

and use the properties of the Riemann invariant region for (1.1). Finally, we find a partial differential equation to calculate the bound equations, $F = F(\rho, m, q)$, of the invariant region for (1.1); here $k = \triangle t / 2\triangle x$, $\overline{\rho}$, $\overline{m}$ and $\overline{q}$ are the mean-integrals of ρ, m and q, respectively.

Thus, we show that there does not exist any bounded invariant region for (1.1) with an ideal gas state. This means that the equations for nonisentropic gas dynamics are quite different from the isentropic case.

This paper is organized as follows. In Section 2, we study the Riemann solution of system (1.1) and calculate the mean-integral of the conserved quantity. In Section 3, we mainly discuss the properties of the Riemann invariant region for (1.1) and prove the nonexistence of Riemann invariants for (1.1) with an ideal gas state. The paper concludes with a discussion in Section 4.

https://doi.org/10.1017/S144618111600033X Published online by Cambridge University Press

2. The mean-integral conserved quantity

The Riemann problem which consists of system (1.1) and the Riemann initial data (1.3) was solved by Smoller [9]. For brevity, we state the solutions as

$$\begin{cases} \text{1-family} & \begin{cases} \dfrac{\rho_r}{\rho_l} = e^{-x} \\ \dfrac{s_r - s_l}{c_v} = \begin{cases} 0 & x \geq 0 \\ \gamma x + \log \dfrac{1-\beta e^x}{e^x - \beta} & x \leq 0 \end{cases} \\ \dfrac{u_r - u_l}{c_l} = \begin{cases} \dfrac{2}{\gamma - 1}(1 - e^{-\tau\gamma x}) & x \geq 0 \\ (e^x - 1)\sqrt{\dfrac{1-\beta}{e^x - \beta}} & x \leq 0 \end{cases} \end{cases} \\ \text{2-family} & \dfrac{\rho_r}{\rho_l} = e^x, \quad \dfrac{s_r - s_l}{c_v} = -\gamma x, \quad u_r = u_l \\ \text{3-family} & \begin{cases} \dfrac{\rho_r}{\rho_l} = e^{x} \\ \dfrac{s_r - s_l}{c_v} = \begin{cases} 0 & x \geq 0 \\ -\gamma x + \log \dfrac{e^x - \beta}{1-\beta e^x} & x \leq 0 \end{cases} \\ \dfrac{u_r - u_l}{c_l} = \begin{cases} \dfrac{2}{\gamma - 1}(e^{\tau\gamma x} - 1) & x \geq 0 \\ (1 - e^{-x})\sqrt{\dfrac{1-\beta}{e^{-x} - \beta}} & x \leq 0. \end{cases} \end{cases} \end{cases}$$

To investigate the mean-integral for the conserved quantity of (1.1) and (1.3), we set $m = \rho u$ and $q = \rho E$.

For

$$E = \frac{1}{2}u^2 + \frac{1}{\gamma - 1}\frac{p}{\rho},$$

we obtain

$$p = (\gamma - 1)\Big(q - \frac{1}{2}\frac{m^2}{\rho}\Big). \tag{2.1}$$

Substituting (2.1) into system (1.1), the system (1.1) reduces to

$$\begin{cases} \rho_t + m_x = 0 \\ m_t + \Big[\dfrac{3-\gamma}{2}\dfrac{m^2}{\rho} + (\gamma - 1)q\Big]_x = 0 \\ q_t + \Big[\gamma \dfrac{mq}{\rho} - \dfrac{\gamma - 1}{2}\dfrac{m^3}{\rho^2}\Big]_x = 0. \end{cases}$$

Defining the mean-integral of U as

$$\overline{U}(x,t) = \frac{1}{2l}\int_{-l}^{l} U(x,h)\,dx = (\overline{\rho}(x,t), \overline{m}(x,t), \overline{q}(x,t)),$$

we have the following theorem.

https://doi.org/10.1017/S144618111600033X Published online by Cambridge University Press

THEOREM 2.1.

$$\begin{cases} \overline{\rho} = \frac{1}{2}(\rho_l + \rho_r) + k(m_l - m_r) \\ \overline{m} = \frac{1}{2}(m_l + m_r) + k(p_l - p_r) + k(\rho_l {u_l}^2 - \rho_r {u_r}^2) \\ \overline{q} = \frac{1}{2}(q_l + q_r) + k(q_l u_l - q_r u_r) + k(p_l u_l - p_r u_r), \end{cases} \tag{2.2}$$

where $k = h/2l$.

PROOF. We suppose that S_i, R_i and J_2 are the ith shock waves, the rarefaction waves and the contact discontinuity, respectively. U_i are the ith middle state, where $i = 1, 2$.

(1) If the Riemann solutions are

$$U_l \xrightarrow{S_1} U_1 \xrightarrow{J_2} U_2 \xrightarrow{S_3} U_r,$$

then

$$\begin{aligned}
\bar{\rho} &= \frac{1}{2l}\int_{-l}^{l} \rho(x,h)\,dx \\
&= \frac{1}{2l}\Big(\int_{-l}^{\sigma_1 h} \rho_l\,dx + \int_{\sigma_1 h}^{u_m h} \rho_1\,dx + \int_{u_m h}^{\sigma_2 h} \rho_2\,dx + \int_{\sigma_2 h}^{l} \rho_r\,dx\Big) \\
&= \frac{1}{2}(\rho_l + \rho_r) + k(\rho_l u_l - \rho_r u_r), \\
\bar{m} &= \frac{1}{2l}\int_{-l}^{l} m(x,h)\,dx \\
&= \frac{1}{2l}\Big(\int_{-l}^{\sigma_1 h} m_l\,dx + \int_{\sigma_1 h}^{u_m h} m_1\,dx + \int_{u_m h}^{\sigma_2 h} m_2\,dx + \int_{\sigma_2 h}^{l} m_r\,dx\Big) \\
&= \frac{1}{2}(m_l + m_r) + k(p_l - p_r) + k(m_l u_l - m_r u_r), \\
\bar{q} &= \frac{1}{2l}\int_{-l}^{l} n(x,h)\,dx \\
&= \frac{1}{2l}\Big(\int_{-l}^{\sigma_1 h} q_l\,dx + \int_{\sigma_1 h}^{u_m h} q_1\,dx + \int_{u_m h}^{\sigma_2 h} q_2\,dx + \int_{\sigma_2 h}^{l} q_r\,dx\Big) \\
&= \frac{1}{2}(q_l + q_r) + k(p_l u_l - p_r u_r) + k(q_l u_l - q_r u_r).
\end{aligned}$$

(2) If the Riemann solutions are

$$U_l \xrightarrow{R_1} U_1 \xrightarrow{J_2} U_2 \xrightarrow{S_3} U_r,$$

then

$$\begin{aligned}
\bar{\rho} &= \frac{1}{2l}\int_{-l}^{l} \rho(x,h)\,dx \\
&= \frac{1}{2l}\Big(\int_{-l}^{(u_l - c_l)h} \rho_l\,dx + \int_{(u_l - c_l)h}^{(u_m - c_1)h} \rho(x)\,dx + \int_{(u_m - c_1)h}^{u_m h} \rho_{m,1}\,dx + \int_{u_m h}^{\sigma h} \rho_2\,dx + \int_{\sigma h}^{l} \rho_r\,dx\Big) \\
&= \frac{1}{2}(\rho_l + \rho_r) + k[(u_l - c_l)\rho_l + \rho_1 c_1] + \frac{1}{2l} I - k\rho_r u_r,
\end{aligned}$$

https://doi.org/10.1017/S144618111600033X Published online by Cambridge University Press

where $x/h = u - c$, $c^2 = c_l^2 e^{-2\tau y}$, $I = \int_{u_l - c_l}^{u_m - c_{m,1}} \rho(x,h)\,dx$. Since

$$\frac{x}{h} = u_l + \frac{2}{\gamma - 1} c_l - \beta c_l e^{-\tau y},$$

we obtain

$$\begin{aligned}\frac{1}{2l} I &= \frac{1}{2l} \int_{(u_l - c_l)h}^{(u_m - c_1)h} \rho\Big(\frac{x}{h}\Big)\,dx \\ &= \frac{1}{2l} \int_0^{(1/\tau)\log(c_l/c_1)} \rho_l e^{-y/\gamma} h(-\beta c_l) e^{-\tau y}(-\tau)\,dy \\ &= k \int_0^{\log(c_l/c_1)} \beta \rho_l c_l e^{-\beta s}\,ds \quad (\text{with } y = s/\tau) \\ &= k(\rho_l c_l - \rho_1 c_1).\end{aligned}$$

Thus

$$\begin{aligned}\bar\rho &= \tfrac{1}{2}(\rho_l + \rho_r) + k[(u_l - c_l)\rho_l + \rho_1 c_1] + k\rho_l(c_l - c_1) - k\rho_r u_r \\ &= \tfrac{1}{2}(\rho_l + \rho_r) + k(\rho_l u_l - \rho_r u_r),\end{aligned}$$

which completes the proof of the theorem. □

3. The invariant region for the equation of gas dynamics

We suppose that the weak solutions of (1.1) exist in a bounded invariant region Σ and that $U = (\rho,\ m,\ q) \in \Sigma$ is a point which belongs to the neighbourhood of U_l: that is,

$$F(\rho, m, q) \le C.$$

Depending on the convexity of Σ, $F(\bar\rho, \bar m, \bar q)$ reaches a maximum value at (ρ_l, m_l, q_l). From (1.2) and (2.2),

$$\begin{cases} \overline{\rho} = \dfrac{\rho_l + \rho}{2} + \dfrac{k}{2}(m_l - m) \\ \overline{m} = \dfrac{m_l + m}{2} + \dfrac{k}{2}\left\{\dfrac{3-\gamma}{2}\dfrac{m_l{}^2}{\rho_l} + (\gamma - 1) q_l - \dfrac{3-\gamma}{2}\dfrac{m^2}{\rho} - (\gamma - 1) q\right\} \\ \overline{q} = \dfrac{q_l + q}{2} + \dfrac{k}{2}\left\{\gamma \dfrac{m_l q_l}{\rho_l} - \dfrac{\gamma - 1}{2}\dfrac{m_l{}^3}{\rho_l{}^2} - \gamma\dfrac{mq}{\rho} + \dfrac{\gamma - 1}{2}\dfrac{m^3}{\rho^2}\right\}. \end{cases} \tag{3.1}$$

According to the extremum principle and the Lagrangian multiplier method, there exists $\lambda \in R$ such that

$$\begin{cases} \dfrac{\partial F(\overline{\rho}, \overline{m}, \overline{q})}{\partial \rho}\Big|_{U_l = U} - \lambda \dfrac{\partial F(\rho, m, q)}{\partial \rho} = 0 \\ \dfrac{\partial F(\overline{\rho}, \overline{m}, \overline{q})}{\partial m}\Big|_{U_l = U} - \lambda \dfrac{\partial F(\rho, m, q)}{\partial m} = 0 \\ \dfrac{\partial F(\overline{\rho}, \overline{m}, \overline{q})}{\partial q}\Big|_{U_l = U} - \lambda \dfrac{\partial F(\rho, m, q)}{\partial q} = 0. \end{cases}$$

https://doi.org/10.1017/S144618111600033X Published online by Cambridge University Press

Eliminating λ yields

$$\begin{cases} F_q(F_\rho\overline{\rho}_\rho + F_m\overline{m}_\rho + F_q\overline{q}_\rho) = F_\rho(F_q\overline{\rho}_q + F_q\overline{m}_q + F_q\overline{q}_q) \\ F_q(F_\rho\overline{\rho}_m + F_m\overline{m}_m + F_q\overline{q}_m) = F_m(F_\rho\overline{\rho}_q + F_m\overline{m}_q + F_q\overline{q}_q). \end{cases} \tag{3.2}$$

Substituting (3.1) into (3.2), and with an arbitrary value of k,

$$\begin{cases} \left[\gamma\dfrac{mq}{\rho^2} - (\gamma-1)\dfrac{m^3}{\rho^3}\right]{F_q}^2 + \dfrac{3-\gamma}{2}\dfrac{m^2}{\rho^2}F_mF_q + \gamma\dfrac{m}{\rho}F_\rho F_q + (\gamma-1)F_\rho F_m = 0 \\ \left[\dfrac{3}{2}(\gamma-1)\dfrac{m^2}{\rho^2} - \gamma\dfrac{q}{\rho}\right]{F_q}^2 + (2\gamma-3)\dfrac{m}{\rho}F_mF_q - F_\rho F_q + (\gamma-1){F_m}^2 = 0. \end{cases} \tag{3.3}$$

Then

$$\frac{\gamma-1}{2}\left[\frac{m^3}{\rho^3}{F_q}^2 + 3\frac{m^2}{\rho^2}F_mF_q + 2\frac{m}{\rho}F_\rho F_q + 2F_\rho F_m + 2\frac{m}{\rho}{F_m}^2\right] = 0.$$

For $\gamma > 1$, set $t = m/\rho$ to yield $t^3{F_q}^2 + 3t^2F_mF_q + 2tF_\rho F_q + 2F_\rho F_m + 2t{F_m}^2 = 0$, that is,

$$(tF_q + F_m)(t^2F_q + 2tF_m + 2F_\rho) = 0. \tag{3.4}$$

We solve equation (3.4) for the following two cases.

(I) If $tF_q + F_m = 0$, substituting into (3.3) gives

$$F_\rho = \left[\frac{\gamma+1}{2}\frac{m^2}{\rho^2} - \gamma\frac{q}{\rho}\right]F_q,$$

and we conclude that

$$\begin{cases} F_m = \dfrac{m}{\rho}F_q \\ F_\rho = \left[\dfrac{\gamma+1}{2}\dfrac{m^2}{\rho^2} - \gamma\dfrac{q}{\rho}\right]F_q. \end{cases} \tag{3.5}$$

Depending on the first equation of (3.5) and taking ρ as a constant,

$$F(\rho, m, q) = G\left(\rho, q - \frac{m^2}{2\rho}\right). \tag{3.6}$$

Substituting (3.6) into the second equation of (3.5) yields

$$F(\rho, m, q) = f\left(\rho^{-\gamma}\left(q - \frac{m^2}{2\rho}\right)\right).$$

(II) If $t^2F_q + 2tF_m + 2F_\rho = 0$, substituting into equation (3.3) yields

$$\begin{cases} F_m = \left[-\dfrac{m}{\rho} \pm \sqrt{\dfrac{\gamma}{\gamma-1}\left(\dfrac{q}{\rho} - \dfrac{m^2}{2\rho^2}\right)}\right]F_q \\ F_\rho = \left[\dfrac{1}{2}\dfrac{m^2}{\rho^2} \mp \sqrt{\dfrac{\gamma}{\gamma-1}\left(\dfrac{q}{\rho} - \dfrac{m^2}{2\rho^2}\right)}\dfrac{m}{\rho}\right]F_q. \end{cases} \tag{3.7}$$

https://doi.org/10.1017/S144618111600033X Published online by Cambridge University Press

Depending on the first equation of (3.7) and taking ρ as a constant,

$$F(\rho, m, q) = G\Big(\rho, 2\sqrt{y} \pm \sqrt{\frac{\gamma}{\gamma-1}}\frac{m}{\rho}\Big), \tag{3.8}$$

where $y = q - m^2/2\rho$. By setting

$$z = 2\sqrt{y} \mp \sqrt{\frac{\gamma}{\gamma-1}}\frac{m}{\rho}$$

and substituting (3.8) into the second equation of (3.7),

$$G_\rho = \Big[\mp\sqrt{\frac{\gamma}{\gamma-1}} m\rho^{-3/2} \mp \sqrt{\frac{\gamma}{\gamma-1}}\frac{m}{\rho^2}\Big]G_z.$$

Then

$$\Big[\mp\sqrt{\frac{\gamma}{\gamma-1}} m\rho^{-3/2} \mp \sqrt{\frac{\gamma}{\gamma-1}}\frac{m}{\rho^2}\Big]$$

cannot be presented as a function of ρ and z, because z is a function of q. Thus, if the nonisentropic gas dynamic system (1.1) has a bounded invariant region, then the boundary equation must be

$$F(\rho, m, q) = f(\rho^{-\gamma} y) \equiv C,$$

where C is a constant. In other words, $q = (2\rho)^{-1}m^2 + C\rho^\gamma$.

The function $q = q(\rho, m)$ is convex in the (ρ, m, q)-space, which means that, for the convex invariant region of (1.1), $q \geq (2\rho)^{-1}m^2 + C\rho^\gamma$. It is obvious that such an invariant region is not bounded for ρ, m and q, which leads to the following main theorem of our paper.

THEOREM 3.1. *A bounded invariant region does not exist for the equations of nonisentropic gas dynamics with an ideal gas.*

Without a bounded invariant region, we cannot obtain the uniform bounded estimation, which is the first step of the compensated compactness method. The result indicates that there is a big gap between the nonisentropic case and the isentropic case.

4. Conclusion

We have shown that a bounded invariant region does not exist for one-dimensional equations of nonisentropic gas dynamics with an ideal gas. This indicates that the compensated compactness theory encounters bottlenecks when solving equations of nonisentropic gas dynamics. In a future work, we will investigate the equations of nonisentropic gas dynamics with a nonideal gas. We will propose a necessary and sufficient condition about the gas state $p(\rho, s)$, to prove that, for a special gas state, a bounded invariant region for equations of nonisentropic gas dynamics does exist.

https://doi.org/10.1017/S144618111600033X Published online by Cambridge University Press

Acknowledgements

Weifeng Jiang would like to express gratitude to his junior fellow apprentice Jun Li, who helped him during the writing of this manuscript. This work was supported by the National Natural Science Foundation of China (No.11171340).

References

[1] G. Barles and P. E. Souganidis, "Convergence of approximation schemes for fully nonlinear second order equations", *Asymototic. Anal.* **4** (1991) 271–283; doi:10.3233/ASY-1991-4305.

[2] R. J. DiPerna, "Convergence of the viscosity method for isentropic gas dynamics", *Commun. Math. Phys.* **91** (1983) 1–30; doi:10.1007/BF01206047.

[3] R. J. DiPerna, "Convergence of approximate solutions to conservation laws", *Arch. Ration. Mech. Anal.* **82** (1983) 27–70; doi:10.1016/B978-0-12-493280-7.50018-8.

[4] X. X. Ding, G. Q. Chen and P. Z. Luo, "A supplement to the papers 'Convergence of the Lax–Friedrichs scheme for isentropic gas dynamics' (II)–(III)", *Acta Math. Sci.* **9** (1989) 43–44; https://www.researchgate.net/publication/269014104_A_supplementto_the_papers_Convergence_of_the_Lax-Friedrichs_scheme_forisentropic_gas_dynamics_II_III.

[5] X. X. Ding, G. Q. Chen and P. Z. Luo, "Convergence of the fractional step Lax–Friedrichs scheme and Godunov scheme for the isentropic system of gas dynamics", *Commun. Math. Phys. Anal.* **121** (1989) 63–84; doi:10.1007/BF01218624.

[6] F. Huang and Z. Wang, "Convergence of viscosity solutions for isothermal gas dynamics", *SIAM J. Math. Anal.* **34** (2002) 595–610; doi:10.1007/s00205-004-0344-3.

[7] P. L. Lions, B. Perthame B and P. E. Souganidis, "Existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates", *Commun. Pure Appl. Math.* **49** (1996) 599–638; doi:10.1002/(SICI)1097-0312(199606)49:6<599::AID-CPA2>3.0.CO;2-5.

[8] P. L. Lions, B. Perthame and E. Tadmor, "Kinetic formulation of the isentropic gas dynamics and p-systems'", *Commun. Math. Phys. Anal.* **163** (1994) 415–431; doi:10.1007/BF02102014.

[9] J. Smoller, *Shock waves and reaction-diffusion equations*, 2nd edn (Springer, New York, 1983).

[10] L. Tartar, *Systems of nonlinear partial differential equations,* NATO Science Series C, vol. 1 (Springer, Amsterdam, 1983) 263–285.