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Adjoint L-Values and Primes of Congruence for Hilbert Modular Forms

EKNATH GHATE
School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, 400 005 Mumbai, India.
e-mail: eghate@math.tifr.res.in

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Abstract. Let f be a primitive Hilbert modular cusp form of arbitrary level and parallel weight k, defined over a totally real number field F. We define a finite set of primes $\mathcal{S}$ that depends on the weight and level of f, the field F, and the torsion in the boundary cohomology groups of the Borel–Serre compactification of the underlying Hilbert-Blumenthal variety. We show that, outside $\mathcal{S}$, any prime that divides the algebraic part of the value at $s = 1$ of the adjoint L-function of f is a congruence prime for f. In special cases we identify the 'boundary primes' in terms of expressions of the form $N_{F/\mathbb{Q}}(\epsilon^{k-1} - 1)$, where ϵ is a totally positive unit of F.

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1. Introduction

The purpose of this paper is to establish a relationship between the primes of congruence of a Hilbert modular cusp form f, and the primes dividing the algebraic part of the value of its adjoint L-function $L(s, \mathrm{Ad}(f))$ at $s = 1$.

Such matters were first investigated by Doi and Hida [3], Hida [10, 11], and Ribet [19], in the elliptic modular case. Let us state their results. Let $f \in S_k(\Gamma_0(N), \psi)$ be a normalized newform that is a common eigenform of all the Hecke operators. Let $L(s, \mathrm{Ad}(f))$ denote the imprimitive adjoint L-function attached to f, let $\Gamma(s, \mathrm{Ad}(f))$ denote the associated Γ-factor, and let $W(f)$ be the complex constant that arises in the functional equation of the standard L-function attached to f (see the text for exact definitions). Then, in [10], Hida proved:

THEOREM 1. *There is a large number field K, such that the following holds: For each prime $\wp \mid p$ in K with $p > k - 2$ and $p \nmid 6N$, there exists periods $\Omega(f, +)$ and $\Omega(f, -) \in \mathbb{C}^\times / \mathcal{O}_{(\wp)}{}^\times$, such that if*

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$$\wp \,\Big|\, \frac{W(f)\,\Gamma(1,\mathrm{Ad}(f))\,L(1,\mathrm{Ad}(f))}{\Omega(f,+)\Omega(f,-)}, \tag{1}$$

then $\wp$ is a congruence prime for f.[★]

Here 'large' means that K contains the number fields generated by the Fourier coefficients of all the normalized cusp forms in $S_k(\Gamma_0(N),\psi)$ which are eigenforms for all the Hecke operators T_q, for $q \nmid N$. Also $\mathcal{O}_{(\wp)}$ is a valuation ring of K, and it is implicit in the statement of the Theorem that the quotient in (1) is an element of $\mathcal{O}_{(\wp)}$. Finally, we remind the reader that $\wp$ is said to be a congruence prime for $f = \sum_{m=1}^{\infty} a(m,f)\,q^m$ if there exists another normalized cusp form $g = \sum_{m=1}^{\infty} b(m,f)\,q^m \in S_k(\Gamma_0(N),\psi)$, which is an eigenform for all the Hecke operators T_q, such that $a(m,f) \equiv b(m,f) \bmod \wp$, for all m.

A converse to Theorem 1 was established by Hida [11] and Ribet [19]. They proved the following theorem:

THEOREM 2. *Let f be as in Theorem 1, and let $\wp \mid p$ be a prime of K, with $p > k-2$ and $p \nmid 6N$. If $\wp$ is a congruence prime for f, then*

$$\wp \,\Big|\, \frac{W(f)\,\Gamma(1,\mathrm{Ad}(f))\,L(1,\mathrm{Ad}(f))}{\Omega(f,+)\Omega(f,-)}.$$

Call a prime $\wp | p$ of K ordinary for f if $\wp \nmid a(p,f)$. In fact in [11] Hida established Theorem 2 under the additional hypothesis that $\wp$ is ordinary for f. This hypothesis was later removed by Ribet in [19].

In this paper we generalize Theorem 1 to the Hilbert modular situation. Let F be a totally real field, and let f be a Hilbert modular cusp form, defined over F, of parallel weight. Assume that f is a normalized newform and a common eigenform of all the Hecke operators. We will define a finite set of primes $\mathcal{S}$ in a large number field K and show that:

THEOREM 5.[★★] *For each prime $\wp$ of K with $\wp \notin \mathcal{S}$, there are periods $\Omega(f,\pm\epsilon) \in \mathbb{C}^\times / \mathcal{O}_{(\wp)}{}^\times$, such that if*

$$\wp \,\Big|\, \frac{W(f)\,\Gamma(1,\mathrm{Ad}(f))\,L(1,\mathrm{Ad}(f))}{\Omega(f,\epsilon)\Omega(f,-\epsilon)},$$

then $\wp$ is a congruence prime for f.

Though this theorem is rather general, in any specific situation it would be useful to have as much information about the set $\mathcal{S}$ as possible. To this end, we shall show that the set $\mathcal{S}$ may be decomposed as follows:

[★]The statement here is slightly different from what is proved in [10]; it is the version exposed, for instance, in Hida's recent book [17]. Also, in [12], the theorem was extended to include all primes $\wp \mid p$ with $p \geqslant 5$ and $\wp$ ordinary for f.

[★★]The numbering used in the Introduction coincides with the numbering used in the main body of the paper.

$$\mathcal{S} = \mathcal{S}_{\text{weight}} \cup \mathcal{S}_{\text{level}} \cup \mathcal{S}_{\text{elliptic}} \cup \mathcal{S}_F \cup \mathcal{S}_{\text{invariant}} \cup \mathcal{S}_\partial.$$

The sets $\mathcal{S}_{\text{weight}}$, $\mathcal{S}_{\text{level}}$ and $\mathcal{S}_{\text{elliptic}}$ are excluded from Theorem 5 purely for technical reasons (explained in the text). When $F = \mathbb{Q}$, these sets consist of the primes $p \leqslant k-2$, $p \mid N$, and $p = 2, 3$ in Theorem 1. Over more general totally real fields F, we must also exclude the set $\mathcal{S}_F$, consisting of the primes dividing the class number and the discriminant of F. Additionally, the set $\mathcal{S}_{\text{invariant}}$ must be excluded when $[F\colon \mathbb{Q}]$ is even and the weight of f is $(2, 2, \dots, 2)$.

But perhaps the most interesting set of primes in $\mathcal{S}$ that must be excluded are the primes in $\mathcal{S}_\partial$. We will refer to these primes informally as the 'boundary primes'. They are, more precisely, the primes of torsion in the cohomology groups with integral coefficients of the boundary of the Borel–Serre compactification of the underlying Hilbert–Blumenthal variety. In general these primes are somewhat difficult to describe explicitly, but in the case $F = \mathbb{Q}$ this is not so: they are the primes $p \leqslant k-2$ and p dividing N (and so have already been accounted for by the primes in $\mathcal{S}_{\text{weight}}$ and $\mathcal{S}_{\text{level}}$ in this case).

The boundary primes arise as obstructions to a key result which we need in order to establish Theorem 5. This result, Theorem 3 in Section 3.3, is a duality theorem for the parabolic cohomology groups of the underlying Hilbert–Blumenthal variety, with integral coefficients. With it in hand, the proof of Theorem 5 follows quickly. In fact our proof essentially imitates the method of proof of Theorem 1 outlined in [10]. We have also benefited from some ideas in [23], where Urban established an analog of Theorem 1 for cusp forms over imaginary quadratic fields.

The question remains as to how explicitly one can describe the set $\mathcal{S}_\partial$ of boundary primes. As mentioned above, the case $F = \mathbb{Q}$ is easy, and, in any case, was already treated by Hida in [10]. We are further able to treat some special cases (see § 3.4). For instance, if $[F\colon \mathbb{Q}] = 2$, F has strict class number 1, and if f has level 1, and parallel weight $(2, 2)$, then we are able to explicitly determine the boundary primes in terms of a generator of the totally positive unit group of F (see Proposition 4). Moreover, when the weight is bigger than 2, we are able to draw on a result of Hida (Theorem 3.12 in [15]) which allows us to describe explicitly the *ordinary* boundary primes (see Corollary 1). As a result we can obtain the more precise result:

COROLLARY 2. *Let F be a real quadratic field of strict class number* 1. *Let ϵ_1 denote a generator of the group of totally positive units of F, and let D_F denote the discriminant of F. Let f be a normalized newform, and a common eigenform of the Hecke operators, of level 1 and weight* (k, k). *Then there exits a large number field K such that the following holds:*

If $\wp \mid p$ is a prime of K such that $p > k-2$, $p \nmid 30 \cdot D_F \cdot N_{F/\mathbb{Q}}(\epsilon_1^{k-1} - 1)$ and

$$\begin{array}{ll} \wp \notin \mathcal{S}_{\text{invariant}}, & \text{if } k = 2, \\ \wp \text{ is ordinary for } f, & \text{if } k > 2, \end{array}$$

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then there exist periods $\Omega(f, \pm\epsilon) \in \mathbb{C}^\times/\mathcal{O}_{(\wp)}{}^\times$ *such that if*

$$\wp \,\Big|\, \frac{\Gamma(1, \mathrm{Ad}(f))\, L(1, \mathrm{Ad}(f))}{\Omega(f, \epsilon)\Omega(f, -\epsilon)},$$

then $\wp$ *is a congruence prime for* f.

A similar explicit result for level 1, weight $(2, 2, \ldots, 2)$ cusp forms over totally real fields of strict class number 1 and odd degree is also derived:

COROLLARY 4. *Let* F *be a totally real field of strict class number* 1, *odd degree* d, *and discriminant* D_F. *Let* $\epsilon_1, \epsilon_2, \ldots, \epsilon_{d-1}$ *be a basis of the group of totally positive units of* F, *and let* U_F *denote the product of the primes which divide each* $N_{F/\mathbb{Q}}(\epsilon_j - 1)$, *for* $j = 1, 2, \ldots d-1$. *Let* E_F *denote the product of all the primes* p *such that the maximal totally real subfield of the* pth *cyclotomic field is contained in* F.

Let f *be a normalized newform, and a common eigenform of the Hecke operators, of level* 1 *and weight* $(2, 2, \ldots, 2)$. *Then there exits a large number field* K *such that the following holds:*

If $\wp \mid p$ *is a prime of* K *such that* $p \nmid E_F \cdot D_F \cdot U_F$, *then there exist periods* $\Omega(f, \pm\epsilon) \in \mathbb{C}^\times/\mathcal{O}_{(\wp)}{}^\times$ *such that if*

$$\wp \,\Big|\, \frac{\Gamma(1, \mathrm{Ad}(f))\, L(1, \mathrm{Ad}(f))}{\Omega(f, \epsilon)\Omega(f, -\epsilon)},$$

then $\wp$ *is a congruence prime for* f.

In a sequel to this paper [8] we will investigate the analog of Theorem 2 in the Hilbert modular situation. That is, under certain conditions, we will establish that almost all (ordinary) congruence primes are captured by the adjoint L-value. To do this we use a freeness criterion invented by Taylor and Wiles (see Fujiwara [6] and Diamond [2]) to show that under certain conditions suitable localizations of the above-mentioned parabolic cohomology groups are free as Hecke modules. The converse divisibility then follows quickly.

Finally I should mention that one motivation for generalizing Theorems 1 and 2 to the Hilbert modular situation, is that such theorems would provide a possible approach to tackling some conjectures of Doi, Hida and Ishii [4] on *twisted* elliptic modular adjoint L-values and primes of congruence between base-change and non-base-change Hilbert modular cusp forms (see [7] for more details in the real quadratic case).

2. Preliminaries

Useful general references for some of the material in this section are the papers [13,16] and the book [24].

2.1. SOME NOTATION

Fix a totally real field F of degree d. Let $\mathcal{O}_F$ denote the ring of integers of F. Let I_F denote the set of embeddings of F into $\mathbb{R}$.

Let $\mathrm{G} = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$. Then let $\mathrm{G}_f = \mathrm{G}(\mathbb{A}_f)$ denote the finite part of $\mathrm{G}(\mathbb{A})$, where $\mathbb{A} = \mathbb{A}_f \times \mathbb{R}$ denotes the ring of adeles over $\mathbb{Q}$. Let $\mathrm{G}_\infty = \mathrm{G}(\mathbb{R})$, and let $\mathrm{G}_{\infty+}$ denote the set of those elements in G_∞ which have positive determinant at each place $\sigma \in I_F$. Let $\mathrm{G}(\mathbb{Q})_+ = \mathrm{G}(\mathbb{Q}) \cap \mathrm{G}_{\infty+}$.

Let K_f be an open, compact subgroup of $\mathrm{G}(\mathbb{A}_f)$, let $\mathrm{K}_\infty = \prod \mathrm{O}_2(\mathbb{R})$ denote the standard maximal compact subgroup of $\mathrm{G}(\mathbb{R})$, and let $\mathrm{K}_{\infty+} = \prod \mathrm{SO}_2(\mathbb{R}) \subset \mathrm{G}_{\infty+}$ denote the connected component of K_∞ containing the identity element. Let $\mathrm{K} = \mathrm{K}_f\mathrm{K}_\infty$.

Let Z denote the center of G, and let Z_∞ denote the center of G_∞.

2.2. HILBERT–BLUMENTHAL VARIETIES

We keep the notation of the previous section. Set

$$Y(\mathrm{K}) = \mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A})/\mathrm{K}_f\mathrm{K}_{\infty+}\mathrm{Z}_\infty.$$

Then $Y(\mathrm{K})$ is the set of complex points of a quasi-projective variety of dimension d defined over $\mathbb{Q}$, which, following standard usage, will be referred to as a Hilbert–Blumenthal variety.

By the strong approximation theorem one may find $t_i \in \mathrm{G}(\mathbb{A})$ of the form $t_i = \begin{pmatrix} a_i & 0 \\ 0 & 1 \end{pmatrix}$ with $(a_i)_\infty = 1$, where $(a_i)_\infty$ is the infinite part of the idele a_i, such that

$$\mathrm{G}(\mathbb{A}) = \coprod_{i=1}^{h} \mathrm{G}(\mathbb{Q})\, t_i\, \mathrm{K}_f\, \mathrm{G}_{\infty+}.$$

Note that when $\det(\mathrm{K}_f) = \hat{\mathcal{O}}_F^\times$, with $\hat{\mathcal{O}}_F = \prod_{\mathfrak{p}} \mathcal{O}_{F_\mathfrak{p}}$, then $h = \left|Cl_F^+\right|$ is just the strict class number of F, where

$$Cl_F^+ := F^\times\backslash\mathbb{A}_F^\times/\det(\mathrm{K}_f)F_{\infty+}^\times$$

is the strict class group of F.

Now set $\Gamma_i = \mathrm{G}(\mathbb{Q})_+ \cap t_i\, \mathrm{K}_f\, \mathrm{G}_{\infty_+}\, t_i^{-1}$. Let $\mathcal{Z} = \mathrm{G}_{\infty+}/\mathrm{K}_{\infty+}\mathrm{Z}_\infty$ denote the d-fold product of the Poincaré upper half plane H on which a discrete subgroup Γ of $\mathrm{G}(\mathbb{Q})_+$ acts in the usual way via fractional linear transformations. Let $z_0 = (\sqrt{-1}, \sqrt{-1}, \dots, \sqrt{-1})$ denote the standard 'base point' in $\mathcal{Z}$. Then the map $\gamma t_i u_f g_\infty \mapsto g_\infty(z_0)$, for $\gamma \in \mathrm{G}(\mathbb{Q})$, $u_f \in \mathrm{K}_f$, and $g_\infty \in \mathrm{G}_{\infty+}$, induces a decomposition

$$Y(\mathrm{K}) = \coprod_{i=1}^{h} \Gamma_i\backslash\mathcal{Z} \tag{2}$$

of $Y(\mathrm{K})$ into its connected components.

Let $\mathfrak{N} \subset \mathcal{O}_F$ be an ideal. From Section 2.4 onwards we will restrict to the case when $\mathrm{K}_f = \mathrm{K}_0(\mathfrak{N})$ is the level $\mathfrak{N}$ compact-open subgroup of $\mathrm{G}(\mathbb{A}_f)$:

$$\mathrm{K}_0(\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\hat{\mathcal{O}}_F) \;\middle|\; c \equiv 0 \pmod{\mathfrak{N}\hat{\mathcal{O}}_F} \right\}.$$

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In this case we will write $Y(\mathfrak{N})$ instead of $Y(\mathrm{K})$, and $\Gamma_i(\mathfrak{N})$ instead of Γ_i. We have: $\Gamma_i(\mathfrak{N})$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F) \,\middle|\, a \in \mathcal{O}_F, b \in \mathfrak{A}_i, c \in \mathfrak{N}\mathfrak{A}_i^{-1}, d \in \mathcal{O}_F,\ ad - bc \in \mathcal{O}_{F+}^{\times} \right\},$$

where $\mathfrak{A}_i = a_i\mathcal{O}_F$ is the fractional ideal generated by a_i, and $\mathcal{O}_{F+}^{\times}$ is the ring of totally positive units in F.

Note that without loss of generality we may make the following two assumptions on the a_i for $i = 1, \dots, h$. Firstly we may assume that $(a_i)_{\mathfrak{N}} = 1$, where $(a_i)_{\mathfrak{N}}$ is the $\mathfrak{N}$-part of the adèle a_i, and secondly, that the fractional ideals $\mathfrak{A}_i = a_i\mathcal{O}_F$ are all integral ideals. We keep these assumptions for the rest of the paper.

2.3. COMPACTIFICATIONS OF $Y(\mathrm{K})$

There are various compactifications of $Y(\mathrm{K})$. One such, which we will not have occasion to use in this paper, is the Bailey–Borel–Satake compactification. It is constructed by adding finitely many points, called cusps, to $Y(\mathrm{K})$. For instance, the number of cusps required to compactify $\mathrm{SL}_2(\mathcal{O}_F)\backslash\mathcal{Z}$ is equal to the class number of F (see Proposition 1.1 of [24]).

Another compactification of $Y(\mathrm{K})$ is the Borel–Serre compactification [1]. It will play an important role in this paper, and, following [15] and [23], we describe it briefly now.

Let $X = (\mathbb{R}_+^{\times})^{I_F} \times \mathbb{R}^{I_F}$. Thus if $(y, x) \in X$ then $x = (x_\sigma)$ and $y = (y_\sigma)$ are indexed by the embeddings σ of F into $\mathbb{R}$. Let us set $N(y) = \prod_{\sigma \in I_F} y_\sigma$. Consider for each $t \in \mathbb{R}$ with $t > 0$, the set

$$X_t = \{(y, x) \in X \mid N(y) = t\}.$$

Note that for each $t > 0$ one has $X_t \tilde{=} X_1$ where the identification is induced by the map $(y, x) \mapsto (y/t^{1/d}, x)$. Thus $X \tilde{=} X_1 \times \mathbb{R}_+^{\times}$. We now add a copy of X_1 to X at $t = \infty$ to get a space X^*.

Let B_∞ denote the standard Borel of upper triangular matrices in $\mathrm{G}(\mathbb{Q})$. We first construct a space $\mathcal{Z}^*$, by adding a boundary to $\mathcal{Z}$, whose connected components are in bijection with the set $\mathrm{G}(\mathbb{Q})/B_\infty = \mathbb{P}^1(F)$ of Borel subgroups of $\mathrm{G}(\mathbb{Q})$. Indeed, if $B = \alpha B_\infty \alpha^{-1}$ is a Borel subgroup of $\mathrm{G}(\mathbb{Q})$, where $\alpha \in \mathrm{G}(\mathbb{Q})_+$ may be taken to be totally positive, then the Iwasawa decomposition corresponding to B_∞ of $\alpha^{-1}g$, with $g \in \mathrm{G}_{\infty+}$, permits us to identify X with $\mathcal{Z}$ via the map ι_α, where

$$\iota_\alpha : (y, x) \mapsto \alpha \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \mathrm{K}_{\infty_+} \mathrm{Z}_\infty. \tag{3}$$

The compactification X^* of X described above allows us to adjoin, via ι_α, for each α (and, hence, B) as above, a boundary, say ∂_B, to $\mathcal{Z}$. We set $\partial\mathcal{Z}^* = \coprod_B \partial_B$.

If Γ is an arithmetic subgroup of $\mathrm{G}(\mathbb{Q})_+$, then one may check that Γ acts properly discontinuously on $\mathcal{Z}^*$. Moreover, $\Gamma\backslash\mathcal{Z}^*$ is a compactification of $\Gamma\backslash\mathcal{Z}$, whose bound-

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ary, $\partial(\Gamma\backslash\mathcal{Z}^*)$, is a finite union of spaces $\Gamma_B\backslash\partial_B$ (where $\Gamma_B = \Gamma \cap B$) indexed by the Γ-conjugacy classes of Borel subgroups, B, of $\mathrm{G}(\mathbb{Q})$. Now the Borel–Serre compactification of $Y(\mathrm{K})$ is $Y(\mathrm{K})^* := \coprod_{i=1}^h \Gamma_i\backslash\mathcal{Z}^*$. Note that for K_f sufficiently deep, this is a smooth manifold, with smooth boundary:

$$\partial(Y(\mathrm{K})^*) = \coprod_{i=1}^{h} \partial(\Gamma_i\backslash\mathcal{Z}^*).$$

2.4. HILBERT MODULAR FORMS

Let $n = \sum_{\sigma\in I_F} n_\sigma\,\sigma \in \mathbb{Z}[I_F]$ with each $n_\sigma \geqslant 0$. Let $t = \sum_{\sigma\in I_F}\sigma \in \mathbb{Z}[I_F]$ and let $k = n + 2t$. The data n and k will represent the weights of cusp forms described below. For simplicity, we will assume in this paper that the weights are 'parallel', namely that $n_\sigma = n_\tau$ (and so $k_\sigma = k_\tau$), for any $\sigma, \tau \in I_F$. Thus n and k will also sometimes be used to denote this common value.

Fix an ideal $\mathfrak{N}$ of $\mathcal{O}_F$. We assume from now on that $\mathrm{K}_f = \mathrm{K}_0(\mathfrak{N})$ (see § 2.2).

Let $\mathbb{A}_F = \mathbb{A}\otimes_{\mathbb{Q}} F$ denote the ring of adèles of F. Identify the center $\mathrm{Z}(\mathbb{A})$ of $\mathrm{G}(\mathbb{A})$ with the idele group $\mathbb{A}_F^\times$. Similarly identify $\mathrm{Z}(\mathbb{Q})$ with $F^\times$. Let $\chi\colon F^\times\backslash\mathbb{A}_F^\times \to \mathbb{C}$ be a fixed Hecke character whose conductor divides $\mathfrak{N}$ and whose infinity type is $-n$.

The character χ, restricted to $\hat{\mathcal{O}}_F^\times$, induces a finite order character $\chi_{\mathfrak{N}}$ of $\hat{\mathcal{O}}_F^\times/(1+\mathfrak{N}\hat{\mathcal{O}}_F)$. We also write $\chi_{\mathfrak{N}}$ for the character of $\mathrm{K}_0(\mathfrak{N})$ defined as follows: for $u_f = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{K}_0(\mathfrak{N})$, let

$$\chi_{\mathfrak{N}}(u_f) = \chi_{\mathfrak{N}}(d_{\mathfrak{N}}) = \prod_{\mathfrak{p}\mid\mathfrak{N}} \chi(d_{\mathfrak{p}}).$$

A Hilbert modular cusp form of weight n, level $\mathfrak{N}$, central character χ, and holomorphy type $J \subset I_F$ is a function $f\colon \mathrm{G}(\mathbb{A}) \to \mathbb{C}$ satisfying the following properties:

- $f(\gamma g) = f(g)$ for all $\gamma \in \mathrm{G}(\mathbb{Q})$,
- $f(zg) = \chi(z)f(g)$ for all $z \in \mathrm{Z}(\mathbb{A})$,
- $f(gu_f u_\infty) = \chi_{\mathfrak{N}}(u_f)f(g)\exp(2\pi i(\sum_{\sigma\in J} k_\sigma\theta_\sigma - \sum_{\sigma\notin J} k_\sigma\theta_\sigma))$ where

  $$u_\infty = \begin{pmatrix} \cos(2\pi\theta_\sigma) & \sin(2\pi\theta_\sigma) \\ -\sin(2\pi\theta_\sigma) & \cos(2\pi\theta_\sigma)\end{pmatrix} \in \mathrm{K}_{\infty+},$$

  and where $u_f \in \mathrm{K}_0(\mathfrak{N})$,
- $D_\sigma f = ((n_\sigma^2/2) + n_\sigma)f$, where D_σ is the Casimir operator at $\sigma \in I_F$,
- f has vanishing constant terms, that is, for each $g \in \mathrm{G}(\mathbb{A})$, $\int_{\mathrm{U}(\mathbb{Q})\backslash\mathrm{U}(\mathbb{A})} f(ug)\,\mathrm{d}u = 0$, where U is the unipotent radical of B_∞, the standard Borel subgroup of upper triangular matrices in G.

Let us denote the space of such forms by $S_{k,J}(\mathfrak{N}, \chi)$.

2.5. FOURIER EXPANSIONS

Recall that any $f \in S_{k,J}(\mathfrak{N}, \chi)$ has a Fourier expansion which we describe now. Define $W\colon (\mathbb{R}_+^\times)^{I_F} \to \mathbb{C}$ by $W(y) = \prod_{\sigma \in I_F} \exp(-2\pi y_\sigma)$, where $y = (y_\sigma)_{\sigma \in I_F}$. Let $|\ |_F$ denote the modulus character on $\mathbb{A}_F^\times$. Let ϑ be an idele which generates the different of $F/\mathbb{Q}$. Let $\mathrm{e}_F\colon F\backslash\mathbb{A}_F \to \mathbb{C}$ denote the usual additive character of $\mathbb{A}_F$. Then

$$f\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} = |y|_F \sum_{\xi \in F^\times,\, [\xi]=J} c(\xi y \vartheta, f) W(\xi y_\infty) \mathrm{e}_F(\xi x),$$

where we have written $[\xi]$ for $\{\sigma \in I_F | \xi^\sigma > 0\}$. Here, the Fourier coefficients $c(g, f)$ only depend on the fractional ideal generated by the finite part g_f of the idele g. Thus if $g_f \mathcal{O}_F = \mathfrak{m}$, then we may write $c(\mathfrak{m}, f)$ without ambiguity. Moreover, one may check that $c(\mathfrak{m}, f)$ vanishes outside the set of integral ideals.

There is a natural integral structure on the space of cusp forms coming from Fourier expansion. For a subring A of $\mathbb{C}$ containing the values of the finite order character $\chi_{\mathfrak{N}}$, let us denote by $S_{k,J}(\mathfrak{N}, \chi, A)$ the set of elements of $S_{k,J}(\mathfrak{N}, \chi)$ whose Fourier coefficients lie in A.

2.6. HECKE ALGEBRAS

Let, momentarily, K_f be an arbitrary open compact subgroup of $\mathrm{G}_f = \mathrm{G}(\mathbb{A}_f)$. Let $\Delta \subset \mathrm{G}_f$ be a semigroup, such that $\Delta \supset \mathrm{K}_f$. Let $\mathrm{K}_f\backslash\Delta/\mathrm{K}_f$ denote the space of double cosets of K_f in Δ. Define the Hecke algebra $h(\Delta, \mathrm{K}_f) = \mathbb{Z}[\mathrm{K}_f\backslash\Delta/\mathrm{K}_f]$ to be the free abelian group with basis the set of double cosets of K_f in Δ. For a double coset $\mathrm{K}_f g \mathrm{K}_f \in \mathrm{K}_f\backslash\Delta/\mathrm{K}_f$, we denote the corresponding basis element by $[\mathrm{K}_f g \mathrm{K}_f]$. The algebra structure on $h(\Delta, \mathrm{K}_f)$ is given by 'convolution':

$$[\mathrm{K}_f g_1 \mathrm{K}_f] \cdot [\mathrm{K}_f g_2 \mathrm{K}_f] = \sum_{m=3}^{M} v_m [\mathrm{K}_f g_m \mathrm{K}_f],$$

where

$$\mathrm{K}_f g_1 \mathrm{K}_f g_2 \mathrm{K}_f = \coprod_{m=3}^{M} \mathrm{K}_f g_m \mathrm{K}_f,$$

and

$$v_m = \left| \frac{\mathrm{K}_f g_1 \mathrm{K}_f \cap g_m \mathrm{K}_f g_2^{-1} \mathrm{K}_f}{\mathrm{K}_f} \right| \in \mathbb{N}$$

for $m = 3, \ldots, M$.

Now let

$$\hat{R}(\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\hat{\mathcal{O}}_F) \,\middle|\, d_\mathfrak{p} \in \mathcal{O}_{F\mathfrak{p}}^\times,\ c_\mathfrak{p} \in \mathfrak{N}\mathcal{O}_{F\mathfrak{p}},\ \text{for all } \mathfrak{p} \mid \mathfrak{N} \right\},$$

and let $\Delta_0(\mathfrak{N}) = \hat{R}(\mathfrak{N}) \cap \mathrm{G}(\mathbb{A}_f)$. Let us describe the Hecke algebra $h(\Delta_0(\mathfrak{N}), \mathrm{K}_0(\mathfrak{N}))$. For each ideal $\mathfrak{m} \subset \mathcal{O}_F$ let $T_\mathfrak{m} = \sum_g [\mathrm{K}_0(\mathfrak{N}) g \mathrm{K}_0(\mathfrak{N})]$, where the sum is taken over all distinct double cosets with $g \in \Delta_0(\mathfrak{N})$ with $\det(g)\mathcal{O}_F = \mathfrak{m}$. Note that

$$T_\mathfrak{p} = \left[\mathrm{K}_0(\mathfrak{N}) \begin{pmatrix} \varpi_\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix} \mathrm{K}_0(\mathfrak{N})\right].$$

Also, for each $\mathfrak{p}$ with $\mathfrak{p} \nmid \mathfrak{N}$ let

$$S_\mathfrak{p} = \left[\mathrm{K}_0(\mathfrak{N}) \begin{pmatrix} \varpi_\mathfrak{p} & 0 \\ 0 & \varpi_\mathfrak{p} \end{pmatrix} \mathrm{K}_0(\mathfrak{N})\right].$$

The following Proposition is well known:

PROPOSITION 1. *$h(\Delta_0(\mathfrak{N}), \mathrm{K}_0(\mathfrak{N}))$ is a commutative algebra with* 1. *It is generated over $\mathbb{Z}$ by the Hecke operators $T_\mathfrak{p}$ as $\mathfrak{p}$ varies through all prime ideals of $\mathcal{O}_F$, and by $S_\mathfrak{p}$ for $\mathfrak{p} \nmid \mathfrak{N}$.*

The Hecke algebra $h(\Delta_0(\mathfrak{N}), \mathrm{K}_0(\mathfrak{N}))$ acts on the space $S_{k,J}(\mathfrak{N}, \chi, A)$ of cusp forms over A. Let us recall this action. Extend the definition of the character $\chi_\mathfrak{N}$ from $\mathrm{K}_0(\mathfrak{N})$ to $\Delta_0(\mathfrak{N})$: for $g \in \Delta_0(\mathfrak{N})$, set $\chi_\mathfrak{N}(g) = \prod_{\mathfrak{p} \mid \mathfrak{N}} \chi(d_\mathfrak{p})$. Then, for $f\colon \mathrm{G}(\mathbb{A}) \to \mathbb{C} \in S_{k,J}(\mathfrak{N}, \chi)$ and $[\mathrm{K}_0(\mathfrak{N}) g \mathrm{K}_0(\mathfrak{N})] \in h(\Delta_0(\mathfrak{N}), \mathrm{K}_0(\mathfrak{N}))$, define the linear operator $[\mathrm{K}_0(\mathfrak{N}) g \mathrm{K}_0(\mathfrak{N})]\colon S_{k,J}(\mathfrak{N}, \chi, A) \to S_{k,J}(\mathfrak{N}, \chi, A)$ by

$$f\big|[\mathrm{K}_0(\mathfrak{N}) g \mathrm{K}_0(\mathfrak{N})](x) = \sum_i \chi_\mathfrak{N}^{-1}(g_i) f(x g_i),$$

where we have decomposed $\mathrm{K}_0(\mathfrak{N}) g \mathrm{K}_0(\mathfrak{N}) = \coprod_i g_i \mathrm{K}_0(\mathfrak{N})$.

A cusp form $f \in S_{k,I_F}(\mathfrak{N}, \chi)$ is said to be primitive if it is a newform, it is normalized ($c(\mathcal{O}_F, f) = 1$), and it is an eigenfunction of all the Hecke operators (i.e. $T_\mathfrak{m} f = c(\mathfrak{m}, f) f$, for all $T_\mathfrak{m}$). Let $\mathcal{B}$ denote the set of normalized cusp forms that are eigenforms for all the Hecke operators $T_\mathfrak{m}$ (thus $\mathcal{B}$ contains oldforms as well as newforms).

There is also an action of the group of 'complex conjugations' $\mathcal{C} := \mathrm{K}_\infty/\mathrm{K}_{\infty+} = \{\pm 1\}^{I_F}$ on $\sum_J S_{k,J}(\mathfrak{N}, \chi)$. In fact, for any two subsets J and J' of I_F, there is a unique element $c(J, J') \in \mathcal{C}$ which maps $S_{k,J}(\mathfrak{N}, \chi)$ to $S_{k,J'}(\mathfrak{N}, \chi)$ as an isomorphism of Hecke modules.

Let $h_k(\mathfrak{N}, \chi, A)$ be the subalgebra of $\mathrm{End}_A(S_{k,J}(\mathfrak{N}, \chi, A))$ generated by $T_\mathfrak{p}$ (for all $\mathfrak{p}$) and $S_\mathfrak{p}$ (for $\mathfrak{p} \nmid \mathfrak{N}$). Note that this algebra is independent of J.

2.7. RELATION WITH CLASSICAL CUSP FORMS

Each $f \in S_{k,J}(\mathfrak{N}, \chi)$ may be realized as a tuple of functions (f_i) on $\mathcal{Z}$ satisfying the usual transformation property with respect to the congruence subgroups $\Gamma_i(\mathfrak{N})$ above. To see this let $f_i'\colon \mathrm{G}(\mathbb{R}) \to \mathbb{C}$ be defined via $f_i'(g_\infty) = f(t_i g_\infty)$, for $g_\infty \in \mathrm{G}(\mathbb{R})$. Since the map $d \mapsto d_\mathfrak{N} = \prod_{\mathfrak{p} \mid \mathfrak{N}} d$ induces an isomorphism $(\mathcal{O}_F/\mathfrak{N}\mathcal{O}_F)^\times \xrightarrow{\sim} \hat{\mathcal{O}}_F^\times/(1 + \mathfrak{N}\hat{\mathcal{O}}_F)$, we may think of $\chi_\mathfrak{N}$ as a character of $(\mathcal{O}_F/\mathfrak{N}\mathcal{O}_F)^\times$. Then we see that

- $f_i'(\gamma g) = \chi_{\mathfrak{N}}(d)^{-1} f(g)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_i(\mathfrak{N})$,
- $f_i'(zg) = \chi_\infty(z) f(g)$ for all $z \in \mathrm{Z}_\infty$,
- $f_i'(gu_\infty) = f_i'(g)\exp(2\pi i(\sum_{\sigma\in J} k_\sigma\theta_\sigma - \sum_{\sigma\notin J} k_\sigma\theta_\sigma))$ where

  $$u_\infty = \begin{pmatrix} \cos(2\pi\theta_\sigma) & \sin(2\pi\theta_\sigma) \\ -\sin(2\pi\theta_\sigma) & \cos(2\pi\theta_\sigma) \end{pmatrix} \in \mathrm{K}_{\infty+},$$

- $D_\sigma f_i' = ((n_\sigma^2/2) + n_\sigma) f_i'$, where D_σ is the Casimir operator at $\sigma \in I_F$,
- f_i' satisfies a cuspidal condition.

Let us denote the space of cusp forms f_i' satisfying the above conditions by $S_{k,J}(\Gamma_i(\mathfrak{N}), \chi_{\mathfrak{N}}^{-1})$. Then (cf. [16], Equation 3.5) the map $f \mapsto (f_i')_{i=1}^h$ induces an isomorphism

$$\bigoplus_\chi S_{k,J}(\mathfrak{N}, \chi) \xrightarrow{\sim} \bigoplus_{i=1}^h S_{k,J}(\Gamma_i(\mathfrak{N}), \chi_{\mathfrak{N}}^{-1}), \tag{4}$$

where the left-hand sum runs over all central characters χ whose restriction to $\hat{\mathcal{O}}_F^\times$ is $\chi_{\mathfrak{N}}$, and whose infinity type is $-n$.

We now realize the functions in $S_{k,J}(\Gamma_i(\mathfrak{N}), \chi_{\mathfrak{N}}^{-1})$ as functions on $\mathcal{Z}$. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$ and $\tau \in \mathbb{C}$ let $j(\gamma, \tau) = c\tau + d$ denote the standard automorphy factor. Now let $\alpha = (\alpha_\sigma)_{\sigma\in I_F} \in \mathrm{G}_{\infty+}$ and $z = (z_\sigma)_{\sigma\in I_F} \in \mathcal{Z}$, and set

$$j_{k,J}(\alpha, z) = \prod_{\sigma\in J} j(\alpha_\sigma, z_\sigma)^{k_\sigma} \prod_{\sigma\notin J} j(\alpha_\sigma, \overline{z_\sigma})^{k_\sigma}.$$

Finally define $f_i\colon \mathcal{Z} \to \mathbb{C}$ by

$$f_i(z) = f_i'(g_\infty)\, j_{k,J}(g_\infty, z_0),$$

where $g_\infty \in \mathrm{G}_{\infty+}$ with $\det(g_\infty) = 1$ is chosen such that

$$g_\infty z_0 = z. \tag{5}$$

One may check that f_i is well-defined, and that for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_i(\mathfrak{N})$,

$$f_i(\gamma z) = \chi_{\mathfrak{N}}(d)^{-1} j_{k,J}(\gamma, z) f_i(z).$$

Moreover, the fact that f is an eigenfunction of the Casimir operators, along with the fact that f transforms under $\mathrm{K}_{\infty+}$ in the manner prescribed above, ensures that each f_i is holomorphic at $\sigma \in J$ and anti-holomorphic at $\sigma \notin J$ (cf. [16], page 460).

Finally, the Fourier expansion of f induces the usual Fourier expansion of classical Hilbert modular cusp forms. Choosing the idele $g = 1/\sqrt{y}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$ in (5) above, one may easily compute that each f_i has Fourier expansion

$$f_i(z) = |a_i|_F \sum_{\xi\in F^\times,\, [\xi]=J} c(\xi a_i \vartheta, f)\, W(\xi y_\infty)\, \mathrm{e}_F(\xi x_\infty),$$

When $J = I_F$ and $a_i = 1$ this reduces to the usual Fourier expansion of holomorphic Hilbert modular cusp forms:

$$f_1(z) = \sum_{\xi \in \vartheta^{-1},\ \xi \gg 0} c(\xi\vartheta, f)\ \exp(2\pi i \operatorname{Tr}_{F/\mathbb{Q}}(\xi z)).$$

3. A Duality Theorem

We now describe the relevant cohomology groups that will play a role in this paper, and discuss duality theorems for these cohomology groups in the middle dimension.

3.1. COHOMOLOGY

Let K denote a fixed large number field which contains all the conjugates of

- the totally real field F,
- the number fields generated by the Fourier coefficients of cusp forms $f \in \mathcal{B}$ (see § 2.6), and,
- the number field generated by the values of the finite order character $\chi_{\mathfrak{N}}$.

Fix once and for all an embedding of K into $\mathbb{C}$, via which $\mathbb{C}$ can be regarded as a K-algebra. Let $\mathcal{O}_K$ denote the ring of integers of K, and let $\hat{\mathcal{O}}_K = \prod_{\wp} \mathcal{O}_{\wp}$ denote the product of all the completions of $\mathcal{O}_K$.

Let $L(n, A)$ denote the module of all polynomials with coefficients in A, in the variables $\{X_\sigma, Y_\sigma \mid \text{for all } \sigma \in I_F\}$, which are homogeneous of degree n_σ in each pair X_σ, Y_σ.

Note that each $\sigma \in I_F$ induces a map $\hat{\mathcal{O}}_F \to \hat{\mathcal{O}}_K$. Now set $L_i(n, \mathcal{O}_K) = t_i L(n, \hat{\mathcal{O}}_K) \cap L(n, K)$, where $t_i L(n, \hat{\mathcal{O}}_K)$ denotes all the polynomials of the form $P((X_\sigma, Y_\sigma)^{\iota}(t_i^{\sigma})^{\iota})$, with $P(X_\sigma, Y_\sigma) \in L(n, \hat{\mathcal{O}}_K)$, and ι is the involution $g \mapsto \det(g) g^{-1}$. For an arbitrary $\mathcal{O}_K$-algebra A, let $L_i(n, A) = L_i(n, \mathcal{O}_K) \otimes_{\mathcal{O}_K} A$.

For us, A will denote either $\mathcal{O}_K$, $\mathcal{O}_{(\wp)}$ (a valuation ring of $\mathcal{O}_K$ at the prime $\wp$), $\mathcal{O}_{\wp}$ (a completion of $\mathcal{O}_K$ at the prime $\wp$), K, $K_{\wp}$ (a completion of K at the prime $\wp$), $K_{\wp}/\mathcal{O}_{\wp}$ or $\mathbb{C}$.

Let $\Gamma_i(\mathfrak{N})$ be one of the discrete subgroup of $\mathrm{G}(\mathbb{Q})_+$ as in Section 2.2. We make $L_i(n, A)$ into a $\Gamma_i(\mathfrak{N})$-module via an action on the factor $L_i(n, \mathcal{O}_K)$ as follows. For

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_i(\mathfrak{N}) \quad \text{and} \quad P(X_\sigma, Y_\sigma) \in L_i(n, \mathcal{O}_K)$$

set

$$(\gamma \cdot P)(X_\sigma, Y_\sigma) = \chi_{\mathfrak{N}}(d)^{-1} P((X_\sigma, Y_\sigma)^{\iota}(\gamma^{\sigma})^{-1}). \tag{6}$$

We will write $L_i(n, \chi, A)$ for the module $L_i(n, A)$ to emphasize the $\Gamma_i(\mathfrak{N})$-action, although note that the action depends only on $\chi_{\mathfrak{N}}$. If $\epsilon \in \mathcal{O}_F^{\times}$, then $\chi(\epsilon) = 1$ implies that $\chi_{\mathfrak{N}}(\epsilon) N_{F/\mathbb{Q}}(\epsilon)^{-n} = 1$. This shows that $\Gamma_i(\mathfrak{N}) \cap F^{\times}$ acts trivially on $L_i(n, \chi^{\pm 1}, A)$

and so it is in fact a module for $\overline{\Gamma_i(\mathfrak{N})} = (\Gamma_i(\mathfrak{N}) \cap F^\times)\backslash\Gamma_i(\mathfrak{N})$. If one replaces $\chi_{\mathfrak{N}}(d)^{-1}$ in (6) by $\chi_{\mathfrak{N}}(a)^{-1}$, then one obtains another $\overline{\Gamma_i(\mathfrak{N})}$ module which we shall denote by $L_i(n, \chi^{-1}, A)$.

Assume momentarily that $\overline{\Gamma_i(\mathfrak{N})}$ is torsion-free. Give $L_i(n, \chi^{\pm 1}, A)$ the discrete topology and let $\mathcal{L}_i(n, \chi^{\pm 1}, A)$ denote the sheaf of continuous (therefore locally constant) sections of the covering

$$\Gamma_i(\mathfrak{N})\backslash(\mathcal{Z} \times L_i(n, \chi^{\pm 1}, A)) \to \Gamma_i(\mathfrak{N})\backslash\mathcal{Z}.$$

In this paper we will consider the ordinary sheaf cohomology groups $\mathrm{H}^q(\Gamma_i(\mathfrak{N})\backslash\mathcal{Z}, \mathcal{L}_i(n, \chi^{\pm 1}, A))$, as well as the compactly supported sheaf cohomology groups $\mathrm{H}^q_{\mathrm{c}}(\Gamma_i(\mathfrak{N})\backslash\mathcal{Z}, \mathcal{L}_i(n, \chi^{\pm 1}, A))$. Denote the image of the compactly supported cohomology in the ordinary cohomology under the natural map

$$\mathrm{H}^q_{\mathrm{c}}(\Gamma_i(\mathfrak{N})\backslash\mathcal{Z}, \mathcal{L}_i(n, \chi^{\pm 1}, A)) \xrightarrow{\iota} \mathrm{H}^q(\Gamma_i(\mathfrak{N})\backslash\mathcal{Z}, \mathcal{L}_i(n, \chi^{\pm 1}, A)),$$

by $\mathrm{H}^q_{\mathrm{p}}(\Gamma_i(\mathfrak{N})\backslash\mathcal{Z}, \mathcal{L}_i(n, \chi^{\pm 1}, A))$. This image will be referred to as the parabolic cohomology group, and it will play an important role.

All these cohomology groups may be defined, with some restrictions on the coefficients, when $\overline{\Gamma_i(\mathfrak{N})}$ is not necessarily torsion free. We discuss this now. For each $i = 1, \ldots, h$, choose a normal finite index subgroup $\overline{\Gamma_i} \subset \overline{\Gamma_i(\mathfrak{N})}$, with $\overline{\Gamma_i} = (\Gamma_i \cap F^\times)\backslash\Gamma_i$ torsion free. For each such Γ_i, one has the cohomology groups $\mathrm{H}^i_?(\Gamma_i\backslash\mathcal{Z}, \mathcal{L}_i(n, \chi^{\pm 1}, A))$, for $? = \emptyset$, c, or p. as above. Now assume that

$$[\overline{\Gamma_i(\mathfrak{N})} : \overline{\Gamma_i}] \text{ is invertible in } A, \tag{7}$$

and set

$$\mathrm{H}^i_?(\Gamma_i(\mathfrak{N})\backslash\mathcal{Z}, \mathcal{L}_i(n, \chi^{\pm 1}, A)) = \mathrm{H}^i_?\big(\Gamma_i\backslash\mathcal{Z}, \mathcal{L}_i(n, \chi^{\pm 1}, A)\big)^{\overline{\Gamma_i(\mathfrak{N})}}, \tag{8}$$

for $? = \emptyset$, c, or p. It may be checked that, under (7), the definition (8) is independent of the choice of Γ_i.

Note that (7) always holds if A is a field. If A is not a field, then one may still proceed as follows. Let $\mathcal{S}_{\mathrm{cyclotomic}}$ be the finite set of primes of $\mathbb{Q}$ defined by

$$\mathcal{S}_{\mathrm{cyclotomic}} = \{p : \mathbb{Q}(\mu_p)^+ \subset F\}, \tag{9}$$

where $\mathbb{Q}(\mu_p)^+$ is the maximal totally real subfield of the p-th cyclotomic field $\mathbb{Q}(\mu_p)$. We have:

LEMMA 1. *Let* $\Gamma \subset \mathrm{GL}_2(F)$ *be an arithmetic subgroup.*

(1) *Say* $\gamma \in \Gamma$ *is a torsion element of prime order* ℓ. *Then* $\ell \in \mathcal{S}_{\mathrm{cyclotomic}}$.
(2) *For each* $\ell \notin \mathcal{S}_{\mathrm{cyclotomic}}$, *there is torsion free finite index normal subgroup* $\Gamma' \subset \Gamma$ *such that* $[\Gamma : \Gamma']$ *is prime to* ℓ.

Proof. Since an arbitrary arithmetic subgroup is conjugate, via an element of $\mathrm{GL}_2(F)$, to a subgroup of $\mathrm{GL}_2(\mathcal{O}_F)$, we may assume that $\Gamma \subset \mathrm{GL}_2(\mathcal{O}_F)$. For a prime

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$\mathfrak{p}$ of F, choose a uniformizer π generating $\mathcal{O}_{\mathfrak{p}} = (\pi)$, the completion of $\mathcal{O}_F$ at $\mathfrak{p}$. Then for each $n \geqslant 1$ one has:

$$1 \to \Gamma(\pi^n) \to \mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}}) \to \mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}}/\pi^n) \to 1,$$

where $\Gamma(\pi^n)$ is the kernel of the reduction mod π^n map in the exact sequence above. When n is sufficiently large, it is a fact that $\Gamma(\pi^n)$ is torsion free, and an easy computation shows that

$$[\mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}}) : \Gamma(\pi^n)] = q^{4(n-1)}(q^2 - q)(q^2 - 1) =: f(q), \text{ say,}$$

where $q = p^f$ is the order of the residue field $\mathcal{O}_{\mathfrak{p}}/\pi$. In particular, for n sufficiently large, $\Gamma' = \Gamma \cap \Gamma(\pi^n)$ is a torsion free normal subgroup of Γ of index dividing $f(q)$.

Now suppose that $\gamma \in \Gamma$ has order ℓ. Then for n sufficiently large the image of γ in Γ/Γ' is still an element of order ℓ, so that $\ell \mid f(q)$. Thus

$$N_{F/Q}(\mathfrak{p}) = q \equiv \pm 1 \mod \ell, \tag{10}$$

for all $\mathfrak{p}$ with $\mathfrak{p} \nmid \ell$. Now consider the diagram:

$$\begin{array}{ccccc} F & \hookrightarrow & F(\mu_\ell)^+ & \hookrightarrow & F(\mu_\ell) \\ | & & | & & | \\ \mathbb{Q} & \hookrightarrow & \mathbb{Q}(\mu_\ell)^+ & \hookrightarrow & \mathbb{Q}(\mu_\ell), \end{array}$$

where $F(\mu_\ell)^+$ is the compositum of F and $\mathbb{Q}(\mu_\ell)^+$. Condition (10) says that $\mathfrak{p}$ splits in the extension $F(\mu_\ell)^+/F$ for all but finitely many $\mathfrak{p}$. This implies that $F(\mu_\ell)^+ = F$, and thus $\mathbb{Q}(\mu_\ell)^+ \subset F$, proving the first statement.

For the second, we reverse the above argument. If $\ell \notin \mathcal{S}_{\mathrm{cyclotomic}}$, then $F(\mu_\ell)^+/F$ is a nontrivial extension of F, and so there are infinitely many primes $\mathfrak{p}$ of F which do not split completely in it. For each such $\mathfrak{p}$ one has $q \not\equiv \pm 1 \bmod \ell$. If in addition one takes $\mathfrak{p} \nmid \ell$, then, for n sufficiently large, the torsion free normal subgroup Γ' of Γ constructed above has index $[\Gamma : \Gamma']$ prime to ℓ. $\square$

Let us define

$$\mathcal{S}_{\mathrm{elliptic}} = \{\wp \mid p : \mathbb{Q}(\mu_p)^+ \subset F\} \tag{11}$$

to be the finite set of primes $\wp$ of K lying above the primes in $\mathcal{S}_{\mathrm{cyclotomic}}$. By Lemma 1 we may assume that (7) holds for $A = \mathcal{O}_{(\wp)}$, $\mathcal{O}_\wp$, $K_\wp/\mathcal{O}_\wp$ etc. whenever $\wp \notin \mathcal{S}_{\mathrm{elliptic}}$. Thus the cohomology groups with these coefficients considered above are well defined if we assume that $\mathfrak{p} \notin \mathcal{S}_{\mathrm{elliptic}}$.

Let $\mathcal{L}(n, \chi^{\pm 1}, A)$ be the sheaf on $Y(\mathfrak{N})$ which restricts to $\mathcal{L}_i(n, \chi^{\pm 1}, A)$ on the connected component $\Gamma_i(\mathfrak{N})\backslash\mathcal{Z}$, for $i = 1, 2, \ldots, h$. We now define

$$\mathrm{H}^q_?(Y(\mathfrak{N}), \mathcal{L}(n, \chi^{\pm 1}, A)) = \bigoplus_{i=1}^{h} \mathrm{H}^i_?(\Gamma_i(\mathfrak{N})\backslash\mathcal{Z}, \mathcal{L}_i(n, \chi^{\pm 1}, A)), \tag{12}$$

for $? = \emptyset$, c, or p. Note that, after the discussion above, this cohomology group is defined with suitable restrictions on the coefficients A. In fact the sheaf $\mathcal{L}(n, \chi^{\pm 1}, A)$ and the corresponding cohomology groups of $Y(\mathfrak{N})$ can be defined directly, in which case the decomposition (12) is in fact a statement that can be demonstrated (see § 6 of [13] for the details).

There is a natural action of the Hecke operator $T_{\mathfrak{m}}$ on the cohomology groups on the left-hand side of (12). For its definition, and its effect on the decomposition (12), the reader may consult Section 4 of [16] or Section 7 of [13]. We also note that the group of complex conjugations $\mathcal{C}$ and the strict class group Cl_F^+ act on the cohomology as well (for the former action see [13] p. 306, § 2; the latter action is induced by the action of the center $\mathrm{Z}(\mathbb{A}) = \mathbb{A}_F^\times$).

3.2. COHOMOLOGY OF THE BOUNDARY

In Section 2.3 we described the Borel–Serre compactification $Y(\mathfrak{N})^*$ of $Y(\mathfrak{N})$. There are natural inclusions

$$Y(\mathfrak{N}) \overset{\imath}{\hookrightarrow} Y(\mathfrak{N})^* \overset{j}{\hookleftarrow} \partial(Y(\mathfrak{N})^*),$$

where $\partial(Y(\mathfrak{N})^*)$ is the boundary of $Y(\mathfrak{N})^*$. Let $\imath_*$ denote direct image, and j^* denote inverse image. Then, for the sheaf $\mathcal{L}(n, \chi^{\pm 1}, A)$ on $Y(\mathfrak{N})$ described in the previous section, one obtains the boundary cohomology group:

$$\mathrm{H}^q(\partial(Y(\mathfrak{N})^*),\ j^*\imath_*\mathcal{L}(n, \chi^{\pm 1}, A)). \tag{13}$$

Let us describe the sheaf $j^*\imath_*\mathcal{L}$, where $\mathcal{L} = \mathcal{L}(n, \chi^{\pm 1}, A)$, more concretely. It suffices to describe the sheaf $j^*\imath_*\mathcal{L}_i$ on $\partial(\Gamma_i(\mathfrak{N})\backslash\mathcal{Z}^*)$, where $\mathcal{L}_i = \mathcal{L}_i(n, \chi^{\pm 1}, A)$. Assume temporarily that $\overline{\Gamma_i(\mathfrak{N})}$ is torsion free. Since $\Gamma_i(\mathfrak{N})\backslash\mathcal{Z}^*$ has fundamental group $\overline{\Gamma_i(\mathfrak{N})}$, a local system construction similar to the one described above yields a sheaf $\mathcal{L}_i^*$ on $\Gamma_i(\mathfrak{N})\backslash\mathcal{Z}^*$. A comparison of stalks shows that this sheaf is nothing but $\imath_*\mathcal{L}_i$. Now j induces an inclusion on each boundary component $\Gamma_i(\mathfrak{N})_B\backslash\partial_B \hookrightarrow \Gamma_i(\mathfrak{N})\backslash\mathcal{Z}^*$, which, by functoriality of π_1, induces a map (the natural inclusion)

$$\overline{\Gamma_i(\mathfrak{N})_B} \to \overline{\Gamma_i(\mathfrak{N})}. \tag{14}$$

Thus any local system on $\Gamma_i(\mathfrak{N})\backslash\mathcal{Z}^*$ gives rise, via (14), to a local system on $\Gamma_i(\mathfrak{N})_B\backslash\partial_B$. A comparison of stalks shows that the local system so obtained from the sheaf $\imath_*\mathcal{L}_i$, is the sheaf $j^*i_*\mathcal{L}_i$. When $\Gamma_i(\mathfrak{N})$ is not torsion free, then the usual procedure of 'taking invariants' described in the previous section, still allows one to define the boundary cohomology group (13) with some restrictions on the coefficients. In particular these groups are well defined when $A = \mathcal{O}_\wp$, as long as $\wp \notin \mathcal{S}_{\text{elliptic}}$.

All the cohomology groups considered so far are related via the boundary long exact sequence:

$$\begin{array}{ccccc} \mathrm{H}^q_c(Y(\mathfrak{N}), \mathcal{L}) & \to & \mathrm{H}^q(Y(\mathfrak{N}), \mathcal{L}) & \to & \mathrm{H}^q(\partial(Y(\mathfrak{N})^*), \mathcal{L}) \\ \| & & \| & & \| \\ \bigoplus_{i=1}^h \mathrm{H}^q_c(\Gamma_i(\mathfrak{N})\backslash\mathcal{Z}, \mathcal{L}_i) & \to & \bigoplus_{i=1}^h \mathrm{H}^q(\Gamma_i(\mathfrak{N})\backslash\mathcal{Z}), \mathcal{L}_i) & \to & \bigoplus_{i=1}^h \mathrm{H}^q(\partial(\Gamma_i(\mathfrak{N})\backslash\mathcal{Z}), \mathcal{L}_i), \end{array} \tag{15}$$

where we have written $\mathcal{L}$ for $\mathcal{L}(n, \chi^{\pm 1}, A)$, and $\mathcal{L}_i$ for $\mathcal{L}_i(n, \chi^{\pm 1}, A)$, and where, for simplicity, we have dropped $\imath_*$ and $j^*\imath_*$ from the notation.

3.3. POINCARÉ DUALITY

Fix $i \in \{1, \dots, h\}$. Consider the pairing

$$\langle\, ,\rangle :\ L_i(n, \chi, A) \otimes L_i(n, \chi^{-1}, A) \to A,$$

defined via

$$\left\langle \bigotimes_{\sigma} \sum_{l} (u_\sigma)_l X_\sigma^{n-l} Y_\sigma^l, \bigotimes_{\sigma} \sum_{l} (v_\sigma)_l X_\sigma^{n-l} Y_\sigma^l \right\rangle \mapsto |a_i|_F^n \prod_{\sigma \in I_F} \sum_l \frac{(-1)^l (u_\sigma)_l (v_\sigma)_{n-l}}{\binom{n}{l}}.$$

Here we assume that A is one of the $\mathcal{O}_K$-algebras mentioned above, with the restriction that the primes $p \leqslant n$ are invertible in it, so that the expression $\binom{n}{l}^{-1}$ makes sense. Thus, when $A = \mathcal{O}_{(\wp)}$, $\mathcal{O}_\wp$, or $K_\wp/\mathcal{O}_\wp$, we assume that $\wp \notin \mathcal{S}_{\text{weight}}$, where

$$\mathcal{S}_{\text{weight}} = \left\{ \wp \,\middle|\, p \,:\, p \leqslant n = k - 2 \right\}. \tag{16}$$

If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_i(\mathfrak{N})$, then $ad \equiv \epsilon \mod \mathfrak{N}$ for some $\epsilon \in \mathcal{O}_{F+}^\times$. Since $\chi_{\mathfrak{N}}(\epsilon) = N_{F/\mathbb{Q}}(\epsilon)^n = 1$, we see that $\chi_{\mathfrak{N}}(a) = \chi_{\mathfrak{N}}(d)^{-1}$. Using this one may check that the pairing $\langle\, ,\rangle$ is $\Gamma_i(\mathfrak{N})$-equivariant, and that it is perfect. It induces the following pairings on cohomology via cup product, which are again denoted by $\langle\, ,\rangle$

$$\langle\, ,\rangle : \mathrm{H}^q(Y(\mathfrak{N}), \mathcal{L}(n, \chi, A)) \otimes \mathrm{H}_{\mathrm{c}}^{2d-q}(Y(\mathfrak{N}), \mathcal{L}(n, \chi^{-1}, A)) \to A, \tag{17}$$

$$\langle\, ,\rangle : \mathrm{H}_{\mathrm{c}}^q(Y(\mathfrak{N}), \mathcal{L}(n, \chi, A)) \otimes \mathrm{H}^{2d-q}(Y(\mathfrak{N}), \mathcal{L}(n, \chi^{-1}, A)) \to A. \tag{18}$$

In order to state the next Proposition, for an $\mathcal{O}_\wp$ module M of finite type, we let M' denote the maximal torsion free quotient of M. Then we have,

PROPOSITION 2 (Poincaré duality). *Say $q = d$ is the middle dimension. Then if $A = K_\wp$ or $\mathbb{C}$, the pairing* (17) *is perfect. That is, the morphisms induced by* (17)*:*

$$\mathrm{H}^d(Y(\mathfrak{N}), \mathcal{L}(n, \chi, A)) \to \mathrm{Hom}_A(\mathrm{H}_{\mathrm{c}}^d(Y(\mathfrak{N}), \mathcal{L}(n, \chi^{-1}, A))),$$

$$\mathrm{H}_{\mathrm{c}}^d(Y(\mathfrak{N}), \mathcal{L}(n, \chi^{-1}, A)) \to \mathrm{Hom}_A(\mathrm{H}^d(Y(\mathfrak{N}), \mathcal{L}(n, \chi, A))),$$

are isomorphisms. A similar statement holds for the pairing (18).

When $A = \mathcal{O}_\wp$, and $\wp \notin \mathcal{S}_{\text{weight}} \cup \mathcal{S}_{\text{elliptic}}$, then the morphisms induced by (17) *are isomorphisms modulo torsion. That is:*

$$\mathrm{H}^d(Y(\mathfrak{N}), \mathcal{L}(n, \chi, A))' \to \mathrm{Hom}_A(\mathrm{H}_c^d(Y(\mathfrak{N}), \mathcal{L}(n, \chi^{-1}, A))', A),$$

$$\mathrm{H}_c^d(Y(\mathfrak{N}), \mathcal{L}(n, \chi^{-1}, A))' \to \mathrm{Hom}_A(\mathrm{H}^d(Y(\mathfrak{N}), \mathcal{L}(n, \chi, A))', A)$$

are isomorphisms. Again, an analogous statement holds for the pairing (18).

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Proof. If $Y(\mathfrak{N})_{/\mathbb{Q}}$ is smooth, the statements follow from ordinary Poincaré duality. Even if $Y(\mathfrak{N})$ is not smooth, we may still deduce the perfectness of the pairings, with the usual restrictions on the coefficients (cf. [23], Remark after Theorem 1.6). Indeed, if $\mathfrak{p} \notin \mathcal{S}_{\text{elliptic}}$, by Lemma 1, we may choose torsion free finite index normal subgroups $\overline{\Gamma_i}$ of $\overline{\Gamma_i(\mathfrak{N})}$ with the property that $[\overline{\Gamma_i(\mathfrak{N})} : \overline{\Gamma_i}]$ is invertible in A. Since the perfectness of the pairings is true for the subgroups Γ_i, the theorem follows for the $\Gamma_i(\mathfrak{N})$ as well by 'taking invariants'. $\square$

We now derive a duality statement for the parabolic cohomology groups with integral coefficients $A = \mathcal{O}_\wp$. First note that the two diagrams

$$\begin{array}{ccccc}
\mathrm{H}^d_{\mathrm{c}}(Y(\mathfrak{N}), \mathcal{L}(n,\chi,A)) & \otimes & \mathrm{H}^d_{\mathrm{c}}(Y(\mathfrak{N}), \mathcal{L}(n,\chi^{-1},A)) & \to & A \\
\downarrow \imath & & \| & & \| \\
\mathrm{H}^d(Y(\mathfrak{N}), \mathcal{L}(n,\chi,A)) & \otimes & \mathrm{H}^d_{\mathrm{c}}(Y(\mathfrak{N}), \mathcal{L}(n,\chi^{-1},A)) & \to & A,
\end{array}$$

and

$$\begin{array}{ccccc}
\mathrm{H}^d_{\mathrm{c}}(Y(\mathfrak{N}), \mathcal{L}(n,\chi,A)) & \otimes & \mathrm{H}^d_{\mathrm{c}}(Y(\mathfrak{N}), \mathcal{L}(n,\chi^{-1},A)) & \to & A \\
\| & & \downarrow \imath & & \| \\
\mathrm{H}^d_{\mathrm{c}}(Y(\mathfrak{N}), \mathcal{L}(n,\chi,A)) & \otimes & \mathrm{H}^d(Y(\mathfrak{N}), \mathcal{L}(n,\chi^{-1},A)) & \to & A,
\end{array}$$

commute. Consequently, we get a (well-defined) induced pairing which we again denote by $\langle\, , \rangle$:

$$\langle\, , \rangle : \ \mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L}(n,\chi,A)) \otimes \mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L}(n,\chi^{-1},A)) \to A. \tag{19}$$

DEFINITION. *Let $\mathcal{S}_\partial$ be the finite set of primes $\wp$ of K for which the $\mathcal{O}_\wp$-modules $\mathrm{H}^d(\partial(\Gamma_i(\mathfrak{N})\backslash\mathcal{Z}), \mathcal{L}_i)$ have torsion, for i varying from 1 through h, and $\mathcal{L}_i = \mathcal{L}_i(n, \chi^{\pm 1}, \mathcal{O}_\wp)$. Also let*

$$\mathcal{S}_{\text{duality}} = \mathcal{S}_{\text{weight}} \cup \mathcal{S}_{\text{elliptic}} \cup \mathcal{S}_\partial. \tag{20}$$

Note that $\mathcal{S}_{\text{duality}}$ is a finite set of primes of K.

THEOREM 3 (Poincaré duality for parabolic cohomology). *If $\wp$ is a prime of K such that $\wp \notin \mathcal{S}_{\text{duality}}$, and $A = \mathcal{O}_\wp$ is the $\wp$-adic completion of the ring of integers $\mathcal{O}_K$ of K, then the pairing* (19) *above is perfect modulo torsion. That is, the induced morphisms*

$$\mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L}(n,\chi,A))' \to \mathrm{Hom}_A(\mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L}(n,\chi^{-1},A))', A), \tag{21}$$

$$\mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L}(n,\chi^{-1},A))' \to \mathrm{Hom}_A(\mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L}(n,\chi,A))', A), \tag{22}$$

are isomorphisms.

Proof. Fix $\wp \notin \mathcal{S}_{\text{duality}}$. First note that since $\wp \notin \mathcal{S}_{\text{elliptic}} \cup \mathcal{S}_{\text{weight}}$, the morphisms (21) and (22) are well defined. We only prove (21) is an isomorphism, since the proof for (22) is similar.

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First, one may easily check that the injectivity (modulo torsion) of the morphism (21) follows from the injectivity (modulo torsion) of the maps in Proposition 2 (note that we have assumed $\wp \notin \mathcal{S}_{\text{elliptic}}$). So we must only show the surjectivity. To this end, let $\alpha \in \mathrm{Hom}_{\mathcal{O}_\wp}(\mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L}(n, \chi^{-1}, \mathcal{O}_\wp)), \mathcal{O}_\wp)$. Thanks to Proposition 2, on composing α with the map

$$\mathrm{H}^d_{\mathrm{c}}(Y(\mathfrak{N}), \mathcal{L}(n, \chi^{-1}, \mathcal{O}_\wp)) \xrightarrow{i} \mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L}(n, \chi^{-1}, \mathcal{O}_\wp)),$$

we can find an element $x \in \mathrm{H}^d(Y(\mathfrak{N}), \mathcal{L}(n, \chi, \mathcal{O}_\wp))$ such that $\langle x, y\rangle = (\alpha \circ i)(y)$, for all $y \in \mathrm{H}^d_{\mathrm{c}}(Y(\mathfrak{N}), \mathcal{L}(n, \chi^{-1}, \mathcal{O}_\wp))$. Since after tensoring with $K_\wp$ the morphism (21) is an isomorphism, we may assume that

$$x \otimes 1 \in \mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L}(n, \chi, \mathcal{O}_\wp)) \otimes K_\wp = \mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L}(n, \chi, K_\wp)).$$

Let $\bar{z}$ denote taking the class of $z \in \mathrm{H}^d$ modulo $\mathrm{H}^d_{\mathrm{p}}$, over $\mathcal{O}_\wp$ or $K_\wp$. Then

$$\bar{x} \otimes 1 = \overline{x \otimes 1} = 0,$$

so that $\bar{x}$ is a torsion element in

$$\frac{\mathrm{H}^d(Y(\mathfrak{N}), \mathcal{L}(n, \chi, \mathcal{O}_\wp))}{\mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L}(n, \chi, \mathcal{O}_\wp))}. \tag{23}$$

By the boundary exact sequence (15) we have

$$\frac{\mathrm{H}^d(Y(\mathfrak{N}), \mathcal{L})}{\mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L})} \subset \mathrm{H}^d(\partial(Y(\mathfrak{N})^*), \mathcal{L}),$$

with $\mathcal{L} = \mathcal{L}(n, \chi, \mathcal{O}_\wp)$. Thus since $\wp \notin \mathcal{S}_\partial$, (23) is a torsion-free $\mathcal{O}_\wp$-module. Thus $\bar{x} = 0$. That is, $x \in \mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L}(n, \chi, \mathcal{O}_\wp))$, as desired. □

Remark 1. One expects that the only maximal ideals of the Hecke algebra that occur in the support of the boundary cohomology group are all Eisenstein. Thus, since for the purpose of investigating congruences it suffices to work locally, one could possibly also work outside the set of primes of K which are Eisenstein. Such matters are currently being investigated by M. Dimitrov in his thesis under the supervision of Prof. J. Tilouine.

3.4. EXPLICIT COMPUTATION OF THE SET $\mathcal{S}_\partial$

It would be nice to be able to compute the set $\mathcal{S}_\partial$ explicitly. Here we give a general method for doing this, and then work out some special cases. Recall that $\mathcal{S}_\partial$ is the set of primes $\wp$ for which the $\mathcal{O}_\wp$-modules

$$\mathrm{H}^d(\partial(\Gamma_i(\mathfrak{N})\backslash\mathcal{Z}), \mathcal{L}_i)$$

have torsion, as i varies between 1 and h, and $\mathcal{L}_i = \mathcal{L}_i(n, \chi^{\pm 1}, \mathcal{O}_\wp)$. We only treat the case $\mathcal{L}_i(n, \chi, \mathcal{O}_\wp)$, since the discussion for $\mathcal{L}_i(n, \chi^{-1}, \mathcal{O}_\wp)$ is similar.

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Recall that the sheaf $\mathcal{L}_i$ on the boundary arise from a local system construction (see § 3.2). Thus, if $\wp \notin \mathcal{S}_{\text{elliptic}}$, we have

$$\mathrm{H}^d(\partial(\Gamma_i(\mathfrak{N})\backslash\mathcal{Z}), \mathcal{L}_i(n, \chi, \mathcal{O}_\wp)) = \oplus_B \mathrm{H}^d(\overline{\Gamma_i(\mathfrak{N})}_B, L_i(n, \chi, \mathcal{O}_\wp)),$$

where the sum runs over the set of representatives of the $\Gamma_i(\mathfrak{N})$-conjugacy classes of Borel subgroups B, and the bar means image in $\mathrm{PGL}_2(F)$. Thus, if we ignore the primes in $\mathcal{S}_{\text{elliptic}}$, which we will eventually exclude anyway, $\mathcal{S}_\partial$ is exactly the set of primes $\wp$ for which the $\mathcal{O}_\wp$-modules

$$\mathrm{H}^d(\overline{\Gamma_i(\mathfrak{N})}_B, L_i(n, \chi, \mathcal{O}_\wp))$$

have torsion, for some i and B (of course the B's depend on i!).

Let now $\Gamma = \Gamma_i(\mathfrak{N})$ for some i, and write $L(n, \chi, A)$ for the corresponding $\overline{\Gamma}$-module $L_i(n, \chi, A)$. Let B be a Borel subgroup as above. We can make a reduction. The exact sequence of $\overline{\Gamma}_B$ modules

$$0 \to L(n, \chi, \mathcal{O}_\wp) \to L(n, \chi, K_\wp) \to L(n, \chi, K_\wp/\mathcal{O}_\wp) \to 0,$$

gives the following long exact sequence of group cohomology groups:

$$\begin{aligned} \cdots \longrightarrow \mathrm{H}^{d-1}(\overline{\Gamma}_B, L(n, \chi, K_\wp)) \longrightarrow \mathrm{H}^{d-1}(\overline{\Gamma}_B, L(n, \chi, K_\wp/\mathcal{O}_\wp)) \\ \longrightarrow \mathrm{H}^d(\overline{\Gamma}_B, L(n, \chi, \mathcal{O}_\wp)) \longrightarrow \mathrm{H}^d(\overline{\Gamma}_B, L(n, \chi, K_\wp)) \longrightarrow \cdots \end{aligned}$$

From this it is easy to check that $\mathrm{H}^d(\overline{\Gamma}_B, L(n, \chi, \mathcal{O}_\wp))$ is torsion-free if and only if $\mathrm{H}^{d-1}(\overline{\Gamma}_B, L(n, \chi, K_\wp/\mathcal{O}_\wp))$ is divisible. Thus it suffices to investigate the divisibility of the cohomology groups $\mathrm{H}^{d-1}(\overline{\Gamma}_B, L(n, \chi, K_\wp/\mathcal{O}_\wp))$.

Let $C(\Gamma)$ denotes the set of cusps of Γ. Then

$$C(\Gamma) = \Gamma\backslash\mathrm{GL}_2(F)/B_\infty = \Gamma\backslash\mathbb{P}^1(F),$$

where B_∞, the Borel of upper triangular matrices, is the stabilizer of the cusp $\infty = [1:0] \in \mathbb{P}^1(F)$. Fix a set of representatives, $g_1, \ldots, g_r \in \mathrm{GL}_2(F)$, of the double coset space, where $r = r(\Gamma)$ is the number of cusps of Γ. Then $\{g_j B_\infty g_j{}^{-1} \mid j = 1, \ldots, r\}$ is a set of representatives of the Γ-conjugacy classes of Borel subgroups. Setting $B = g_j B_\infty g_j{}^{-1}$, we have

$$\mathrm{H}^{d-1}(\overline{\Gamma}_B, L(n, \chi, K_\wp/\mathcal{O}_\wp)) = \mathrm{H}^{d-1}(\overline{(g_j^{-1}\Gamma g_j)}_{B_\infty}, L(n, \chi, K_\wp/\mathcal{O}_\wp)).$$

This allows us to restrict our computations to the Borel at ∞.

Now write $B_\infty = TU$, where T is the standard torus, and U is the unipotent radical of B_∞. Then, for an arbitrary arithmetic subgroup $\Gamma' \subset \mathrm{GL}_2(F)_+$, we have the exact sequence

$$1 \to \Gamma'_U \to \Gamma'_B \to \Gamma'_T \to 1,$$

where $\Gamma'_T := \Gamma' \cap T$ and $\Gamma'_U := \Gamma' \cap U$. Projecting this to $\mathrm{PGL}_2(F)$ and noting that $\Gamma'_U \cap F^\times = \{1\}$, one obtains

$$1 \to \Gamma'_U \to \overline{\Gamma}'_B \to \overline{\Gamma}'_T \to 1, \tag{24}$$

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where $\overline{\Gamma}'_B = \Gamma'_B/\Gamma'_B \cap F^\times$ and $\overline{\Gamma}'_T = \Gamma'_T/\Gamma'_T \cap F^\times$. The Hochschild–Serre spectral sequence for the $\overline{\Gamma}'_B$-module $L(n, \chi, K_\wp/\mathcal{O}_\wp)$ is

$$E_2^{p,q} = \mathrm{H}^p(\overline{\Gamma}'_T, \mathrm{H}^q(\Gamma'_U, L(n, \chi, K_\wp/\mathcal{O}_\wp))) \Rightarrow \mathrm{H}^n(\overline{\Gamma}'_{B_\infty}, L(n, \chi, K_\wp/\mathcal{O}_\wp)).$$

In $\mathrm{PGL}_2(F)$ we have $\overline{\Gamma}'_T \xrightarrow{\sim} \mathbb{Z}^{d-1}$ and $\Gamma'_U \xrightarrow{\sim} \mathbb{Z}^d$, so that these groups have cohomological dimension $d-1$ and d respectively. Consequently, it is easy to check that all the differentials in the E_d-plane are 0: that is the spectral sequence degenerates at E_d. Thus, to show that

$$\mathrm{H}^{d-1}(\overline{(g_j{}^{-1}\Gamma_i(\mathfrak{N})g_j)}_{B_\infty}, L_i(n, \chi, K_\wp/\mathcal{O}_\wp))$$

is divisible, it suffices to show that each term in its filtration

$$\frac{F^p\mathrm{H}^{d-1}}{F^{p+1}\mathrm{H}^{d-1}} = E_\infty^{p,d-1-p} = E_d^{p,d-1-p}$$

is divisible.

Clearly, in general, this is a difficult task. However, some special cases can be dealt with, which we discuss now.

3.4.1. $F = \mathbb{Q}$

The case $F = \mathbb{Q}$ has already been dealt with by Hida in [10], but for the sake of completeness we review his computations here.

Note that $\mathbb{Q}$ has strict class number 1, so there is only one congruence subgroup, $\Gamma_i(\mathfrak{N})$, to consider. Moreover we may choose $t_1 = 1$. Thus if we let N be the positive integer which generates the ideal $\mathfrak{N} \subset \mathbb{Z}$, we see that $\Gamma_1(\mathfrak{N})$ is just the usual congruence subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \bmod N \right\}.$$

Also, from (11), we have that $\mathcal{S}_{\text{elliptic}} = \{\wp \mid p \,:\, p = 2 \text{ or } 3\}$, and we ignore these primes in our computations.

The following Proposition has been distilled from the discussion on page 232–233 of [10]:

PROPOSITION 3. *If $n = 0$, then $\mathcal{S}_\partial = \emptyset$. For $n > 0$, let $\mathcal{S}_1$ to be the set of primes $\wp$ lying above the set $\{p \,:\, p \leqslant n = k-2$ or $p \mid N\}$. Then $\mathcal{S}_\partial \subset \mathcal{S}_1$.*

Proof. Note that we may choose the set of representatives, $g_1, \ldots, g_r$, of the cusps of $\Gamma_0(N)$ to lie in $\mathrm{SL}_2(\mathbb{Z})$. Then, up to the center $\{\pm 1\}$, the group $g_j^{-1}\Gamma_0\,(N)g_j \cap B_\infty \xrightarrow{\sim} \mathbb{Z}$ is generated by an element of the form $\pi_j = \begin{pmatrix} 1 & m_j \\ 0 & 1 \end{pmatrix}$, for some $m_j \in \mathbb{Z}$. Write $L(n, \mathcal{O}_\wp)$ for $L_1(n, \chi^{\pm 1}, \mathcal{O}_\wp)$. Then we must investigate the torsion in

$$\mathrm{H}^1(\overline{\Gamma_0(N)}_B, L(n, \mathcal{O}_\wp)) = \frac{L(n, \mathcal{O}_\wp)}{(\pi_j - 1)L(n, \mathcal{O}_\wp)}, \tag{25}$$

for $B = g_j B_\infty g_j^{-1}$.

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When $n = 0$, the action is trivial, and this last module is just $\mathcal{O}_{\wp}$, which is clearly torsion-free.

When $n > 0$ then a direct computation shows that (25) has torsion only for those $\wp$ whose residue characteristic p satisfies $p \leqslant n$ or $p \mid m_j$. But, since the principal congruence subgroup $\Gamma(N)$ is a normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and $\Gamma(N) \subset \Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$, one may easily check that each $m_j \mid N$. $\square$

3.4.2. *Real Quadratic Fields*

Let F be a real quadratic field. So $d = [F\colon \mathbb{Q}] = 2$. Let ϵ_0 denote the fundamental unit of F, and let ϵ_1 denote a generator of the group of totally positive units $\mathcal{O}_{F+}^{\times}$. Thus,

$$\epsilon_1 = \begin{cases} \epsilon_0 & \text{if } N_{F/\mathbb{Q}}(\epsilon_0) = 1, \\ \epsilon_0^2 & \text{if } N_{F/\mathbb{Q}}(\epsilon_0) = -1. \end{cases}$$

By (11), we have $\mathcal{S}_{\mathrm{elliptic}} = \{\wp \mid p \,:\, p = 2, 3 \text{ or } 5\}$, and we ignore such primes as usual! (In fact when $F \neq \mathbb{Q}(\sqrt{5})$, one may take $\mathcal{S}_{\mathrm{elliptic}} = \{\wp \mid p \,:\, p = 2 \text{ or } 3\}$).

We first treat the case when the weight $n = 0$. Even in this case, we make the following restrictive assumption, which will at least allow us to compute.

ASSUMPTION 1 Assume that

- F has strict class number $h = 1$, and,
- the level $\mathfrak{N} = 1$.

In view of this assumption, we must only consider the congruence subgroup

$$\Gamma = \Gamma_1(\mathfrak{N}) = \{\gamma \in \mathrm{GL}_2(\mathcal{O}_{\mathrm{F}}) \mid \det(\gamma) \in \mathcal{O}_{\mathrm{F}+}^{\times}\}. \tag{26}$$

This Γ has only one cusp, and we may choose the representative $g_1 = 1$. Now (working modulo the center; we will omit bars from the notation) we see that

$$\Gamma_T := \Gamma \cap T = \left\{ \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \mid \epsilon \in \mathcal{O}_{F+}^{\times} \right\}$$

and

$$\Gamma_U := \Gamma \cap U = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathcal{O}_F \right\}.$$

We have:

PROPOSITION 4. Let F *and* Γ *satisfy the hypothesis of Assumption* 1 *above. Let* $n = 0$. *Let* $\mathcal{S}_2$ *be the set of primes of* K *lying above the set* $\{p \,:\, p \mid N_{F/\mathbb{Q}}(\epsilon_1 - 1)\}$. *Then* $\mathcal{S}_{\partial} \subset \mathcal{S}_2$.

Proof. Fix $\wp \notin \mathcal{S}_2$. We show $\wp \notin \mathcal{S}_{\partial}$. That is, we show $\mathrm{H}^2(\Gamma_{B_\infty}, L(n, \mathcal{O}_{\wp}))$ is torsion-free, when $n = 0$. Here, and for the rest of this subsection, we write $L(n, A)$ for $L_1(n, \chi^{\pm 1}, A)$, noting that $\chi_{\mathfrak{N}} = 1$. By the general remarks at the beginning of

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Section 3.4, it suffices to show that the $\mathcal{O}_\wp$-modules $E_2^{1,0}$ and $E_2^{0,1}$ are divisible. Note that since we are in the case $n = 0$, the Γ-action is trivial. We have

$$\begin{aligned} E_2^{1,0} &= \mathrm{H}^1(\Gamma_T, \mathrm{H}^0(\Gamma_U, K_\wp/\mathcal{O}_\wp)) \\ &= \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, K_\wp/\mathcal{O}_\wp) \\ &= K_\wp/\mathcal{O}_\wp, \end{aligned}$$

which is clearly divisible. Moreover

$$\begin{aligned} E_2^{0,1} &= \mathrm{H}^0(\Gamma_T, \mathrm{H}^1(\Gamma_U, K_\wp/\mathcal{O}_\wp)) \\ &= \mathrm{Hom}_{\mathbb{Z}}(\Gamma_U, K_\wp/\mathcal{O}_\wp)^{\Gamma_T}, \end{aligned}$$

where $\gamma \in \Gamma_T$ acts on $f \in \mathrm{Hom}_{\mathbb{Z}}(\Gamma_U, K_\wp/\mathcal{O}_\wp)$ via

$$(\gamma \cdot f)(g) = \gamma \cdot f(\gamma^{-1} \cdot g) = f(\gamma^{-1} g \gamma).$$

Recall that p is the the residue characteristic of $\wp$. We claim that, since $p \nmid N_{F/\mathbb{Q}}(\epsilon_1 - 1)$, there are no invariant elements under this action. Indeed if f is such a homomorphism we have (identifying Γ_U with $\mathcal{O}_F$ in an obvious manner) that

$$f(\epsilon b) = f(b), \tag{27}$$

for all $\epsilon \in \mathcal{O}_{F+}^\times$ and all $b \in \mathcal{O}_F$. Fix an integral basis $\{\omega_1, \omega_2\}$ of $\mathcal{O}_F$, and write $\epsilon_1 \omega_i = \sum_j a_{ij} \omega_j$. Then Equations (27) imply that $(A - \mathrm{Id})F = 0$, where $A = (a_{ij})$ and $F = (f(\omega_1), f(\omega_2))^t$. Thus $\det(A - \mathrm{Id}) = N_{F/\mathbb{Q}}(\epsilon_1 - 1)$ annihilates the $f(\omega_i)$. Since f takes values in the $\wp$-torsion module $K_\wp/\mathcal{O}_\wp$, we see that $f = 0$, since $p \nmid N_{F/\mathbb{Q}}(\epsilon_1 - 1)$. Thus $E_2^{1,0} = 0$ for such $\wp$.

In sum, when $n = 0$, and the residue characteristic p of $\wp$ satisfies $p \nmid N_{F/\mathbb{Q}}(\epsilon_1 - 1)$, then $\mathrm{H}^1(\Gamma_B, L(n, K_\wp/\mathcal{O}_\wp)) = E_2^{1,0}$ is divisible. $\square$

Remark 2. When the strict class number of F is not 1, the other $\Gamma_i(\mathfrak{N})$ in the decomposition (2) must be treated. Moreover, even if the strict class number is 1, and the level $\mathfrak{N}$ is not full, there may be more than one cusp. This must be taken into account, further complicating the computation of the boundary cohomology groups. However the essential idea is already to be found in the above computations.

We now consider the case $n > 0$, although again, for simplicity, we suppose that Assumption 1 above holds.

Since the computations are still too difficult, we appeal to Hida's general theory in [15] via which one can extract some information for the ordinary primes. This theory is fairly elaborate and we only make some brief remarks here, referring the reader to Section 3 of [15] for further information.

Let F denote, momentarily, an arbitrary totally real field (of strict class number 1). Fix a prime p of $\mathbb{Q}$, and let $\Sigma = \{\mathfrak{p} \mid p\}$ denote the set of primes of F lying over p. Let $\iota = (\iota_\mathfrak{p})_{\mathfrak{p}\in\Sigma} \in \{0, \infty\}^\Sigma$ and let $r = (r_\mathfrak{p})_{\mathfrak{p}\in\Sigma} \in \mathbb{N}^\Sigma$, where $r_\mathfrak{p} := \mathrm{rank}_{\mathbb{Z}_p} \mathcal{O}_{F\mathfrak{p}}$. Set $\{\iota r\} = \sum_{\iota_\mathfrak{p} \,\mathrm{with}\, \iota_\mathfrak{p} = \infty} r_\mathfrak{p}$.

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Fix an embedding, j, of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_p$. Since we have fixed an embedding of K into $\bar{\mathbb{Q}}$, such an embedding singles out a prime $\wp$ of K. Conversely, a choice of a prime $\wp$ of K yields an embedding of K into $\bar{\mathbb{Q}}_p$ (which may be extended to an embedding of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_p$).

Each $\sigma \in I_F$ determines a unique prime $\mathfrak{p}$ of F which makes

$$F \xrightarrow{\sigma} \bar{\mathbb{Q}} \xrightarrow{j} \bar{\mathbb{Q}}_p$$

a $\mathfrak{p}$-adically continuous embedding of F into $\bar{\mathbb{Q}}_p$. For each $\mathfrak{p} \in \Sigma$, let $I(\mathfrak{p}) = \{\sigma \in I_F \mid \wp \cap \sigma(F) = \sigma(\mathfrak{p})\} \subset I_F$ denote the set of all such $\mathfrak{p}$-adically continuous embeddings of F into $\bar{\mathbb{Q}}_p$ (with respect to j).

For each $\imath$ as above, let $I(\imath) = \cup_{\mathfrak{p} \text{ with } \imath_\mathfrak{p}=\infty} I(\mathfrak{p})$. Set $n^\imath = \sum_{\sigma \in I(\imath)} n_\sigma \sigma - \sum_{\sigma \notin I(\imath)} n_\sigma \sigma$. Let $K_\wp/\mathcal{O}_\wp(n^\imath)$ denote the module $K_\wp/\mathcal{O}_\wp$, on which $\epsilon \in \mathcal{O}_F^\times$ acts via: $\epsilon \cdot a = \epsilon^{n^\imath} a$.

Let $\Gamma' = \mathrm{SL}_2(\mathcal{O}_F)$, and let $\mathrm{H}^q_{\mathrm{ord}}(\partial(\Gamma' \backslash \mathcal{Z}^*), L(n, K_\wp/\mathcal{O}_\wp))$ denote the ordinary part of the boundary cohomology group, as defined in [15]. We just recall here that the ordinary part is the maximal $\mathcal{O}_\wp$-submodule of the boundary cohomology group on which the Hecke operator T_p acts as an automorphism. Then Hida proves (cf. [15], Theorem 3.12):

THEOREM 4. *Fix F and Γ' as above. Then*

$$\begin{aligned}&\mathrm{H}^q_{\mathrm{ord}}(\partial(\Gamma' \backslash \mathcal{Z}^*), L(n, K_\wp/\mathcal{O}_\wp))\\ &\quad = \bigoplus_{i=1}^{q} \mathrm{H}^{q-i}\left(\mathcal{O}_F^\times, \bigoplus_{\imath \text{ with } \{\imath r\}=i} \mathrm{Hom}_{\mathbb{Z}_p}\Big(\bigwedge_{\mathfrak{p}\in\Sigma \text{ with } \imath_\mathfrak{p}=\infty} \wedge^{r_\mathfrak{p}} \mathcal{O}_{F\mathfrak{p}},\ K_\wp/\mathcal{O}_\wp(n^\imath)\Big)\right),\end{aligned}$$

In the real quadratic situation, we obtain:

COROLLARY 1. *Let F be a real quadratic field of strict class number 1, and let $\Gamma' = \mathrm{SL}_2(\mathcal{O}_F)$. Suppose that $n > 0$. Let $\mathcal{S}'_{\partial,\mathrm{ord}}$ denote the set of primes of torsion in the ordinary part of the middle dimensional boundary cohomology group for Γ'. Let $\mathcal{S}_3$ be the set of primes of K lying above the set*

$$\{p : p \mid N_{F/\mathbb{Q}}(\pm\epsilon_0^{2(n+1)} - 1)\},$$

where the sign is $N_{F/\mathbb{Q}}(\epsilon_0)^n$. Then $\mathcal{S}'_{\partial,\mathrm{ord}} \subset \mathcal{S}_3$.

Proof. Let $\wp \mid p$ be a prime of K and assume that $\wp \notin \mathcal{S}_3$. We show that $\wp \notin \mathcal{S}'_{\partial,\mathrm{ord}}$. It is enough to show $\mathrm{H}^q_{\mathrm{ord}}(\partial(\Gamma' \backslash \mathcal{Z}^*), L(n, K_\wp/\mathcal{O}_\wp))$ is divisible when $q = d - 1 = 1$. By Theorem 4, if p is inert or ramifies in F, $\mathrm{H}^1_{\mathrm{ord}} = 0$ vacuously. This is because, in either case, if $\mathfrak{p} \mid p$, we have $r_\mathfrak{p} = 2$, and so there is no sum over $\imath$ to consider.

Now suppose that $p = \mathfrak{p}_1\mathfrak{p}_2$ splits in F. Say $I_F = \{\sigma_1, \sigma_2\}$, and assume that j is chosen so that $I(\mathfrak{p}_k) = \{\sigma_k\}$, for $k = 1, 2$. Then, since $i = 1$, and $r_{\mathfrak{p}_1} = r_{\mathfrak{p}_2} = 1$, only $\imath = (\infty, 0)$ and $(0, \infty)$ contribute to the cohomology, in Theorem 4.

When $\imath = (\infty, 0)$, this contribution is

$$\mathrm{H}^0(\mathcal{O}_F^\times, \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{O}_{F\mathfrak{p}_1}, K_\wp/\mathcal{O}_\wp(n'))). \tag{28}$$

Note that $\epsilon^{-n\sigma_2} = \pm\epsilon^{n\sigma_1}$, where the sign is given by $N_{F/\mathbb{Q}}(\epsilon)^n$. Let us identify $\mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{O}_{F\mathfrak{p}_1}, K_\wp/\mathcal{O}_\wp(n'))$ with $K_\wp/\mathcal{O}_\wp$ via $f \mapsto f(1)$. Note, by page 287 of [15], the action of $\epsilon^{-1} \in \mathcal{O}_F^\times$ on $\mathcal{O}_{F\mathfrak{p}_1}$ is just multiplication by ϵ^2. Thus, as an $\mathcal{O}_F^\times$-module, $\mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{O}_{F\mathfrak{p}_1}, K_\wp/\mathcal{O}_\wp(n'))$ is just $K_\wp/\mathcal{O}_\wp(n\sigma_1 - n\sigma_2 + 2\sigma_1) = K_\wp/\mathcal{O}_\wp((2n+2)\sigma_1)$, up to the above mentioned sign. Thus since $\mathfrak{p} \notin \mathcal{S}_3$ we see that (28) vanishes.

A similar computation for $\iota = (0, \infty)$, shows that

$$\mathrm{H}^0(\mathcal{O}_F^\times, \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{O}_{F\mathfrak{p}_2}, K_\wp/\mathcal{O}_\wp(n'))) = 0.$$

Thus, $\mathrm{H}^1_{\mathrm{ord}}(\partial(\Gamma'\backslash\mathcal{Z}^*), L(n, K_\wp/\mathcal{O}_\wp)) = 0$ is indeed divisible. □

Remark 3. Note that the computations for $n > 0$ are slightly different since Hida's theory is for subgroups of $\mathrm{SL}_2(F)$, whereas we should really be taking Γ as in (26). If, in the case of $n = 0$, we had taken $\Gamma' = \mathrm{SL}_2(\mathcal{O}_F)$ instead of $\Gamma = \mathrm{GL}_2(\mathcal{O}_F)_+$ in Proposition 4, then, since

$$\Gamma'_T = \left\{ \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \middle| \epsilon \in \mathcal{O}_F^\times \right\},$$

acts on $\Gamma'_U = \mathcal{O}_F$ by multiplication by ϵ^2, we would have gotten the obstruction $N_{F/Q}(\epsilon_0^2 - 1)$ there. This matches well with Corollary 1 when $n = 0$. On the other hand, since it is $\Gamma = \mathrm{GL}_2(\mathcal{O}_F)_+$ that we really wish to work with, it is clear that the (ordinary) torsion in the boundary for $n > 0$ will occur at primes $\wp | p$ with $p \mid N_{F/\mathbb{Q}}(\epsilon_1^{n+1} - 1) = N_{F/\mathbb{Q}}(\epsilon_1^{k-1} - 1)$.

3.4.3. *Totally Real Cubic Fields*

Let now $[F : \mathbb{Q}] = 3$. We only treat the case $n = 0$ here. As in the quadratic case, we assume that $h = 1$ and $\mathfrak{N} = \mathcal{O}_F$ (cf. Assumption 1). As before, this allows us to deal with only the analog of the level one congruence subgroup (26), and with only the standard Borel B_∞. Note that now $\overline{\Gamma}_T = \mathcal{O}_{F+}^\times \xrightarrow{\sim} \mathbb{Z}^2$ and $\Gamma_U = \mathcal{O}_F \xrightarrow{\sim} \mathbb{Z}^3$. Let ϵ_1 and ϵ_2 denote a basis of the group of totally positive units.

As usual, we ignore the set of primes in $\mathcal{S}_{\mathrm{elliptic}}$ which in this case turns out to be $\{\wp \mid p : p = 2, 3, \text{ or } 7\}$. (If $F \neq \mathbb{Q}(\mu_7)^+$, then we may take $\mathcal{S}_{\mathrm{elliptic}} = \{\wp \mid p : p = 2 \text{ or } 3\}$). We have

PROPOSITION 5. *Let F be a totally real cubic field of class number* 1, *and let Γ be as in* (26). *Let $\mathcal{S}_4$ be the set of primes of K lying above the set*

$$\{p : p \mid N_{F/\mathbb{Q}}(\epsilon_1 - 1) \text{ and } p \mid N_{F/\mathbb{Q}}(\epsilon_2 - 1)\}.$$

Then, if $n = 0$, $\mathcal{S}_\partial \subset \mathcal{S}_4$.

Proof. Fix $\mathcal{O}_\wp$ with $\wp \mid p$ such that $\wp \notin \mathcal{S}_4$. We show $\wp \notin \mathcal{S}_\partial$. After the remarks at the beginning of Section 3.4, it suffices to show that each of the three term $E_3^{p,q}$

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$(p+q=2)$ in the filtration of $\mathrm{H}^2(\Gamma_B, L(n, K_\wp/\mathcal{O}_\wp))$, with $n=0$, is divisible. Note that again we write $L(n, A)$ for $L_1(n, \chi^\pm, A)$, since $\chi_{\mathfrak{N}}=1$.

We have $E_3^{2,0}=E_2^{2,0}/\mathrm{image}(d_2^{0,1}: E_2^{0,1}\to E_2^{2,0})$. But

$$E_2^{2,0}=\mathrm{Hom}_{\mathbb{Z}}(\wedge^2\mathbb{Z}^2, K_\wp/\mathcal{O}_\wp)=K_\wp/\mathcal{O}_\wp,$$

and a quotient of a divisible is divisible, so that $E_3^{2,0}$ is divisible.

Secondly, $E_3^{1,1}=E_2^{1,1}=\mathrm{H}^1(\Gamma_T, M)$, for $M=\mathrm{H}^1(\Gamma_U, K_\wp/\mathcal{O}_\wp)$. Since $\mathcal{O}_{F+}^\times=\langle\epsilon_1\rangle\times\langle\epsilon_2\rangle$, the inflation-restriction sequence yields:

$$0\to\mathrm{H}^1(\langle\epsilon_1\rangle, M^{\langle\epsilon_2\rangle})\to\mathrm{H}^1(\Gamma_T, M)\to\mathrm{H}^1(\langle\epsilon_2\rangle, M)^{\langle\epsilon_1\rangle}\to 0. \tag{29}$$

Now we have that $p\nmid N_{F/\mathbb{Q}}(\epsilon_i-1)$, for i $=$ 1 or 2. Suppose $i=2$. Then we have $M^{\langle\epsilon_2\rangle}=0$, just as in the real quadratic case. Thus, the first term of the exact sequence (29) vanishes. Similarly, the last term of (29) vanishes since it is just $(M/(\epsilon_2-1)M)^{\langle\epsilon_1\rangle}$, and is thus contained in $M/(\epsilon_2-1)M=0$, which vanishes in turn since $p\nmid N_{F/\mathbb{Q}}(\epsilon_2-1)$. A symmetric argument applies if $i=1$. Consequently, $E_3^{1,1}=0$.

Finally, $E_3^{0,2}=\ker\left(d_2^{0,2}: E_2^{0,2}\to E_2^{2,1}\right)$. We claim

$$E_2^{0,2}\ =\ \mathrm{H}^0(\Gamma_T, \mathrm{H}^2(\Gamma_U, K_\wp/\mathcal{O}_\wp))\ =\ 0.$$

To see this, note that

$$\mathrm{H}^2(\Gamma_U, K_\wp/\mathcal{O}_\wp)=\mathrm{Hom}_{\mathbb{Z}}(\wedge^2\mathcal{O}_F, K_\wp/\mathcal{O}_\wp).$$

If f is a homomorphism in this latter space, invariant under some $\epsilon\in\Gamma_T$, then we must have

$$(\epsilon^{-1}\cdot f)(a\wedge b)=f(\epsilon a\wedge\epsilon b)=f(a\wedge b). \tag{30}$$

Now a familiar argument applies: let $\{\omega_1, \omega_2, \omega_3\}$ be an integral basis for $\mathcal{O}_F$, and write $\epsilon\omega_i=\sum_j a_{ij}\omega_j$. Then if B denotes the matrix of 2×2 cofactors (minors with signs) of $A=(a_{ij})$, the Equations (30) show that $\det(B-\mathrm{Id})$ annihilates each component of the vector $(f(\omega_2\wedge\omega_3), f(\omega_3\wedge\omega_1), f(\omega_1\wedge\omega_2))^t$. But note that $B=A^{-1}$, since $\det(A)=1$. Consequently choosing ϵ to be ϵ_1 or ϵ_2, so that

$$p\nmid\det(A^{-1}-\mathrm{Id})=-\det(A-\mathrm{Id})=-N_{F/\mathbb{Q}}(\epsilon-1),$$

we see that $f=0$. Thus $E_2^{0,2}=0$, and so $E_3^{0,2}=0$ as well. □

3.4.4. *Totally Real Fields of Arbitrary Degree*

The referee has suggested that, at least in the case $n=0$, our computations of the previous section should generalize. This is indeed the case and we present this now.

So let now $[F:\mathbb{Q}]=d$ be arbitrary. We assume that $n=0$, and as in previous cases, that $h=1$ and $\mathfrak{N}=\mathcal{O}_F$ (cf. Assumption 1). Therefore, as before, Γ is the level one congruence subgroup (26), and we need only work with the standard Borel B_∞. Note that now $\overline{\Gamma}_T=\mathcal{O}_{F+}^\times\xrightarrow{\sim}\mathbb{Z}^{d-1}$ and $\Gamma_U=\mathcal{O}_F\xrightarrow{\sim}\mathbb{Z}^d$. Let $\epsilon_1, \ldots, \epsilon_{d-1}$ denote a basis of the group of totally positive units of $\mathcal{O}_F$.

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PROPOSITION 6. *Let F be a totally real field of class number* 1, *and let* Γ *be as in* (26). *Let $\mathcal{S}_5$ be the set of primes of K lying above the set*

$$\bigcap_{i=1}^{d-1} \{p : p \mid N_{F/\mathbb{Q}}(\epsilon_i - 1)\}.$$

Then, if $n = 0$, $\mathcal{S}_\partial \subset \mathcal{S}_5$.

Before we give the proof we start with a simple lemma:

LEMMA 2. *Let G be a finitely generated group and M a G-module. Say there is a g in the center of G, such that $g - 1\colon M \to M$ is an automorphism. Then $\mathrm{H}^i(G, M) = 0$ for all $i \geqslant 0$.*

Proof. Note that g induces a map on cohomology $g\colon \mathrm{H}^i(G, M) \to \mathrm{H}^i(G, M)$ which is well known to be the identity map (cf. [20], Chapter VII, Proposition 3). Thus $g - 1$ acts as 0 on cohomology. On the other hand, by our hypothesis on g, the map $g - 1$ is an automorphism on cohomology as well. The Lemma follows. $\square$

Proof of Proposition 6. Fix $\mathcal{O}_\wp$ with $\wp \mid p$ such that $\wp \notin \mathcal{S}_5$. We show $\wp \notin \mathcal{S}_\partial$. As before, after the remarks at the beginning of Section 3.4, it suffices to show that each of the terms $E_d^{p,q}$ $(p + q = d - 1)$ in the filtration of $\mathrm{H}^{d-1}(\Gamma_B, K_\wp/\mathcal{O}_\wp)$ (recall $n = 0$ and $\chi_{\mathfrak{N}} = 1$), is divisible. We will show that all steps in the filtration vanish, except the top one, which is divisible.

To do this we show that the spectral sequence degenerates at the E_2 term. Indeed, $E_2^{p,q} = \mathrm{H}^p(\overline{\Gamma}_T, M(q))$, where

$$M(q) := \mathrm{H}^q(\Gamma_U, K_\wp/\mathcal{O}_\wp) = \mathrm{Hom}_{\mathbb{Z}}(\wedge^q \mathcal{O}_F, K_\wp/\mathcal{O}_\wp),$$

since the action of Γ_U on $K_\wp/\mathcal{O}_\wp$ is trivial. When $q > 0$, the action of $\overline{\Gamma}_T$ on $M(q)$ is non-trivial. By hypothesis, we can find $\epsilon = \epsilon_i$, for some i, such that $p \nmid N_{F/\mathbb{Q}}(\epsilon - 1)$. Now $\epsilon - 1$ acts on $M(q)$ as a matrix whose determinant is a power of $N_{F/Q}(\epsilon - 1)$. This follows from a 19th century theorem attributed to J. J. Sylvester (see, for instance, pages 87-89 of [22]) which says that the determinant of the matrix of $s \times s$ minors of a matrix of size $t \geqslant s$ is just a power of the original determinant. In particular, $\epsilon - 1$ acts as an automorphism on $M(q)$ for each $q > 0$. By Lemma 2 we see that for $q > 0$, $E_2^{p,q} = 0$ for all $p \geqslant 0$. Thus the spectral sequence degenerates at the E_2 term, and an analysis of the filtration of $\mathrm{H}^d(\overline{\Gamma}_B, \mathcal{O}_\wp)$ now shows that $\mathrm{H}^{d-1}(\overline{\Gamma}_B, K_\wp/\mathcal{O}_\wp) = E_\infty^{d-1,0} = E_2^{d-1,0} = \mathrm{Hom}_{\mathbb{Z}}(\wedge^{d-1}\mathbb{Z}^{d-1}, K_\wp/\mathcal{O}_\wp) = K_\wp/\mathcal{O}_\wp$ is divisible, as desired. $\square$

Remark 4. The set $\bigcap_{i=1}^{d-1} \{p : p \mid N_{F/\mathbb{Q}}(\epsilon_i - 1)\}$ may depend on the choice of basis $\epsilon_1, \dots, \epsilon_{d-1}$ of $\mathcal{O}_{F+}^\times$. However, the identity $\epsilon\epsilon' - 1 = \epsilon(\epsilon' - 1) + (\epsilon - 1)$ shows that the set $\bigcap_{i=1}^{d-1} \{\mathfrak{p} \subset \mathcal{O}_F : \mathfrak{p} \mid \epsilon_i - 1\}$ is independent of the basis. Thus it would probably be better to take for $\mathcal{S}_5$ the set of primes of K lying over this latter set.

Remark 5. As in the case when F is a real quadratic field, we could possibly use Theorem 4 to compute the torsion in the ordinary part of the boundary cohomology, when the weight $n > 0$, for arbitrary d. We have, however, refrained from presenting the details.

3.5. A MODIFIED PAIRING

We have investigated in some detail the perfectness of the pairing (19):

$$\langle\, ,\rangle:\ \mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L}(n,\chi,A)) \otimes \mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L}(n,\chi^{-1},A)) \to A,$$

for various coefficients A. However, note that the sheaves $\mathcal{L}(n,\chi,A)$ and $\mathcal{L}(n,\chi^{-1},A)$ that occur in the modules on both sides are different. For this and other reasons we now modify the cup product pairing (19) slightly.

Fix an idele $v \in \mathbb{A}_F^\times$, such that $v\mathcal{O}_F = \mathfrak{N}$ and $(v)_\infty = 1$, and define $\tau = \begin{pmatrix} 0 & -1 \\ v & 0 \end{pmatrix} \in \mathrm{G}(\mathbb{A})$. Then τ induces an isomorphism

$$\tau: \mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L}(n,\chi,A)) \to \mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L}(n,\chi^{-1},A)) \tag{31}$$

defined as follows. Let $s_i = \begin{pmatrix} 1 & 0 \\ 0 & a_i^{-1} \end{pmatrix}$, for the strict class group representatives a_i $(i = 1, \ldots, h)$ fixed in Section 2.2. Then

$$\mathrm{G}(\mathbb{A}) = \coprod_{i=1}^{h} \mathrm{G}(\mathbb{Q})\, s_i\, \mathrm{K}_0(\mathfrak{N})\, \mathrm{G}_{\infty+}$$

and, since $\Gamma_i(\mathfrak{N}) = \mathrm{G}(\mathbb{Q}) \cap s_i \mathrm{K}_0(\mathfrak{N}) s_i^{-1} \mathrm{G}_{\infty_+}$, the decomposition (2) remains unchanged using the representatives s_i.

Fix i. Since $a_i^{-1} v \in \mathbb{A}_F^\times$, we may pick $j \in \{1, \ldots, h\}$, $q_i \in F_+^\times$ and $w_i \in \hat{\mathcal{O}}_F^{\times}$ such that $a_i^{-1} v = q_i a_j w_i$. Let $\alpha_i = \begin{pmatrix} 0 & -1 \\ q_i & 0 \end{pmatrix}$, and $u_i = \begin{pmatrix} w_i & 0 \\ 0 & 1 \end{pmatrix}$. Then $s_i \tau = \alpha_i t_j u_i$ and $\alpha_i^{-1} \Gamma_i(\mathfrak{N}) \alpha_i = \Gamma_j(\mathfrak{N})$. These choices (for each i) induce the following commutative diagram

$$\begin{array}{ccc} Y(\mathfrak{N}) & \longrightarrow & \coprod_{i=1}^{h} \Gamma_i(\mathfrak{N})\backslash \mathcal{Z} \\ \downarrow & & \downarrow \\ Y(\mathfrak{N}) & \longrightarrow & \coprod_{j=1}^{h} \Gamma_j(\mathfrak{N})\backslash \mathcal{Z} \end{array}$$

where the first vertical map is $g \mapsto g\tau$, the second vertical map (on the ith connected component) is $z \mapsto \alpha_i^{-1} z$, and the horizontal maps are, as usual, $\gamma s_i k g_\infty \mapsto g_\infty(z_0)$, respectively, $\gamma' t_j k' g'_\infty \mapsto g'_\infty(z_0)$, where $\gamma, \gamma' \in \mathrm{G}(\mathbb{Q})$, $k, k' \in \mathrm{K}_0(\mathfrak{N})$, and $g_\infty, g'_\infty \in \mathrm{G}_{\infty+}$.

Now consider the following diagram

$$\begin{array}{ccccc} \Gamma_i(\mathfrak{N}) & \times & L_i(n,\chi,A) & \longrightarrow & L_i(n,\chi,A) \\ \uparrow & & \downarrow & & \downarrow \\ \Gamma_j(\mathfrak{N}) & \times & L_j(n,\chi^{-1},A) & \longrightarrow & L_j(n,\chi^{-1},A) \end{array}$$

where the upward pointing arrow is $\gamma \mapsto \alpha_i \gamma \alpha_i^{-1}$, the downward pointing arrow is $P \mapsto \alpha^{-1} P$, and the horizontal arrows are the usual actions of $\Gamma_i(\mathfrak{N})$, respectively $\Gamma_j(\mathfrak{N})$ (see the discussion surrounding Equation (6)). In this diagram $L_i = L_i(n, \chi, A)$ is defined using s_i. Since $t_i = a_i s_i$, where a_i is considered as a scalar matrix, the modules L_i obtained using t_i or s_i are isomorphic as $\Gamma_i(\mathfrak{N})$-modules, since they differ by a factor of $|a_i|_F^n$. When $A = \mathcal{O}_\wp$ or $K_\wp/\mathcal{O}_\wp$ we may always assume that the p-part of the idele a_i is trivial ($(a_i)_p = 1$) so that the two integral structures on L_i, arising from the two choices of representatives, are the same.

Now, if $\gamma \in \Gamma_j(\mathfrak{N})$, then $d_{\alpha_i \gamma \alpha_i^{-1}} = a_\gamma$, so that $\chi_{\mathfrak{N}}(d_{\alpha_i \gamma \alpha_i^{-1}}) = \chi_{\mathfrak{N}}(a_\gamma)$. This shows that the above diagram is compatible (in the sense of [20], Chapter 7, § 5). Thus there is an induced map on cohomology

$$\mathrm{H}^d(\Gamma_i(\mathfrak{N}), L_i(n, \chi, A)) \mapsto \mathrm{H}^d(\Gamma_j(\mathfrak{N}), L_j(n, \chi^{-1}, A)).$$

Taking the direct sum over all i, and restricting to the parabolic part, gives us the map τ in (31). We note here that if the $\Gamma_i(\mathfrak{N})$ are not torsion free, we still obtain the map (31) when $A = \mathcal{O}_\wp$ for $\wp \notin \mathcal{S}_{\text{elliptic}}$ by the usual procedure of 'taking invariants'.

Define a new pairing

$$[\,,]:\ \mathrm{H}_{\mathrm{p}}^d(Y(\mathfrak{N}), \mathcal{L}(n, \chi, A)) \otimes \mathrm{H}_{\mathrm{p}}^d(Y(\mathfrak{N}), \mathcal{L}(n, \chi, A)) \to A, \tag{32}$$

by setting $[x, y] = \langle x, \tau(y) \rangle$. The advantage of doing this is two-fold: for one, we may now choose x and y both in $\mathrm{H}_{\mathrm{p}}^d(Y(\mathfrak{N}), \mathcal{L}(n, \chi, A))$. Secondly, this new pairing satisfies the following condition with respect to the Hecke operators $T_{\mathfrak{m}}$: $[T_{\mathfrak{m}} x, y] = [x, T_{\mathfrak{m}} y]$. Also, note that since τ preserves the rational (or integral) structure coming from the coefficients, the paring $[\,,]$ is perfect, exactly when the original cup product pairing $\langle\,,\rangle$ is.

The matrix τ also induces a map

$$\tau: S_{k,I_F}(\mathfrak{N}, \chi) \to S_{k,I_F}(\mathfrak{N}, \chi^{-1}),$$

given by

$$f|_\tau(g) = \chi^{-1}(\det g) f(g\tau).$$

When f is primitive, one has

$$f|_\tau = W(f) f_c, \tag{33}$$

where $f_c \in S_{k,I_F}(\mathfrak{N}, \chi^{-1})$ is the cusp form whose Fourier coefficients satisfy $c(\mathfrak{m}, f_c) = \overline{c(\mathfrak{m}, f)}$, and $W(f)$ is a complex number, with $|W(f)| = 1$. Note that $W(f) = 1$ when f is of level 1.

4. Periods of Cusp Forms

In this section, we define 'integral' periods attached to cusp forms $f \in S_{k,I_F}(\mathfrak{N}, \chi)$. These are defined by measuring the difference in the two different integral structures

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put on the cuspidal cohomology groups; one of which arises from the coefficients of these cohomology groups, and the other, via the Eichler–Shimura–Harder isomorphism, from the integral structure defined by Fourier expansion of cusp forms.

4.1. EICHLER–SHIMURA–HARDER ISOMORPHISM

The following isomorphism, due to Harder [9], and worked out explicitly by Hida in [16] (see also Theorem 1.1 in [15]) is a generalization of the classical Eichler–Shimura isomorphism relating the space of cusp forms to cuspidal cohomology groups:

$$\delta_i : \oplus_J S_{k,J}(\Gamma_i(\mathfrak{N}), \chi_{\mathfrak{N}}^{-1}) \xrightarrow{\sim} \mathrm{H}^d_{\mathrm{cusp}}(\Gamma_i(\mathfrak{N})\backslash \mathcal{Z}, \mathcal{L}_i(n, \chi, \mathbb{C})),$$

where the sum is over all holomorphy types J. Here we recall that the sheaf $\mathcal{L}_i(n, \chi, \mathbb{C})$ depends only on $\chi_{\mathfrak{N}}$, the restriction of χ to $\hat{\mathcal{O}}_F^\times$. Taking the direct sum over all i, and using (4), we obtain the isomorphism

$$\delta = \oplus\delta_i : \oplus_\chi \oplus_J S_{k,J}(\mathfrak{N}, \chi) \xrightarrow{\sim} \mathrm{H}^d_{\mathrm{cusp}}(Y(\mathfrak{N}), \mathcal{L}(n, \chi, \mathbb{C})).$$

Note that there are natural actions of Cl_F^+, the strict class group of F, the Hecke operators $T_{\mathfrak{m}}$, and the group $\mathcal{C} = \mathrm{K}_\infty/\mathrm{K}_{\infty+}$ of complex conjugations, on both sides. All these actions commute, and the isomorphism δ is equivariant with respect to these actions.

When $n > 0$, or d is odd and $n = 0$, the cuspidal cohomology groups appearing above are equal to the parabolic cohomology groups, and from now onwards they are identified. But when d is even and $n = 0$, there are ‘invariant cohomology classes’, denoted by Inv, which are parabolic but not cuspidal: in this case $\mathrm{H}^d_{\mathrm{cusp}}$ is the orthogonal compliment of Inv in $\mathrm{H}^d_{\mathrm{p}}$. Thus in this case we will have to modify our argument slightly (see Remark 6 below).

4.2. PERIODS ATTACHED TO f

4.2.1. *Eichler–Shimura Periods*

Fix a primitive holomorphic cusp form $f \in S_{k,I_F}(\mathfrak{N}, \chi)$ (that is $J = I_F$). Let

$$\hat{\mathcal{C}} = \{\epsilon : \mathrm{K}_\infty/\mathrm{K}_{\infty+} \to \{\pm 1\}\}.$$

Let us write $[f]$, respectively $[\epsilon]$, respectively $[\chi]$, for an eigenspace with respect to the Hecke algebra homomorphism corresponding to f, respectively, the character $\epsilon \in \hat{\mathcal{C}}$, respectively the character χ thought of as a character of Cl_F^+. Then if δ_ϵ denotes δ followed by projection to the ϵ eigenspace, we have the following isomorphism induced by δ

$$\delta_\epsilon : S_{k,J}(\mathfrak{N}, \chi)[f] \xrightarrow{\sim} \mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L}(n, \chi, \mathbb{C}))[f, \epsilon, \chi].$$

It is well known (see [16]) that if A is a p.i.d., then the maximal torsion-free quotient of the A-module

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$$\mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L}(n, \chi, A))[f, \epsilon, \chi]$$

is free of rank 1 over A. We let $\eta(f, \epsilon, A)$ denote a generator. Recall that K is a large Galois extension of $\mathbb{Q}$ that contains all the conjugates of F as well as all the Hecke fields. Let now $A = \mathcal{O}_{(\wp)}$ be a valuation ring of K at the prime $\wp$. Then we define the periods

$$\Omega(f, \epsilon, \mathcal{O}_{(\wp)}) \in \mathbb{C}^\times/\mathcal{O}_{(\wp)}{}^\times$$

by

$$\delta_\epsilon(f) = \Omega(f, \epsilon, \mathcal{O}_{(\wp)})\, \eta(f, \epsilon, \mathcal{O}_{(\wp)}). \tag{34}$$

These periods were originally defined by Eichler (in the case $F = \mathbb{Q}$) and Shimura, and they govern the behaviour of the standard L-function attached to f within the critical strip. They will also play a role in describing the value of the adjoint L-function attached to f which we will describe in Section 5 below.

4.2.2. *Hida Periods*

We now define another period that was used by Hida in [10] in the case when $F = \mathbb{Q}$. However we will not use this period in this paper, and we only mention it for completeness sake. In any case, we show that it is related in a simple manner to the Eichler–Shimura periods defined in Section 4.2.1.

We have, in the notation of Section 4.2.1, the embeddings $\mathcal{O}_{(\wp)} \to K \to \mathbb{C}$, which induce maps

$$\mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L}(n, \chi, \mathcal{O}_{(\wp)}))[f, \chi] \to \mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L}(n, \chi, \mathbb{C}))[f, \chi],$$

whose image we call $L_f(\mathcal{O}_{(\wp)})$. Now the latter space is 2^d-dimensional over $\mathbb{C}$, and has as basis the set $\{\delta(f_J) \mid J \subset I_F\}$, where $f_J \in S_{k,J}(\mathfrak{N}, \chi)$ is the cusp form defined by $c(I_F, J) \cdot f = f_J$, where $c(I_F, J) \in \mathcal{C}$ (cf. § 2.6). On the other hand, $L_f(\mathcal{O}_{(\wp)})$ is a free $\mathcal{O}_{(\wp)}$-module of the same rank, say spanned by the basis $\{\eta_j(f, \mathcal{O}_{(\wp)}) \mid j = 1, \ldots, 2^d\}$. Let $U(f, \mathcal{O}_{(\wp)})$ be the change of basis matrix defined via

$$\big(\delta(f_J)\big)_{J\subset I_F} = U(f, \mathcal{O}_{(\wp)})\,\big(\eta_j(f, \mathcal{O}_{(\wp)})\big)_{j=1,\ldots,2^d}$$

and let

$$\Omega(f, \mathcal{O}_{(\wp)}) = \det(U(f, \mathcal{O}_{(\wp)})) \;\in \mathbb{C}^\times/\mathcal{O}_{(\wp)}{}^\times,$$

denote its determinant. The period $\Omega(f, \mathcal{O}_{(\wp)})$ is sometimes referred to as the Hida period. The following Lemma relates it to the Eichler–Shimura periods:

LEMMA 3. *Say $p \neq 2$. Then $\Omega(f, \mathcal{O}_{(\wp)}) = \prod_{\epsilon\in\hat{\mathcal{C}}} \Omega(f, \epsilon, \mathcal{O}_{(\wp)})$, in $\mathbb{C}^\times/\mathcal{O}_{(\wp)}{}^\times$.*

Proof. Up to a power of 2, which can be ignored since we are working in $\mathbb{C}^\times/\mathcal{O}_{(\wp)}{}^\times$ and the residue characteristic p of $\mathcal{O}_{(\wp)}$ is not equal to 2, we may replace

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the basis $(\eta_j(f, \mathcal{O}_{(\wp)}))_{j=1,\dots,2^d}$ by $(\eta(f, \epsilon, \mathcal{O}_{(\wp)}))_{\epsilon\in\hat{\mathcal{C}}}$ in the definition of $\Omega(f, \mathcal{O}_{(\wp)})$. The lemma then follows easily. $\square$

5. Adjoint L-Functions at $s = 1$

In this section we introduce the imprimitive adjoint L-function attached to a primitive holomorphic cusp form $f \in S_{k,I_F}(\mathfrak{N}, \chi)$. Then we quote a well known formula of Shimura relating the value of this L-function at $s = 1$ to the Petersson inner product of f with itself.

5.1. ADJOINT L-FUNCTIONS

We define the imprimitive adjoint L-function $L(s, \mathrm{Ad}(f))$ via its Euler product, following Section 7 of [18]. We note, however, that there are more conceptual descriptions of the adjoint L-function as

- the motivic L-function $L(s, \mathrm{Ad}(\rho_f))$ attached to the adjoint representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$ on traceless two by two matrices, arising from the two dimensional representation ρ_f of $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$ associated to f by Shimura, Ohta, Wiles, Carayol, Taylor and Blasius–Rogawski, or,
- the Langlands L-function $L(s, \pi, r)$ where $\pi = \otimes\pi_\mathfrak{p}$ is the cuspidal automorphic representation of $\mathrm{G}(\mathbb{A})$ corresponding to f, and $r\colon \mathrm{GL}_2(\mathbb{C}) \to \mathrm{SL}_3(\mathbb{C})$ is the adjoint representation of the corresponding L-group, $\mathrm{GL}_2(\mathbb{C})$.

First, fix a prime $\mathfrak{p}$ of F, such that $\mathfrak{p} \nmid \mathfrak{N}$. Define $\alpha_\mathfrak{p}$ and $\beta_\mathfrak{p}$ by

$$1 - c(\mathfrak{p}, f)X - \chi(\mathfrak{p})N_{F/\mathbb{Q}}(\mathfrak{p})X^2 = (1 - \alpha_\mathfrak{p}X)(1 - \beta_\mathfrak{p}X).$$

For such $\mathfrak{p} \nmid \mathfrak{N}$ let

$$L_\mathfrak{p}(s, \mathrm{Ad}(f)) = \left(1 - \frac{\alpha_\mathfrak{p}}{\beta_\mathfrak{p}}N(\mathfrak{p})^{-s}\right)(1 - N(\mathfrak{p})^{-s})\left(1 - \frac{\beta_\mathfrak{p}}{\alpha_\mathfrak{p}}N(\mathfrak{p})^{-s}\right).$$

Now suppose that $\mathfrak{p} \mid \mathfrak{N}$, and that the corresponding local component $\pi_\mathfrak{p}$ of the automorphic representation π corresponding to f is either a minimal principal series, or a minimal special representation. Here minimal means that the conductor of $\pi_\mathfrak{p}$ is minimal among all twists $\pi_\mathfrak{p} \otimes \xi_\mathfrak{p}$, where $\xi_\mathfrak{q}\colon F_\mathfrak{p} \to \mathbb{C}^\times$ is a quasi-character. Then for such $\mathfrak{p} \mid \mathfrak{N}$ let

$$L_\mathfrak{p}(s, \mathrm{Ad}(f)) = \begin{cases} 1 - N(\mathfrak{p})^{-s} & \text{if } \pi_\mathfrak{p} \text{ is principal series and minimal,} \\ 1 - N(\mathfrak{p})^{-s-1} & \text{if } \pi_\mathfrak{p} \text{ is special and minimal.} \end{cases}$$

For all other $\mathfrak{p} \mid \mathfrak{N}$, set $L_\mathfrak{p}(s, \mathrm{Ad}(f)) = 1$. Now let

$$L(s, \mathrm{Ad}(f)) = \prod_\mathfrak{p} L_\mathfrak{p}(s, \mathrm{Ad}(f))^{-1}$$

denote the imprimitive adjoint L-function attached to f.

This is the definition of the imprimitive adjoint L-function given in Equation (7.1) of [18], after correcting for the following typographical errors. In [18], page 241, line 14 there should be an extra Euler factor of $(1-X)$, and here $\mathcal{L}_{\mathfrak{q}}(X)$ should be defined not just for $\pi_{\mathfrak{q}}$ spherical, but for all minimal principal series representations $\pi_{\mathfrak{q}}$. In particular, if $\pi_{\mathfrak{q}}$ is a nonspherical minimal principal series representation, then $\beta_{\mathfrak{q}}=0$, and we recover the Euler factor defined above. Line 15 of the same page should define $\mathcal{L}_{\mathfrak{q}}(X)$ only for *minimal* special representations $\pi_{\mathfrak{q}}$.

We also define the associated Γ-factor by

$$\Gamma(s, \mathrm{Ad}(f)) = \Gamma_{\mathbb{C}}(s+k-1)^d \Gamma_{\mathbb{R}}(s+1)^d,$$

where $\Gamma_{\mathbb{C}}(s)=(2\pi)^{-s}\Gamma(s)$ and $\Gamma_{\mathbb{R}}(s)=\pi^{-s/2}\Gamma(s/2)$ denote the usual real and complex Γ-factors.

5.2. RESIDUE FORMULA

Let f^c denotes the cusp form whose Fourier coefficients satisfy $c(\mathfrak{m}, f^c)=\overline{c(\mathfrak{m}, f)}$. It is well known that $L(s,f,f^c)$, the Rankin–Selberg L-function of f and f^c, has a simple pole at $s=k$. Following Rankin, its residue was first computed by Shimura in [21] by analyzing the integral expression for $L(s,f,f^c)$. Since this residue is essentially $L(1, Ad(f))$, one obtains (see Theorem 7.1 of [18]):

$$\Gamma(1, \mathrm{Ad}(f)) L(1, \mathrm{Ad}(f)) = \frac{2^a}{N_{F/\mathbb{Q}}(\mathfrak{N}) h_F D_F^b} \cdot (f,f), \tag{35}$$

where h_F is the class number of F, D_F is the discriminant of F, a, b are specifiable integers, and (f,f) is the Petersson inner-product of f with itself. For the definition of (f,f) the reader is referred to Equation (7.1) of [14]. We note that the Petersson inner-product that actually occurs in the residue formula (35) above is (f^u, f^u), where f^u is the 'unitarization of f' defined just below (7.2b) in [14]; this is not serious since the two inner products differ by a known power of the discriminant (see [14], (7.2c)). Thus, any ambiguity has been accounted for by introducing the integer b.

6. Main Theorem

Let $f \in S_{k,I_F}(\mathfrak{N}, \chi)$ be a primitive holomorphic cusp form, with Fourier coefficients $c(\mathfrak{m}, f)$. Let $\wp$ be a prime of K. We say that $\wp$ *is a congruence prime for* f if there exists another normalized cusp form $g \in S_{k,I_F}(\mathfrak{N}, \chi)$, that is an eigenform, having Fourier coefficients $c(\mathfrak{m}, g)$, such that

$$c(\mathfrak{m}, f) \equiv c(\mathfrak{m}, g) \pmod{\wp},$$

for all ideals $\mathfrak{m} \subset \mathcal{O}_F$.

Let $\mathcal{S}_F = \{\wp \mid p : p \mid D_F \cdot h_F\}$, let $\mathcal{S}_{\text{level}} = \{\wp \mid p : p \mid N_{F/\mathbb{Q}}(\mathfrak{N})\}$, and when $d=[F:\mathbb{Q}]$ is even and $n=0$, let $\mathcal{S}_{\text{invariant}}$ be defined as in Remark 6 below. Set

$$\mathcal{S} = \mathcal{S}_{\text{duality}} \cup \mathcal{S}_{\text{level}} \cup \mathcal{S}_{\text{F}} \cup \mathcal{S}_{\text{invariant}}. \tag{36}$$

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Note that $\mathcal{S}$ is finite set of primes of K. We have:

THEOREM 5 (Congruence number formula). *Say that* $\wp$ *is a prime of* K *and that* $\wp \notin \mathcal{S}$, *and say* $\epsilon \in \hat{\mathcal{C}}$. *If*

$$\wp \left| \frac{W(f)\,\Gamma(1, \mathrm{Ad}(f))L(1, \mathrm{Ad}(f))}{\Omega(f, \epsilon, \mathcal{O}_{(\wp)})\Omega(f, -\epsilon, \mathcal{O}_{(\wp)})}\right.,$$

then $\wp$ *is a congruence prime for* f.

Proof. For a Dedekind domain $\mathcal{O}_M$ whose quotient field M contains K, let $L(\mathcal{O}_M)$ denote the image of the parabolic cohomology group

$$\mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L}(n, \chi, \mathcal{O}_M))[\chi]$$

in the vector space

$$V(M) := \mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L}(n, \chi, M))[\chi]. \tag{37}$$

For us M will be either K, with $\mathcal{O}_M = \mathcal{O}_K$ or a valuation ring $\mathcal{O}_{(\wp)}$; or M will be a completion $K_\wp$ of K, with $\mathcal{O}_M = \mathcal{O}_\wp$.

Note that $L(\mathcal{O}_M)$ is a lattice in $V(M)$, and we let $d(L(\mathcal{O}_M))$ denote its discriminant with respect to the nondegenerate modified cup product pairing [,] on $V(M)$ (cf. Equation (32)). Recall that

$$\wp \mid d(L(\mathcal{O}_M)) \Longleftrightarrow \wp \in \mathrm{Supp}\left(\frac{L^*(\mathcal{O}_M)}{L(\mathcal{O}_M)}\right), \tag{38}$$

where $L^*(\mathcal{O}_M)$ denotes the lattice which is dual to $L(\mathcal{O}_M)$ with respect to the pairing [,].

Note that $V(M)$ decomposes as

$$V(M) = V(M)[f] \oplus W(M),$$

where

$$V(M)[f] = \mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L}(n, \chi, M))[f, \chi]$$

is the eigenspace of the action of Hecke algebra corresponding to f (and χ), and $W(M)$ is the orthogonal compliment of $V(M)[f]$ with respect to the pairing [,].

Now define

$$L_f(\mathcal{O}_M) := L(\mathcal{O}_M) \cap V(M)[f], \quad L_W(\mathcal{O}_M) := L(\mathcal{O}_M) \cap W(M);$$
$$M_f(\mathcal{O}_M) := \pi_{V[f]}(L(\mathcal{O}_M)), \quad M_W(\mathcal{O}_M) := \pi_W(L(\mathcal{O}_M)),$$

where $\pi_{V[f]}: V \to V[f]$ and $\pi_W: V \to W$ are the two projection maps. Note that $L_f \subset M_f$, respectively $L_W \subset M_W$, are lattices in $V[f]$, respectively, lattices in W. Moreover, one has the following isomorphisms of (finite) $\mathcal{O}_M$-modules:

$$\frac{M_f}{L_f} \xleftarrow{\pi_{V[f]}} \frac{L}{L_f \oplus L_W} \xrightarrow{\pi_W} \frac{M_W}{L_W}. \tag{39}$$

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Let us compute the discriminant $d(L_f(\mathcal{O}_{(\wp)}))$. Following the notation of Section 4.2.1, we have (cf. proof of Lemma 3)

$$\begin{aligned} d(L_f(\mathcal{O}_{(\wp)})) &= \det\Big([\eta_i(f, \mathcal{O}_{(\wp)}), \eta_j(f, \mathcal{O}_{(\wp)})]\Big) \\ &= \det\Big([\eta(f, \epsilon, \mathcal{O}_{(\wp)}), \eta(f, \epsilon', \mathcal{O}_{(\wp)})]\Big), \end{aligned} \tag{40}$$

where we really take the principal ideal generated by the determinants. For $\sigma \in I_F$, let $c_\sigma = c(I_F, I_F \setminus \sigma) \in \mathcal{C}$. Let $a, b \in \mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L}(n, \chi, \mathbb{C}))$. Then, we have

$$[c_\sigma \cdot a, b] = -[a, c_\sigma \cdot b]. \tag{41}$$

Let $-\epsilon \in \hat{\mathcal{C}}$ be the character of $\mathcal{C}$ defined by $(-\epsilon)(c_\sigma) = -(\epsilon(c_\sigma))$ for all $c_\sigma \in \mathcal{C}$. It follows immediately from (41) that

$$[\eta(f, \epsilon, \mathcal{O}_{(\wp)}), \eta(f, \epsilon', \mathcal{O}_{(\wp)})] = 0 \tag{42}$$

unless $\epsilon' = -\epsilon$. Since the residue of characteristic of $\wp$ is odd we may write $L_f(\mathcal{O}_{(\wp)}) = \bigoplus_{\epsilon \in \hat{\mathcal{C}}/\pm 1} L_{f,\epsilon}(\mathcal{O}_{(\wp)})$, where

$$L_{f,\epsilon}(\mathcal{O}_{(\wp)}) = L_f(\mathcal{O}_{(\wp)}) \cap (V(\mathcal{O}_{(\wp)})[\epsilon] \oplus V(\mathcal{O}_{(\wp)})[-\epsilon]).$$

By (42) we see that $L_{f,\epsilon}(\mathcal{O}_{(\wp)})$ pairs trivially with $L_{f,\epsilon'}(\mathcal{O}_{(\wp)})$ unless $\epsilon' = \epsilon$. Moreover, the determinant in (40) breaks into block diagonal form: there are 2^{d-1} two-by-two blocks of the form

$$\begin{pmatrix} 0 & [\eta(f, \epsilon, \mathcal{O}_{(\wp)}), \eta(f, -\epsilon, \mathcal{O}_{(\wp)})] \\ [\eta(f, -\epsilon, \mathcal{O}_{(\wp)}), \eta(f, \epsilon, \mathcal{O}_{(\wp)})] & 0 \end{pmatrix},$$

having determinant (see Equation (34))

$$d(L_{f,\epsilon}(\mathcal{O}_{(\wp)})) = -\frac{[\delta_\epsilon(f), \delta_{-\epsilon}(f)]^2}{(\Omega(f, \epsilon, \mathcal{O}_{(\wp)})\Omega(f, -\epsilon, \mathcal{O}_{(\wp)}))^2}.$$

Now let $F_\infty = \prod_{\sigma \in I_F} c_\sigma$. Then

$$\begin{aligned} &[\delta_\epsilon(f), \delta(f)_{-\epsilon}] \\ &\quad = 2^d \langle F_\infty(\delta(f)), \tau(\delta(f))\rangle \\ &\quad = 2^d\, W(f) \langle F_\infty(\delta(f)), \delta(f_c)\rangle \quad \text{by (33)} \\ &\quad = 2^d\, W(f)\, (f, f), \end{aligned}$$

which, by (35), is just

$$= 2^{d-a}\, W(f)\, N_{F/\mathbb{Q}}(\mathfrak{N})\, h_F\, D_F^b\, \Gamma(1, \mathrm{Ad}(f))\, L(1, \mathrm{Ad}(f)).$$

But, since $\wp \notin \mathcal{S}_F \cup \mathcal{S}_{\mathrm{level}}$, and the residue characteristic of $\wp$ is odd, we may ignore the factor $2^{d-a} N_{F/\mathbb{Q}}(\mathfrak{N})\, h_F\, D_F^b$. Therefore,

$$d(L_{f,\epsilon}(\mathcal{O}_{(\wp)})) = \left(\frac{W(f)\Gamma(1, \mathrm{Ad}(f))L(1, \mathrm{Ad}(f))}{\Omega(f, \epsilon, \mathcal{O}_{(\wp)})\, \Omega(f, -\epsilon, \mathcal{O}_{(\wp)})}\right)^2.$$

Thus, putting things together, we obtain:

$$d(L_f(\mathcal{O}_{(\wp)})) = \prod_{\epsilon \in \hat{\mathcal{C}}/\pm 1} \left(\frac{W(f)\Gamma(1, \mathrm{Ad}(f))L(1, \mathrm{Ad}(f))}{\Omega(f, \epsilon, \mathcal{O}_{(\wp)})\,\Omega(f, -\epsilon, \mathcal{O}_{(\wp)})} \right)^2. \tag{43}$$

Note that

$$\wp \mid d(L_f(\mathcal{O}_K)) \iff \wp \mid d(L_f(\mathcal{O}_{(\wp)})) \iff \wp \mid d(L_f(\mathcal{O}_\wp)). \tag{44}$$

So we may, and now do, work over the completion $K_\wp$.

LEMMA 4. *Assume* $\wp \notin \mathcal{S}$. *Then* $L_f^*(\mathcal{O}_\wp) = M_f(\mathcal{O}_\wp)$.

Proof. Since $\wp \notin \mathcal{S}_{\text{duality}}$, we see by Theorem 3, that the lattice L which is the image of $\mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L}(n, \chi, \mathcal{O}_\wp))$ in $\mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L}(n, \chi, K_\wp))$ is self-dual under the pairing $[\,,]$. For $g \in Cl_F^\times$, we have $[ga, b] = [a, gb]$, for all $a, b \in \mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L}(n, \chi, K_\wp))$. This shows that, for the character χ of $Cl_F^\times$ that we have fixed, $[\,,]$ induces a pairing on the χ eigenspace of the cohomology. Moreover, $\wp \notin S_F$ implies that $\wp \nmid h_F$, so that $L(\mathcal{O}_\wp)$, which is the intersection of the lattice L with the χ eigenspace of the cohomology, is still self dual. It is now a simple exercise in linear algebra to check that the Lemma follows (cf. [10], Equation 4.6). $\square$

By (38), and Lemma 4, we have

$$\begin{aligned} \wp \mid d(L_f(\mathcal{O}_\wp)) &\iff \wp \in \mathrm{Supp}\left(\frac{L_f^*(\mathcal{O}_\wp)}{L_f(\mathcal{O}_\wp)}\right) \\ &\iff \wp \in \mathrm{Supp}\left(\frac{M_f(\mathcal{O}_\wp)}{L_f(\mathcal{O}_\wp)}\right) \\ &\iff \wp \in \mathrm{Supp}\left(\frac{M_f(\mathcal{O}_K)}{L_f(\mathcal{O}_K)}\right). \end{aligned} \tag{45}$$

Thus, by (43) and (45), we conclude that for each $\epsilon \in \hat{\mathcal{C}}$,

$$\wp \;\Big|\; \frac{W(f)\Gamma(1, \mathrm{Ad}(f))L(1, \mathrm{Ad}(f))}{\Omega(f, \epsilon, \mathcal{O}_{(\wp)})\,\Omega(f, -\epsilon, \mathcal{O}_{(\wp)})} \Rightarrow \wp \in \mathrm{Supp}\left(\frac{M_f(\mathcal{O}_K)}{L_f(\mathcal{O}_K)}\right). \tag{46}$$

Now let $k = \mathcal{O}_K/\wp$ be the residue field attached to $\wp$. Let h_f, respectively h_W, denote the images of the Hecke algebra acting on the subspaces $V[f]$, respectively W. By (46) we have that $M_f(\mathcal{O}_K)/L_f(\mathcal{O}_K)) \otimes k \neq 0$. Now h_f acts on $(M_f(\mathcal{O}_K)/L_f(\mathcal{O}_K)) \otimes k$ via the character

$$\begin{aligned} &h_f \to k \\ &T_{\mathfrak{m}} \mapsto c(\mathfrak{m}, f) \pmod{\wp}. \end{aligned}$$

By (39) we have

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$$\frac{M_f(\mathcal{O}_K)}{L_f(\mathcal{O}_K)} \otimes k = \frac{M_W(\mathcal{O}_K)}{L_W(\mathcal{O}_K)} \otimes k,$$

so that there exists a corresponding homomorphism

$$\begin{aligned} &h_W \to k \\ &T_{\mathfrak{m}} \mapsto c(\mathfrak{m}, f) \pmod{\wp}. \end{aligned}$$

By enlarging K (and thus $\mathcal{O}_K$) if necessary we may assume that this homomorphism lifts to a character $h_W \to \mathcal{O}_K$. This shows that there exists a normalized eigenform $g \neq f$, such that $c(\mathfrak{m}, f) \equiv c(\mathfrak{m}, g) \pmod{\wp}$, for all $\mathfrak{m}$, as desired. □

Remark 6. As mentioned earlier, the parabolic and cuspidal cohomology groups coincide, except when the degree d of $F/\mathbb{Q}$ is even, and the weight $n = 0$. In this case we need to modify the argument above slightly. Recall that in this case:

$$\mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L}(0, \chi, \mathbb{C})) = \mathrm{H}^d_{\mathrm{cusp}}(Y(\mathfrak{N}), \mathcal{L}(0, \chi, \mathbb{C})) \oplus \mathrm{Inv}(\mathbb{C}),$$

where $\mathrm{Inv}(\mathbb{C})$ denote the invariant forms. Note further that $\mathrm{Inv}(\mathbb{C})$ is the orthogonal compliment to $\mathrm{H}^d_{\mathrm{cusp}}(Y(\mathfrak{N}), \mathcal{L}(0, \chi, \mathbb{C}))$ under the pairing $[\,,]$. Now there is a natural K-structure on $\mathrm{H}^d_{\mathrm{cusp}}$ coming from the Drinfeld–Manin Theorem (see, for instance, [5], § 7.6), and this is induced by the natural K-structure on $\mathrm{H}^d_{\mathrm{p}}$. Write $\mathrm{Inv}(K)$ for the orthogonal compliment of $\mathrm{H}^d_{\mathrm{cusp}}(Y(\mathfrak{N}), \mathcal{L}(0, \chi, K))$ in $\mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L}(0, \chi, K))$ under $[\,,]$. Then we see that $\mathrm{Inv}(K) \otimes \mathbb{C} = \mathrm{Inv}(\mathbb{C})$. Now define

$$\begin{aligned} &\mathrm{H}^d_{\mathrm{cusp}}(Y(\mathfrak{N}), \mathcal{L}(0, \chi, \mathcal{O}_{(\wp)}) \\ &\quad = \mathrm{H}^d_{\mathrm{cusp}}(Y(\mathfrak{N}), \mathcal{L}(0, \chi, K)) \cap \mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L}(0, \chi, \mathcal{O}_{(\wp)})). \end{aligned}$$

It is easy to check that the self duality of $\mathrm{H}^d_{\mathrm{cusp}}(Y(\mathfrak{N}), \mathcal{L}(0, \chi, \mathcal{O}_{(\wp)})$ under $[\,,]$ follows from the self duality of $\mathrm{H}^d_{\mathrm{p}}(Y(\mathfrak{N}), \mathcal{L}(0, \chi, \mathcal{O}_{(\wp)}))$ under $[\,,]$, at the cost of throwing away finitely many *additional* primes, which we denote by $\mathcal{S}_{\mathrm{invariant}}$. Outside the set $\mathcal{S}_{\mathrm{invariant}}$, the proof of Theorem 5 goes through as before.

6.1. SOME EXPLICIT COROLLARIES

We now utilize the computations made in Section 3.4 of $\mathcal{S}_\partial$ to make Theorem 5 more explicit. Recall that we may write (cf. (20) and (36)):

$$\mathcal{S} = \mathcal{S}_{\mathrm{weight}} \cup \mathcal{S}_{\mathrm{level}} \cup \mathcal{S}_{\mathrm{elliptic}} \cup \mathcal{S}_F \cup \mathcal{S}_{\mathrm{invariant}} \cup \mathcal{S}_\partial.$$

6.1.1. $F = \mathbb{Q}$

Write N for the positive integer that generates the ideal $\mathfrak{N} \subset \mathbb{Z}$. Then we have

$$\mathcal{S}_{\mathrm{weight}} = \{\wp \mid p : p \leqslant k-2\}, \quad \mathcal{S}_{\mathrm{level}} = \{\wp \mid N\},$$

$$\mathcal{S}_{\mathrm{elliptic}} = \{\wp \mid 6\}, \quad \text{and} \quad \mathcal{S}_\partial = \mathcal{S}_{\mathrm{weight}} \cup \mathcal{S}_{\mathrm{level}}$$

by Proposition 3 in Section 3.4.1. Also $\mathcal{S}_F = \mathcal{S}_{\mathrm{invariant}} = \emptyset$.

Thus we see that Theorem 5 just reduces to Theorem 1 of the Introduction (proved by Hida in [10]).

6.1.2. *Real Quadratic Fields*

Now suppose that F is a real quadratic field of strict class number 1, and assume that the level $\mathfrak{N} = 1$ (cf. Assumption 1). In particular since the level is 1, $W(f) = 1$. Under these assumptions we have $\mathcal{S}_{\mathrm{weight}} = \{\wp \mid p : p \leqslant k-2\}$ and $\mathcal{S}_{\mathrm{elliptic}} = \{\wp \mid 30\}$, but now $\mathcal{S}_F = \{\wp \mid D_F\}$ and $\mathcal{S}_{\mathrm{level}} = \emptyset$. Moreover, when $k = 2$, $\mathcal{S}_\partial$ is given by Proposition 4.

When $k > 2$, Corollary 1 in Section 3.4.2 (see also Remark 3) tells us what the torsion in the *ordinary* part of the boundary cohomology can be. Note that the proof of Theorem 5 adapts to the ordinary part of the cohomology. Indeed, since the boundary exact sequence (15) remains valid for the ordinary part of the cohomology, one gets an analog of the duality theorem (Theorem 3) for the ordinary part of parabolic cohomology. Thus, if in the proof of Theorem 5, one replaces the vector space (37) by its ordinary part, one obtains an analog of Theorem 5 for primitive forms f that are ordinary at $\wp$, namely forms for which $\wp \nmid a(p,f)$.

Consequently, we obtain:

COROLLARY 2. *Let F be a real quadratic field of strict class number* 1. *Let ϵ_1 denote a generator of the group of totally positive units of F. Let f be a primitive form of level* 1 *and weight* (k,k). *Let $\wp \mid p$ be a prime of K such that $p > k-2$, $p \nmid 30\cdot D_F \cdot N_{F/\mathbb{Q}}(\epsilon_1^{k-1} - 1)$ and*

$$\begin{cases} \wp \notin \mathcal{S}_{\mathrm{invariant}}, & \text{if } k = 2,\\ \wp \text{ is ordinary for } f, & \text{if } k > 2.\end{cases}$$

Then, if

$$\wp \;\left|\; \frac{\Gamma(1, \mathrm{Ad}(f))\, L(1, \mathrm{Ad}(f))}{\Omega(f,\epsilon)\Omega(f,-\epsilon)}\right.,$$

$\wp$ is a congruence prime for f.

Remark 7. Though the restriction $\wp \notin \mathcal{S}_{\mathrm{invariant}}$ when $k = 2$ is somewhat unpleasant, we remark that this assumption can be avoided in the case when the level is divisible by $p\mathcal{O}_F$, by restricting (as described in weight $k > 2$ above) to the ordinary part of the cohomology. Indeed, since the invariant forms are invariant under the action of $\mathrm{SL}_2(F_\infty)$ (cf. [15], (1.19)), the Hecke operator T_p acts on the invariant forms by degree, which is a nontrivial power of p. Consequently, the ordinary idempotent kills the invariant forms.

6.1.3. *Totally Real Cubic Fields*

Let F be a totally real cubic field. We keep Assumption 1 of the real quadratic case, but restrict to the weight $(2, 2, 2)$ situation. All the bad sets of primes are the same as in the real quadratic situation except that now $\mathcal{S}_{\text{elliptic}} = \{\wp \mid 42\}$, $\mathcal{S}_{\text{invariant}} = \emptyset$, and $\mathcal{S}_{\partial}$ is described by Proposition 5 in Section 3.4.3. We obtain:

COROLLARY 3. *Let F be a totally real cubic field of strict class number* 1 *and let ϵ_1 and ϵ_2 denote generators of the group of totally positive units of F. Let U_F be the product of all the primes that divide both $N_{F/\mathbb{Q}}(\epsilon_1 - 1)$ and $N_{F/\mathbb{Q}}(\epsilon_2 - 1)$. Let f be a normalized newform, that is a common eigenform of the Hecke operators of level* 1 *and weight* $(2, 2, 2)$. *Say $\wp \mid p$ be a prime of K such that $p \nmid 42 \cdot D_F \cdot U_F$. Then, if*

$$\wp \left| \frac{\Gamma(1, \mathrm{Ad}(f))\, L(1, \mathrm{Ad}(f))}{\Omega(f, \epsilon)\Omega(f, -\epsilon)}\right.,$$

$\wp$ is a congruence prime for f.

6.1.4. *Totally Real Fields of Arbitrary Degree*

Finally, let us consider more generally, the case when F is a totally real field of *odd* degree d. We make this parity assumption since, when d is odd, $\mathcal{S}_{\text{invariant}} = \emptyset$. All the other assumptions are as in the cubic case. In particular we assume that the weight is $(2, 2, \ldots, 2)$. All the bad sets of primes are the same as in the cubic situation except for $\mathcal{S}_{\text{elliptic}}$ and $\mathcal{S}_{\partial}$ (the latter is described by Proposition 6 in § 3.4.4). We obtain:

COROLLARY 4. *Let F be a totally real field of strict class number* 1, *odd degree d, and discriminant D_F. Let $\epsilon_1, \epsilon_2, \ldots, \epsilon_{d-1}$ be a basis of the group of totally positive units of F, and let U_F denote the product of the primes which divide each $N_{F/\mathbb{Q}}(\epsilon_j - 1)$, for $j = 1, \ldots, d-1$. Let E_F denote the product of all the primes p such that the maximal totally real subfield of the* pth *cyclotomic field is contained in F.*

Let f be a normalized newform, and a common eigenform of the Hecke operators, of level 1 *and weight* $(2, 2, \ldots, 2)$. *Then there exits a large number field K such that the following is true: for each prime $\wp \mid p$ of K with $p \nmid E_F \cdot D_F \cdot U_F$, if*

$$\wp \left| \frac{\Gamma(1, \mathrm{Ad}(f))\, L(1, \mathrm{Ad}(f))}{\Omega(f, \epsilon)\Omega(f, -\epsilon)}\right.$$

then $\wp$ is a congruence prime for f.

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References

1. Borel, A. and Serre, J.-P.: Corners and arithmetic groups, *Comment. Math. Helv.* **48** (1973), 436–491.
2. Diamond, F.: The Taylor–Wiles construction and multiplicity one, *Invent. Math.* **128**(2) (1997), 379–391.
3. Doi, K. and Hida, H.: Unpublished manuscript, 1978.
4. Doi, K., Hida, H. and Ishii, H.: Discriminants of Hecke fields and twisted adjoint L-values for GL(2), *Invent. Math.* **134**(3) (1998), 547–577.
5. Franke, J.: Harmonic analysis in weighted L_2-spaces, *Ann. Sci. École Norm. Sup.* **31**(2) (1998), 181–279.
6. Fujiwara, K.: Deformation rings and Hecke-algebras in the totally real case, Preprint, 1996.
7. Ghate, E.: Congruences between base-change and non-base-change Hilbert modular forms, In: *Proc. International Conf. Cohomology of Arithmetic Groups, Automorphic Forms and L-functions*, Novosa, New Delhi, 2001, pp. 33–62.
8. Ghate, E.: On the freeness of the integral cohomology groups of Hilbert–Blumenthal varieties as Hecke-modules, Preprint, 2001.
9. Harder, G.: Eisenstein cohomolgy of arithmetic groups, The case GL_2, *Invent. Math.* **89** (1987), 37–118.
10. Hida, H.: Congruence of cusp forms and special values of their zeta functions, *Invent. Math.* **63** (1981), 225–261.
11. Hida, H.: On congruence divisors of cusp forms as factors of the special values of their zeta functions, *Invent. Math.* **64** (1981), 221–262.
12. Hida, H.: Modules of congruence of Hecke algebras and L-functions associated with cusp forms, *Amer. J. Math.* **110**(2) (1988), 323–382.
13. Hida, H.: On p-adic Hecke algebras for GL_2 over totally real fields, *Ann. of Math.* **128**(2) (1988), 295–384.
14. Hida, H.: On p-adic L-functions of $\mathrm{GL}_2 \times \mathrm{GL}(2)$ over totally real fields, *Ann. Inst. Fourier* **41**(2) (1991), 311–391.
15. Hida, H.: p-Ordinary cohomology groups for SL(2) over number fields, *Duke Math. J.* **69** (1993), 259–314.
16. Hida, H.: On the critical values of L-functions of GL(2) and $\mathrm{GL}(2) \times \mathrm{GL}(2)$, *Duke Math. J.* **74** (1994), 431–528.
17. Hida, H.: *Modular Forms and Galois Cohomology*, Cambridge Univ. Press, Cambridge, 2000.
18. Hida, H. and Tilouine, J.: Anti-cyclotomic Katz p-adic L-functions and congruence modules., *Ann. Sci. École Norm. Sup.* **26**(4) (1993), 189–259.
19. Ribet, K.: Mod p Hecke operators and congruences between modular forms, *Invent. Math.* **71**(1) (1983), 193–205.
20. Serre, J.-P.: *Local Fields,* Grad. Texts in Math. 67, Springer-Verlag, Berlin, 1979.

21. Shimura, G.: The special values of the zeta functions associated with Hilbert modular forms, *Duke Math. J.* **45**(3) (1978), 637–679.
22. Turnbull, H.W.: *Theory of Determinants, Matrices and Invariants*, Blackie, Glasgow, 1945.
23. Urban, E.: Formes automorphes cuspidales pour GL(2) sur un corps quadratique imaginare. Valeurs spéciales de fonctions *L* et congruences, *Compositio Math.* **99**(3) (1995), 283–324.
24. van der Geer, G.: *Hilbert Modular Surfaces*, Springer-Verlag, Berlin, York, 1988.