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THE COMMUTATION RELATION $i[Y, Z] = 2Y$ AND THE ABSOLUTELY CONTINUOUS SPECTRUM OF Y

J. V. CORBETT

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Abstract

A relation between positive commutators and absolutely continuous spectrum is obtained. If $i[Y, Z] = 2Y$ holds on a core for Z and if Y is positive then we have a system of imprimitivity for the group $\mathbf{R}_*^+$ on $\mathbf{R}_*^+$, from which it follows that Y has no singular continuous spectrum.

Assume that Y and Z are self-adjoint operators on a separable Hilbert space $\mathcal{H}$ and that

$$i[Y, Z]f = 2Yf \tag{1}$$

for all f belonging to a dense subset D of $\mathcal{H}$. We obtain conditions under which the relation (1) implies that the singular continuous spectrum of Y is empty.

The argument is simple. We first show that if Y is positive and if

$$e^{-iZs}Ye^{iZs}u = e^{2s}Yu \tag{2}$$

for all $u \in D(Y)$ and all $s \in \mathbf{R}$, then the singular continuous spectrum of Y is empty. We then obtain conditions on the subset D that ensure that whenever (1) holds then (2) holds. We also obtain a converse to this, namely, if Y is a positive self-adjoint operator with absolutely continuous spectrum on $[0, \infty)$ and uniform spectral multiplicity then there exists a self-adjoint operator Z such that (1) holds.

THEOREM 1. *Let Y be a positive self-adjoint operator and U_s a unitary representation of the real line, such that for all $u \in D(Y)$ and all $s \in \mathbf{R}$*

$$U_s^{-1}YU_su = e^{-2s}Yu,$$

then if the spectrum of Y is continuous it is absolutely continuous.

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PROOF. For any complex number ω,

$$U_s^{-1}(Y-\omega I)U_s u = e^{-2s}(Y-e^{2s}\omega I)u$$

for all $u \in D(Y)$ and all real s. Therefore, if the imaginary part of ω is non-zero, $(Y-\omega I)$ is invertible, and

$$U_s^{-1}(Y-\omega I)^{-1}U_s = e^{2s}(Y-e^{2s}\omega I)^{-1}.$$

This last equation holds as an operator identity in $B(\mathcal{H})$ for all $s \in \mathbf{R}$.

Assume that the continuous spectrum of Y is non-empty and contains the interval Δ. The spectral projection $E_\Delta(Y)$ is given by Stone's formula

$$E_\Delta(Y) = \lim_{\varepsilon\to 0+} (2\pi i)^{-1}\int_\Delta \left[(Y-\omega I)^{-1} - (Y-\bar{\omega} I)^{-1}\right] d\mu$$

where we have written $\omega = \mu + i\varepsilon$ and $\bar{\omega} = \mu - i\varepsilon$.
Therefore

$$\begin{aligned} U_s^{-1}E_\Delta(Y)U_s &= \lim_{\varepsilon\to 0+} e^{2s}(2\pi i)^{-1}\int_\Delta \left[(Y-e^{2s}\omega)^{-1} - (Y-e^{2s}\bar{\omega})^{-1}\right] d\mu \\ &= \lim_{\varepsilon_0\to 0+} (2\pi i)^{-1}\int_{e^{2s}\Delta} \left[(Y-\eta-i\varepsilon_0)^{-1} - (Y-\eta+i\varepsilon_0)^{-}\right] d\eta \\ &= E_{e^{2s}\Delta}(Y) \end{aligned} \tag{3}$$

where we have put $\eta + i\varepsilon_0 = e^{2s}\omega$.

Let β be any Borel subset in the continuous spectrum of Y, then by the usual construction of Borel subsets from intervals we obtain

$$U_s^{-1}E_\beta(Y)U_s = E_{e^{2s}\beta}(Y). \tag{4}$$

Let $\mathbf{R}_*^+ = (0,\infty)$ denote the multiplicative group of positive real numbers. We obtain a representation V_a of $\mathbf{R}_*^+$ from the representation U_s of $\mathbf{R}$ by putting $a = e^{2s}$ for all $s \in \mathbf{R}$, and observing that

$$V_a = U_{\frac{1}{2}\ln a} \text{ for all } a \in \mathbf{R}_*^+ .$$

By hypothesis Y is positive definite and so its spectrum is contained in $[0,\infty)$. By spectral multiplicity theory, the set of all spectral projections $\{E_\beta(Y);\ \beta$ a Borel subset of $[0,\infty)\}$ has a separating vector Φ. In fact, Φ is a cyclic vector for the commutant of this family of projections.

The measure $v(\Delta) = \langle\Phi, E_\Delta(Y)\Phi\rangle$, defined on the Borel subsets of $\mathbf{R}_*^+$, is equivalent to the Haar measure of $\mathbf{R}_*^+$. To see this, first observe that because Φ is separating if Δ_0 is a Borel subset of $\mathbf{R}_*^+$ such that $v(\Delta_0) = 0$ then $E_{\Delta_0}(Y) = 0$, and therefore

$$\left\langle \Phi, V_a^{-1}E_{\Delta_0}(Y)V_a\Phi\right\rangle = 0 \tag{5}$$

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for all $a \in \mathbf{R}_*^+$. On the other hand when equation (4) is written in terms of the representation V_a of the multiplicative group $\mathbf{R}_*^+$ we obtain $V_a^{-1}E_{\Delta_0}(Y)V_a = E_{a\Delta_0}(Y)$. Therefore

$$v(a\Delta_0) = \left\langle \Phi, E_{a\Delta_0}(Y)\Phi \right\rangle = 0 \tag{6}$$

for all $a \in \mathbf{R}_*^+$. This means that v is a Borel measure on $\mathbf{R}_*^+$ that is quasi-invariant with respect to the action of $\mathbf{R}_*^+$ on itself, and therefore v is equivalent to Haar measure of $\mathbf{R}_*^+$ on $\mathbf{R}_*^+$.

The absolute continuity of the spectrum of Y follows because the Haar measure of $\mathbf{R}_*^+$ on $\mathbf{R}_*^+$ is absolutely continuous with respect to Lebesgue measure. Let the Borel subset S of $\mathbf{R}$ have Lebesgue measure zero, that is, $|S| = 0$. If S is a subset of $\mathbf{R}_*^+$, $v(S) = 0$ and therefore $E_S(Y)\phi = 0$ and $E_S(Y) = 0$ because Φ is separating. If S is not a subset of $\mathbf{R}_*^+$ then $S = S_1 \cup S_2$ where S_2 is a subset of $\mathbf{R}_*^+$ and S_1 lies in the complement of $\mathbf{R}_*^+$. Now $E_S(Y) = E_{S_1}(Y) + E_{S_2}(Y)$ where $E_{S_2}(Y) = 0$ by the argument given above and $E_S(Y) = 0$ by the positivity of Y and the continuity of spectrum of Y.

This theorem shows that the spectral measure class of the positive operator Y is equivalent to the Haar measure of the multiplicative group of the positive reals, $\mathbf{R}_*^+$, on itself. The equation (1) defines a system of imprimitivity of the group $\mathbf{R}_*^+$. The proof is modelled on Mackey's approach to the representations of the canonical commutation relations [4].

DEFINITION. *Let Y be a positive self-adjoint operator in a Hilbert space $\mathcal{H}$. A subset D of $\mathcal{H}$ is said to be a domain of integration for the self-adjoint operator Z with respect to the relation*

$$i[Y, Z] = 2Y \tag{7}$$

if

$$(YZ - ZY)f = -2iYf \tag{8}$$

for all $f \in D$ implies that

$$e^{iZs}Ye^{-iZs}u = e^{-2s}Yu \tag{9}$$

for all $u \in D(Y)$ and all $s \in \mathbf{R}$.

The terminology reflects the fact that equation (8) can be obtained from equation (9) by differentiating with respect to s at $s = 0$. An immediate consequence of this definition and Theorem 1 is the following result:

THEOREM 2. *Let D be a domain of integration for Z and the relation* (7) *and suppose that Y is positive definite, then whenever $i[Y, Z]f = 2Yf$ for all $f \in D$ the singular continuous spectrum of Y is empty.*

The problem of finding a domain of integration for the operator Z and relation (7) is related to the problem of lifting a representation of a Lie algebra as skew-adjoint operators on a Hilbert space to a unitary representation of the corresponding Lie group. Nelson's theorem [5] gives necessary and sufficient conditions for the solution of the general problem, and can be used for our problem. Nevertheless, we present a criterion for D modelled on a result of Kato [2] for the problem of obtaining the Weyl commutation relations from those of Heisenberg (see also Cartier [1]).

THEOREM 3. *Let D be a subset of $D(YZ) \cap D(ZY)$ on which equation* (8) *holds with Y positive. D is a domain of integration for Z and relation* (7) *if D is a core for Z.*

PROOF. Since D is a core for Z there is an $\alpha \neq 0$ such that $(Z - i\alpha)D$ is dense in $\mathcal{H}$. If $\varepsilon > 0$, $(Y + \varepsilon I)$ is strictly positive and symmetric and hence $(Y + \varepsilon)(Z - i\alpha)D$ is dense in $\mathcal{H}$.

Let $f \in D$ and put $u = (Y + \varepsilon)(Z - i\alpha)f$. Then $u = (Z - i(\alpha + 2))(Y + \varepsilon)f + 2i\varepsilon f$, and hence $(Z - i\alpha)^{-1}(Y + \varepsilon)^{-1}u = f = (Y + \varepsilon)^{-1}(Z - i(\alpha + 2))^{-1}(u - 2i\varepsilon f) = (Y + \varepsilon)^{-1}(Z - i(\alpha + 2))^{-1}u + \varepsilon(Y + \varepsilon)^{-1}[(Z - i\alpha)^{-1} - (Z - i(\alpha + 2))^{-1}](Y + \varepsilon)^{-1}u$. But $u \in (Y + \varepsilon)(Z - i\alpha)D$ and thus we have the operator equation

$$(Z - i\alpha)^{-1}(Y + \varepsilon)^{-1} - (Y + \varepsilon)^{-1}(Z - i(\alpha + 2))^{-1} = \varepsilon(Y + \varepsilon)^{-1}\big((Z - i\alpha)^{-1} - (Z - i(\alpha + 2))^{-1}\big)(Y + \varepsilon)^{-1}. \tag{10}$$

We now prove by induction that

$$(Z - i\alpha)^{-n}(Y + \varepsilon)^{-1} - (Y + \varepsilon)^{-1}(Z - i(\alpha + 2))^{-n} = \varepsilon(Y + \varepsilon)^{-1}(Z - i\alpha)^{-n} - (Z - i(\alpha + 2))^{-n}(Y + \varepsilon)^{-1}. \tag{11}$$

for all positive integers n. It is true for $n = 1$; assume it is true for n and write $P_0 = (Z - i\alpha)^{-1}$, $P_2 = (Z - i(\alpha + 2))^{-1}$, and $Q = (Y + \varepsilon)^{-1}$. Then

$$\begin{aligned} P_0^{n+1}Q - QP_2^{n+1} &= P_0^n(P_0Q - QP_2) + (P_0^nQ - QP_2^n)P_2 \\ &= \varepsilon\{P_0^nQ(P_0Q - P_2Q) + (QP_0^n - QP_2^n)QP_2\} \\ &= \varepsilon\{QP_0^{n-1}Q - QP_2^{n+1}Q\}, \end{aligned}$$

on substituting for P_0^nQ and QP_2 in the penultimate line. The argument now goes exactly as in [2]. Use the Neumann series for $(Z - i\beta)^{-1}$ and the fact that $(Z - \omega)^{-1}$ is analytic for $\operatorname{Im}\omega \neq 0$ to extend the validity of (11) from $\omega = i\alpha$ to $\omega = i\beta$ for all real β, $\beta \neq 0$, $\beta \neq -2$.

Multiply equation (11) by $(-i\alpha)^n$ and set $\alpha = n/s$ with $s \neq 0$. $(Z - i\alpha)^n$ becomes $(1 + in^{-1}sZ)^{-n}$ and $(Z - i(\alpha + 2))^{-n}$ becomes $(1 + n^{-1}s(2 + iZ))^{-n}$. Both these expressions have strong limits as n tends to infinity:

$$(1 + in^{-1}sZ)^{-n} \to e^{isZ} \quad \text{and}$$
$$(1 + n^{-1}s(2 + iZ))^{-n} \to e^{-2s}e^{-iZs}.$$

Therefore

$$e^{-iZs}(Y + \varepsilon)^{-1} - (Y + \varepsilon)^{-1}e^{iZs}e^{-2s}$$
$$= \varepsilon(Y + \varepsilon)^{-1}(e^{-iZs} - e^{iZs}e^{-2s})(Y + \varepsilon)^{-1},$$

and, for all $g \in D(Y)$,

$$(Y + \varepsilon)e^{-iZs}g - e^{-iZs}e^{-2s}(Y + \varepsilon)g = \varepsilon(e^{-iZs} - e^{-iZs}e^{-2s})g,$$

or

$$e^{iZs}Ye^{-iZs}g = e^{-2s}Yg.$$

Putting these results together we have the useful corollary of Theorem 3.

COROLLARY. *Let Y and Z be self-adjoint operators on a separable Hilbert space $\mathcal{H}$ and suppose that Y is positive. Let D be a subset of $D(YZ) \cap D(ZY)$ such that for all $f \in D$*

$$i[Y, Z]f = 2Yf$$

and suppose that D is a core for Z. Then the singular continuous spectrum of Y is empty.

We will now use this corollary in a number of examples.

EXAMPLES. 1.

$$\mathcal{H} = L^2([a, b]), \qquad 0 < a < b < \infty,$$
$$Y = -\frac{d^2}{dx^2} \quad \text{on } D(Y), \qquad Z = \frac{1}{2i}\left(x\frac{d}{dx} + \frac{d}{dx}x\right) \quad \text{on } D(Z),$$

where

$$D(Y) = \{f \in \mathcal{H} \mid f \in AC^2[a, b], f(a) = 0 = f(b)\},$$
$$D(Z) = \{f \in \mathcal{H} \mid f \in AC[a, b], xf \in AC[a, b] \text{ and } a\sqrt{f(a)} = \sqrt{b}\, f(b)\},$$
$$AC[a, b] = \{f \in \mathcal{H} \mid f(x) \text{ is absolutely continuous on } [a, b] \text{ and } f'(x) \in \mathcal{H}\},$$
$$AC^2[a, b] = \{f \in \mathcal{H} \mid f \text{ is differentiable, } f' \text{ is absolutely continuous and } f'' \in \mathcal{H}\}.$$

With these domains, Y and Z are self-adjoint and Y is positive. We take $D \subset D(YZ) \cap D(XY)$ to be $C_0^\infty[a, b]$, the set of C^∞ functions with compact support in $[a, b]$ whose support stays away from the end points. Then for all $f \in D$,

$$i[Y, Z]f = 2Yf.$$

We know that the spectrum of Y is not absolutely continuous, but this does not contradict Theorem 3 as D is not a core for Z. For any real number $\alpha \neq 0$, $(Z - i\alpha)D$ is not dense in $L^2[a, b]$, because the function $u(x) = Ax^{\alpha - 1/2}$ is orthogonal to $(Z - i\alpha)D$. In fact this function is orthogonal to $(Z - i\alpha)(D(YZ) \cap D(ZY))$.

2.

$$\mathcal{H} = L^2([a, b]), \qquad 0 < a < b < \infty,$$

Y is the multiplicative operator, $(Yf)(x) = x^2 f(x)$, with $D(Y) = \mathcal{H}$. $Z = -(1/2i)(xd/dx + (d/dx)x)$ on $D(Z)$ as in example (1).

Both Y and Z are self-adjoint, Y is positive, and if we take $D \subset D(YX) \cap D(ZY)$ to be $C_0^\infty[a, b]$ as in example (1), then for all $f \in D$,

$$i[Y, Z]f = 2Yf.$$

The argument of example (1) yields the result that D is not a core for Z, even though we know that the spectrum of Y is absolutely continuous. This shows that the conditions of Theorem 4 are not necessary. What goes wrong in this example is that it is *not* true that $e^{-iZs}Ye^{iZs}f = e^{-2s}Yf$ for all $f \in D(Y)$. This example should be compared with the usual particle in a box counterexample to the uniqueness of the representation for the Heisenberg commutation relations.

3.

$$\mathcal{H} = L^2(0, \infty),$$

Y is the operator of multiplication, $(Yf)(\lambda) = \lambda f(\lambda)$ and

$$D(Y) = \left\{ f \in \mathcal{H} \,\middle|\, \int_0^\infty \lambda^2 |f(\lambda)|^2 \, d\lambda < \infty \right\},$$

$$Z = -\frac{1}{i}\left(\lambda \frac{d}{d\lambda} + \frac{d}{d\lambda}\lambda\right) \quad \text{with domain}$$

$$D(Z) = \Big\{ f \in L^2(0, \infty) \,|\, f \in AC[0, \infty), \lambda f \in AC[0, \infty)$$
$$\text{and } \lim_{a \to 0+} \sqrt{a}\, f(a) = \lim_{b \to \infty} \sqrt{b}\, f(b) \Big\}.$$

The last condition in the description of the domain of Z should be taken to mean that both limits exist and are equal.

With these domains, Y and Z are self-adjoint and Y is positive. Furthermore we know that the spectrum of λ is absolutely continuous. This does follow from Theorem 4 because if D is taken to be $C_0^\infty[0, \infty]$ with the support of the functions staying away from zero and infinity, then D is a core for Z; in fact $(Z - i\alpha)D$ is dense in $L^2([0, \infty))$ for any real $\alpha \neq 0$. This is so because if $(Z - i\alpha)D$ were not dense there must be an element $\omega \neq 0$ that is perpendicular to $(Z - i\alpha)D$, but the only possible ω are of the form $Ax^{\alpha - 1/2}$ which are not in $L^2([0, \infty))$.

4. In non-relativistic quantum theory, the commutation relation (7) arises with $Y = H_0$, the kinetic energy or free Hamiltonian operator, and $Z = A$, the generator of the one parameter group of dilations. In the usual Schrödinger representation for a single particle, $H_0 = \mathbf{p}^2$, $A = \frac{1}{2}(\mathbf{x} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{x})$ with $\mathbf{p}$ representing the canonical momentum operator and $\mathbf{x}$ the canonical position operator. Further, H_0 and A are self-adjoint operators on their natural domains. It is well known that the spectrum of H_0 is $[0, \infty)$ and is purely absolutely continuous. The connection with this paper can be made directly but it is more interesting to notice that in the usual spectral representation of H_0, [3], we have a unitary map U from $L^2(\mathbf{R}^3)$ to $L^2(\mathbf{R}_+, d\lambda; \mathcal{H}')$, where $\mathcal{H}' = L^2(S^2, d\Omega)$, and S^2 is the unit sphere in $\mathbf{R}^3$, and $d\Omega$ its usual surface measure, that sends H_0 to multiplication by λ and A to the operator $Z = -(1/i)(\lambda d/d\lambda + (d/d\lambda)\lambda)$ that is discussed in example (3). Explicitly if $\hat{f}$ denotes the Fourier transform of an element of f of $L^2(\mathbf{R}^3)$ then $(Uf)(\lambda; \omega) = (\sqrt{2})^{-1}\lambda^{1/4}\hat{f}(\lambda^{1/2}\omega)$.

As a result of these last two examples we are led to the following proposition.

PROPOSITION. *Let $\mathcal{H}$ be a separable Hilbert space. If Y is a positive self-adjoint unbounded operator with absolutely continuous spectrum on* $[0, \infty]$ *and uniform spectral multiplicity then there exists a self-adjoint operator Z such that*

$$i[Y, Z]f = 2Yf$$

for all f belonging to a domain of integration Z.

PROOF. By hypothesis, Y has a spectral representation as multiplication by λ a Hilbert space $\mathcal{H} = L^2(\mathbf{R}^+, d\lambda; \mathcal{H}')$ for some constant fibre $\mathcal{H}'$. But by Example 3 the operator $Z_0 = -(\frac{1}{i})(\lambda d/d\lambda + (d/d\lambda)\lambda)$, with domain $D(Z_0)$ given in that example, is self-adjoint and for all $f \in C_0^\infty(\mathbf{R}^+; \mathcal{H}')$

$$i[\lambda, Z_0]f = 2\lambda f.$$

Now the pre-image of Z_0 under the unitary map U of Example 4 gives a self-adjoint operator Z on $D(Z) \subset \mathcal{H}$ such that $i[Y, Z] = 2Y$ on a domain of integration for Z.

This proposition gives a partial converse to Theorem 2 and appears to be useful in non-relativistic scattering theory. We hope to discuss this connection in a subsequent paper.

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School of Mathematics and Physics
Macquarie University
North Ryde
N. S. W. 2113