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# THE COVERING DIMENSION OF WOOD SPACES

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**Abstract.** A Banach space is called (*almost*) *transitive* if the isometry group acts (*almost*) *transitively* on the unit sphere. The main problems around transitivity are the Banach-Mazur conjecture that the only separable and transitive Banach spaces are the Hilbert ones (1930) and the Wood conjecture that $C_0(L)$ cannot be almost transitive in its natural supremum norm unless $L$ is a singleton (1982). In this note we give necessary and sufficient conditions on the locally compact space $L$ for the (almost) transitivity of $C_0(L)$. This will clarify the topological content of Wood's problem.

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**1 Introduction.** A Banach space $X$ is said to be *transitive* if the group $G(X)$ of (linear surjective) isometries of $X$ acts transitively on the unit sphere $S(X)$. The space $X$ is said to be *almost transitive* if the orbits of $G(X)$ are dense in $S(X)$. This means that, given $f, g \in S(X)$ and $\varepsilon > 0$, there exists $T \in G(X)$ such that $\|g - Tf\| \leq \varepsilon$.

There are considerable difficulties in deciding whether certain natural classes of Banach spaces contain an (almost) transitive member or not. For instance, it is not known if all separable transitive Banach spaces are Hilbert spaces. (This is the famous Mazur rotations problem which remains unsolved since 1932 [**1**]; see [**14**, **3**, **2**] for information on the topic.)

Another notorious example is a problem of Wood [**17**] concerning the existence of an almost transitive $C_0(L)$ space, apart from the obvious case in which $L$ is a singleton. As usual, we denote by $C_0(L)$ or $C_0(L, \mathbb{K})$ the Banach space of continuous $\mathbb{K}$-valued functions on the locally compact space $L$ vanishing at infinity, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. (In this paper, only completely regular Hausdorff spaces are considered.) Wood conjectured in [**17**] that all $C_0(L)$ spaces lack almost transitive norm. Greim and Rajalopagan [**8**] proved the conjecture for $\mathbb{K} = \mathbb{R}$. Nevertheless, the problem is wide open for $\mathbb{K} = \mathbb{C}$. (Hence, in the sequel we refer only to complex spaces.) By the Banach-Stone theorem, every isometry of $C_0(L)$ has the form

$$Tf = \sigma(f \circ \varphi),$$

where $\sigma : L \to \mathbb{C}$ is a continuous unimodular function and $\varphi$ is a homeomorphism of $L$. In this way, the properties of the isometry group of $C_0(L)$ can be translated into topological properties of $L$ and vice-versa.

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The object of this note is to study necessary and sufficient (topological) conditions on $L$ for the (almost) transitivity of $C_0(L)$. This will clarify the topological meaning of Wood's problem.

DEFINITION 1. Let us say that (a locally compact space) $L$ is a *Wood space* (resp. an *almost Wood space*) if $C_0(L)$ is a transitive (resp. almost transitive) Banach space with dimension greater than one.

The main results of [**17**, **8**, **4**, **5**] can be summarized as follows.

(1) A Wood space exists if and only if an almost Wood space exists [**8**] if and only if there exists an almost Wood space $L$ whose one-point compactification $\alpha L$ is metrizable [**4**].

(2) The one-point compactification of an almost Wood space is always connected and no almost Wood space is zero-dimensional [**17**].

(3) Let $L$ be a Wood space. Then ($a$) $L$ is not first countable; ($b$) $L$ is countably compact; ($c$) countable unions of compact sets are relatively compact; ($d$) $L$ is an uncountable union of nowhere dense components or connected; ($e$) $L$ is not totally disconnected and non-void open $K_\sigma$ sets and their closures are not zero-dimensional; ($f$) every compact subset of $L$ is an $F$-space; ($g$) the closure of a relatively compact open $K_\sigma$ set is its Stone-Čech compactification; ($h$) every open $K_\sigma$ set contains an open subset whose closure does not coincide with its Stone-Čech compactification; ($i$) every infinite compact subset of $L$ contains a copy of $\beta\mathbb{N}$; ($j$) every non-empty compact $G_\delta$ contains an open set; ($k$) all non-empty open $K_\sigma$ subsets are homeomorphic; ($l$) all non-empty compact $G_\delta$ subsets are homeomorphic [**8**].

**2. The covering dimension of Wood spaces.** Let us start with the following improvement of (3$f$). Throughout the paper, $\dim L$ will denote the covering dimension of the topological space $L$. We refer the reader to [**6**] or [**7**] for definitions and basic dimension theory.

THEOREM 1. *Let $K$ be a compact subset of a Wood space. Then $K$ is an $F$-space of dimension at most one. Moreover, if $K$ has non-empty interior (for instance, if $K$ is $G_\delta$), then* $\dim K = 1$.

LEMMA 1. *Let $L$ be a locally compact space. Then there are unimodular maps $\sigma_1, \sigma_2 : L \to \mathbb{C}$ such that $(\sigma_1 + \sigma_2)/2$ belongs to $C_0(L)$ and $\|(\sigma_1 + \sigma_2)/2\| = 1$.*

*Proof of Lemma 1.* Let $\sigma_1(x) = 1$ for every $x \in L$. To construct $\sigma_2$, take a continuous map $\tau : L \to \mathbb{R}$ such that $\lim \tau(x) = -\pi$ as $x \to \infty$ and $\tau(y) = 0$ for some $y \in L$. Let $\sigma_2(x) = \exp(\tau(x)i)$. Then $(\sigma_1 + \sigma_2)/2 \to 0$ as $x \to \infty$. Clearly $\|(\sigma_1 + \sigma_2)/2\| \leq 1$ and, since $(\sigma_1 + \sigma_2)(y) = 2$, one has $\|(\sigma_1 + \sigma_2)/2\| = 1$, as desired. □

*Proof of Theorem 1.* Let $K$ be a compact subset of a Wood space $L$. We should like every $f \in C(K) = C(K, \mathbb{C})$ with $\|f\|_{C(K)} \leq 1$ to be written as the average of two extreme points of the unit ball of $C(K)$. By a result of Robertson [**13**], this implies that $K$ is an $F$-space and also that $\dim K \leq 1$. Let $f_0 \in C_0(L)$ be an extension of $f$ with $\|f_0\| = 1$. Take $g = (\sigma_1 + \sigma_2)/2$, where $\sigma_1$ and $\sigma_2$ are as in the Lemma. By transitivity of $C_0(L)$ one has $g = Tf_0$ for some isometry $T$ of $C_0(L)$. This implies the existence of a homeomorphism $\varphi$ and a unimodular map $\sigma$ for which

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$$f_0 = \sigma(g \circ \varphi)/2 = \{\sigma(\sigma_1 \circ \varphi) + \sigma(\sigma_2 \circ \varphi)\}/2.$$

Obviously $\sigma(\sigma_1 \circ \varphi)$ and $\sigma(\sigma_2 \circ \varphi)$ are unimodular and, therefore, $f$ is a midpoint of two extreme points of the unit ball of $C(K)$. This proves the first statement. The second one now follows from 3($e$). □

Theorem 1 shows that Wood spaces (if they exist) are locally one-dimensional. Unfortunately, a locally one-dimensional space need not be itself one-dimensional without some additional hypothesis such as paracompactness (or weakly paracompactness; see [**6**, Theorem 3.1.14, p. 214]. Nevertheless, we shall see later that the existence of an almost Wood space implies the existence of a one-dimensional Wood space. This will follow from the following result.

THEOREM 2. *Let $L$ be an almost Wood space such that $\alpha L$ is metrizable. Then* $\dim L = \dim \alpha L = 1$.

*Proof*. Since the covering dimension is monotone for normal spaces it obviously suffices to show that $\dim \alpha L = 1$. We first prove that $\dim K \leq 1$ for every compact subset of $L$. By Theorem 18 of [**11**] (see also [**12**]), $\dim K \leq 1$ if and only if the functions that omit the origin are dense in the unit ball of $C(K)$ or, which is the same, every norm-one $f \in C(K)$ can be approximated by elements of the unit ball that omit the origin.

Fix $f \in C(K)$ with $\|f\| = 1$ and $\varepsilon > 0$ and take a norm-one $g \in C_0(L)$ such that $g(x) \neq 0$ for all $x \in L$. The existence of such a $g$ is guaranteed by the metrizability of $\alpha L$. Let $f_0$ be an extension of $f$ in the unit sphere of $C_0(L)$. By almost transitivity of $C_0(L)$ one can find an isometry $T$ such that $\|f_0 - Tg\| \leq \varepsilon$. Clearly $\|f - (Tg)_{|K}\|_{C(K)} \leq \varepsilon$ and $Tg$ does not vanish on $K$. Therefore $\dim K \leq 1$ for every compact subspace of $L$. Again, metrizability of $\alpha L$ yields a sequence $K_n$ of compact subsets of $L$ that cover $L$. Since $\alpha L = (\cup_n K_n) \cup \{\infty\}$ and $\dim \{\infty\} = 0$, $\dim K_n \leq 1$ the countable sum theorem for the covering dimension yields $\dim \alpha L \leq 1$ ([**6**, Proposition 3.1.7, p. 211]). Since no almost Wood space is zero-dimensional one has $\dim L = \dim \alpha L = 1$, and the proof is complete. □

REMARK 1. It follows from the equivalence theorems [**10**] that metrizable one-point compactifications of almost Wood spaces are also one-dimensional with respect to the small inductive dimension and the large inductive dimension.

REMARK 2. Another interesting consequence of Theorem 3 is that every metrizable one-point compactification of an almost Wood space is homeomorphic to a compact subset of the three-dimensional cube [**10**, Theorem V3, p. 60].

REMARK 3. Let $L$ be a locally compact space. In general the isometries of $C_0(L)$ cannot be extended to (isometries of) $C(\alpha L)$. This is because unimodular maps on $L$ do not converge as $x \to \infty$. But if $C_0(L)$ is a separable almost transitive space (in other words, if $L$ is an almost Wood space with $\alpha L$ metrizable) then the group of the isometries of $C_0(L)$ that extend to $C(\alpha L)$ acts transitively on the unit sphere of $C_0(L)$. Indeed, let $f, g \in C_0(L) \subset C(\alpha L)$ be such that $\|f\| = \|g\| = 1$ and $\varepsilon > 0$. Take norm-one non-vanishing maps $f_1, g_1 \in C(\alpha L)$ such that $\|f - f_1\|_{C(\alpha L)} \leq \varepsilon$, $\|g - g_1\|_{C(\alpha L)} \leq \varepsilon$.

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Put $\sigma(x) = f_1(x)/|f_1(x)|$ and $\tau(x) = g_1(x)/|g_1(x)|$ for all $x \in L$. One has $f_1 = \sigma|f_1|$ and $g_1 = \tau|g_1|$, which yields

$$\|f - \sigma|f|\| \leq 2\varepsilon \text{ and } \|g - \tau|g|\| \leq 2\varepsilon.$$

Let $T$ be an isometry of $C_0(L)$ such that $\||g| - T|f|\| \leq \varepsilon$. We may assume that $Th = h \circ \varphi$ for some homeomorphism $\varphi$ of $L$ (which obviously extends to $\alpha L$ by $\varphi(\infty) = \infty$). The map $\Phi : C(\alpha L) \to C(\alpha L)$ given by

$$\Phi(h) = \tau(\sigma^{-1}h) \circ \varphi$$

is an isometry of $C(\alpha L)$ leaving $C_0(L)$ invariant and satisfying

$$\|g - \Phi f\| \leq 5\varepsilon.$$

Corollary 1. *If an almost Wood space exists, then there exists a one-dimensional Wood space that is an F-space.*

*Proof*. By the result quoted in (1), the hypothesis implies that an almost Wood space $L$ with $\alpha L$ metrizable exists and, therefore, one has $\dim \alpha L = 1$. Let $U$ be a free ultrafilter on $\mathbb{N}$ and let $C_0(L)_U$ be the ultrapower of $C_0(L)$ with respect to $U$. It is well known that $C_0(L)_U$ is isometrically isomorphic to $C_0(W)$, $W$ being a Wood space [**8**, **5**]. To see that $\alpha W$ is one-dimensional, note that $C(\alpha W) = C(\alpha L)_U$ and let us show that every $f \in C(\alpha W)$ can be approximated by non-vanishing functions. Indeed, let $f \in C(\alpha W) = C(\alpha L)_U$ with $\|f\| = 1$ and $\varepsilon > 0$. Write $f = [(f_n)_n]$ for $f_n$ in the unit sphere of $C(\alpha L)$ for all $n$. Reasoning as in the previous Remark, one obtains that

$$\|f_n - \sigma_n|f_n|\| \leq 1/n$$

for suitably chosen unimodular $\sigma_n \in C(\alpha L)$. Hence

$$f = [(f_n)] = [(\sigma_n|f_n|)] = [(\sigma_n)][(|f_n|)] = \sigma|f|,$$

$\sigma \in C(\alpha W)$ being unimodular. This clearly implies that $\alpha W$ is an $F$-space. Finally, put $g(x) = \max\{|f(x)|, \varepsilon\}$ for every $x \in \alpha W$. Obviously, $\sigma g$ omits the origin and $\|f - \sigma g\|_{C(\alpha W)} \leq \varepsilon$, which shows that $\dim \alpha W \leq 1$. □

**3. A characterization of almost Wood spaces.** Recall from [**8**] that $C_0(L)$ is positive transitive if, for any non-negative norm-one $f, g \in C_0(L)$, there is an isometry $T$ such that $g = Tf$. In that case the unimodular function $\sigma$ can be chosen to be identically 1; that is, $g = f \circ \varphi$ for some homeomorphism $\varphi$ of $L$. Hence positive transitivity is a property of the real space $C_0(L)$. Also, $C_0(L)$ is said to allow polar decompositions if for every $f \in C_0(S)$ there is an isometry mapping $|f|$ to $f$. This means that $f = \sigma|f|$ for some unimodular $\sigma$. Clearly, $C_0(L)$ is transitive if and only if it is positive transitive and allows polar decompositions.

Let us say that $C_0(L)$ is almost positive transitive if, given non-negative norm-one $f, g \in C_0(L)$ and $\varepsilon > 0$, there is an isometry $T$ such that $\|g - Tf\| \leq \varepsilon$. Also, we say that $C_0(L)$ *allows nearly polar decompositions* if for every $f$ and $\varepsilon > 0$ one can find $T$ such that $\|f - T|f|\| \leq \varepsilon$, or, in other words, the functions $g$ that admit a decomposition $g = \sigma|g|$, with $\sigma$ unimodular, are dense in $C_0(L)$. Again, $C_0(L)$ is almost transitive if and only if it is almost positive transitive and allows nearly polar decompositions.

It follows from the proof of Theorem 2 that, if the one-point compactification of $L$ is metrizable, then $C_0(L)$ allows nearly polar decompositions if and only if $\dim \alpha L \leq 1$.

We now characterize almost positive transitivity by a similarity property of "chains" in $\alpha L$.

Definition 2. Let us say that $(K_i)_{i=0}^n$ is a chain of *length n in* $\alpha L$ if

(1) each $K_i$ is a non-empty compact subset of $\alpha L$,

(2) $\{K_i\}_{i=0}^n$ cover $\alpha L$,

(3) $K_0$ contains the infinity point of $\alpha L$,

(4) $K_i \cap K_j$ is empty for $|i - j| > 1$.

Theorem 3. *The space $C_0(L)$ is almost positive transitive if and only if $\alpha L$ is connected and, given two chains $(K_i)_{i=0}^n$ and $(F_i)_{i=0}^n$ of the same length, there is an homeomorphism $\varphi$ of $\alpha L$ which*

(a) *leaves fixed the infinity point, and*

(b) *is such that $\varphi(F_i) \subset K_{i-1} \cup K_i \cup K_{i+1}$ for $0 \leq i \leq n$, where $K_{-1}$ and $K_{n+1}$ are assumed to be empty.*

*Proof. Sufficiency.* Let $f, g \in C_0(L) \subset C(\alpha L)$ be non-negative with $\|f\| = \|g\| = 1$. Clearly, $f(\alpha L) = g(\alpha L) = [0, 1]$. Fix $n \in \mathbb{N}$ and let $I_i = [\frac{i}{n}, \frac{i+1}{n}]$ for $0 \leq i \leq n-1$. Let $K_i = f^{-1}(I_i)$ and $F_i = g^{-1}(I_i)$ for $0 \leq i \leq n-1$. Clearly, $(K_i)_{i=0}^{n-1}$ and $(F_i)_{i=0}^{n-1}$ are chains of the same length.

Take a homeomorphism $\varphi : \alpha L \to \alpha L$ satisfying (a) and (b). Let us estimate the distance between $g$ and $f \circ \varphi$. Since $\{F_i\}_{i=0}^{n-1}$ cover $\alpha L$ one has

$$\|g - f \circ \varphi\|_{C(\alpha L)} \leq \max_{0 \leq i \leq n-1} \|g - f \circ \varphi\|_{C(F_i)}.$$

Let $x \in F_i$, then $\varphi(x) \in K_{i-1} \cup K_i \cup K_{i+1}$. Thus,

$$i/n \leq g(x) \leq (i+1)/n \text{ and } (i-1)/n \leq f(\varphi(x)) \leq (i+2)/n,$$

and, therefore, $|g(x) - f(\varphi(x))| \leq 2/n$. Hence taking $n \geq 1/(2\varepsilon)$, one obtains that $\|g - f \circ \varphi\| \leq \varepsilon$, as desired.

*Necessity.* Let $C_0(L)$ be almost positive transitive. Clearly $\alpha L$ must be connected. For if not, there is a zero-one valued map in $C_0(L)$ and, therefore, every non-negative norm-one function in $C_0(L)$ is zero-one valued, which is clearly impossible.

We now prove the statement about chains. Let $(K_i)_{i=0}^{n+1}$ and $(F_i)_{i=0}^{n+1}$ be chains of the same length. Connectedness of $\alpha L$ together with (4) implies that, in fact,

$$K_i \cap K_j \neq \emptyset \iff |i - j| \leq 1.$$

Let $B_0 = K_0$, $B_i = K_i \cap K_{i+1}$ for $1 \leq i \leq n-1$, and $B_n = K_{n+1}$. Define a map $f$: $\alpha L \to [0, 1]$ putting

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$$f(B_i) = i/n$$

for $0 \leq i \leq n$. By normality, for $1 \leq i \leq n$, one can extend $f$ to the whole of $K_i$ is such a way that $\frac{i-1}{n} \leq f(x) \leq \frac{i}{n}$ for all $x \in K_i$. Clearly, $f \in C_0(L)$ and

$$f^{-1}([(i-1)/n - \varepsilon, i/n + \varepsilon]) \subset K_{i-1} \cup K_i \cup K_{i+1}$$

for $0 \leq i \leq n+1$ and $\varepsilon$ small enough.

Let $(F_i)_{i=0}^{n+1}$ be another chain in $\alpha L$. Proceeding as before one obtains a non-negative norm-one $g \in C_0(L)$ such that $\frac{i-1}{n} \leq g(x) \leq \frac{i}{n}$ for $x \in F_i$. Let $\varphi$ be a homeomorphism of $\alpha L$ such that $\|g - f \circ \varphi\| \leq \varepsilon$. Then, for every $x \in F_i$, one has

$$(i-1)/n - \varepsilon \leq f(\varphi(x)) \leq i/n + \varepsilon,$$

from which it follows that $\varphi(x) \subset K_{i-1} \cup K_i \cup K_{i+1}$. This completes the proof. $\square$

Remark 4. Theorem 3 is false if one omits the connectedness of $\alpha L$ and the condition (4) in the definition of "chain" is replaced (directly) by $K_i \cap K_j \neq \emptyset \iff |i - j| \leq 1$. In that case, $L = \mathbb{N}^*$ (the growth of $\mathbb{N}$ in its Stone-Čech compactification) is easily seen to be a counterexample. See [**16**, 3.31, pp. 80–83].

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