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Operator Estimates for Fredholm Modules

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Abstract. We study estimates of the type

$$\|\phi(D) - \phi(D_0)\|_{E(\mathcal{M},\tau)} \leq C \cdot \|D - D_0\|^{\alpha}, \quad \alpha = \frac{1}{2}, 1$$

where $\phi(t) = t(1+t^2)^{-1/2}$, $D_0 = D_0^*$ is an unbounded linear operator affiliated with a semifinite von Neumann algebra $\mathcal{M}$, $D - D_0$ is a bounded self-adjoint linear operator from $\mathcal{M}$ and $(1+D_0^2)^{-1/2} \in E(\mathcal{M},\tau)$, where $E(\mathcal{M},\tau)$ is a symmetric operator space associated with $\mathcal{M}$. In particular, we prove that $\phi(D) - \phi(D_0)$ belongs to the non-commutative L_p-space for some $p \in (1,\infty)$, provided $(1+D_0^2)^{-1/2}$ belongs to the non-commutative weak L_r-space for some $r \in [1,p)$. In the case $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $1 \leq p \leq 2$, we show that this result continues to hold under the weaker assumption $(1+D_0^2)^{-1/2} \in \mathcal{C}_p$. This may be regarded as an odd counterpart of A. Connes' result for the case of even Fredholm modules.

0 Introduction

In a very general form, one of the basic problems of perturbation theory may be formulated as follows.

I. If F is a continuous function on $(-\infty,\infty)$ under what conditions does the smallness of $D - D_0$ imply that of $F(D) - F(D_0)$?

This paper is intended to study this problem when the function $F = \phi$ and D_0 (respectively, $D - D_0$) is some self-adjoint (respectively, bounded self-adjoint) operator on the infinite-dimensional Hilbert space $\mathcal{H}$ affiliated with $\mathcal{M}$ (respectively, from $\mathcal{M}$). We shall measure "smallness" of $D - D_0$ (respectively, $F(D) - F(D_0)$) in the uniform operator norm (respectively, in the norm of some symmetric operator space associated with $\mathcal{M}$). The difference between norms on the right and left hand sides makes virtually impossible the application of well-known double operator integral techniques from [BSo1–3] and therefore new technique is required even in the simplest situation when $\mathcal{M}$ coincides with the algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$. However, not only this makes the problem of interest. Our choice of F is stipulated by the recent development of the theory of unbounded Fredholm modules [Co1], [Co2] and that of spectral flow [P1], [P2], [CP].

Let $(\mathcal{M},\tau)$ be a semifinite von Neumann algebra on the Hilbert space $\mathcal{H}$ with a fixed faithful and normal semifinite trace, let $E(\mathcal{M},\tau)$ be a rearrangement-invariant Banach space associated with $(\mathcal{M},\tau)$ and the Banach function space E (for the definitions see next section) and let A be a unital Banach $*$-subalgebra of $\mathcal{M}$.

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Definition 0.1 (**[Co1], [Co2], [CP]**) An *odd unbounded* (respectively, *bounded*) *Breuer-Fredholm module* associated with $E(\mathcal{M},\tau)$ and A, is a pair $(\mathcal{M}, D_0)$ (respectively, $(\mathcal{M}, F_0)$) where D_0 (respectively, F_0) is an unbounded (respectively, bounded) self-adjoint operator affiliated with $\mathcal{M}$ (respectively, from $\mathcal{M}$) satisfying:

(1) $(1 + D_0^2)^{-1/2}$ (respectively, $(1 - F_0^2)^{1/2}$) belongs to $E(\mathcal{M},\tau)$; and
(2) $\mathcal{A} := \{a \in A \mid [D_0, a] \in \mathcal{M}\}$ (respectively, $\mathcal{A} := \{a \in A \mid [F_0, a] \in E(\mathcal{M},\tau)\}$) is a dense $*$-subalgebra of A.

In the special case when $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and τ is the standard trace Tr we shall omit the word "Breuer" from the definition and speak about *unbounded* (respectively, *bounded*) *Fredholm modules* $(\mathcal{H}, D_0)$ (respectively, $(\mathcal{H}, F_0)$). In this case and when $E = L_p$ (*i.e.*, when the non-commutative symmetric space $E(\mathcal{M},\tau)$ coincides with the Schatten-von Neumann ideal $\mathcal{C}_p$ of compact operators T such that $\mathrm{Tr}(|T|^p) < \infty$) the "bounded" part of Definition 0.1 is a slight extension of Definition 3 from [Co2, p. 290] (where $\mathcal{A} = A$ and $F_0^2 = 1$, see also [Co1, Appendix 2]). In the special case when $\mathcal{M}$ is a semifinite factor and $E = L_p$, the "unbounded" part of Definition 0.1 coincides with [CP, Definition 2.1]; and, in the case $\mathcal{M} = \mathcal{B}(\mathcal{H})$, it may be considered as an odd counterpart of the notion of an unbounded even p-summable Fredholm module from [Co1, Section 6, Corollary 3 and the remarks thereafter]. Further, if again $(\mathcal{M},\tau) = \big(\mathcal{B}(\mathcal{H}), \mathrm{Tr}\big)$ and if E is Marcinkiewicz function space with the fundamental function $\psi(t) = \log^{1/2}(e^2 + t)$ (see [KPS] and Section 5 below), then the "bounded" part of Definition 0.1 yields the definition of the θ-summable Fredholm module [Co2, Definition 4, p. 291].

Following the line of Connes' results for the even case (see [Co1, I.6]), the importance of the mapping $(\mathcal{H}, D_0) \longrightarrow \big(\mathcal{H}, \mathrm{sgn}(D_0)\big)$ was recognised and outlined in [CP] for the odd case. The smooth approximation of $\mathrm{sgn}(D_0)$ is the map

$$\phi\colon D \longrightarrow D(1 + D^2)^{-1/2}$$

and the latter fact explains our interest in the difference $\phi(D)-\phi(D_0)$. The results presented in this article contribute also to the study of the mapping $(\mathcal{M}, D_0) \longrightarrow \big(\mathcal{M}, \mathrm{sgn}(D_0)\big)$ which was initiated in [CP] for the odd p-summable Breuer-Fredholm modules. At this moment it is far from being clear whether for an arbitrary odd unbounded Breuer-Fredholm module $(\mathcal{M}, D_0)$ associated with $E(\mathcal{M},\tau)$, we may deduce that its bounded counterpart $\big(\mathcal{M}, \mathrm{sgn}(D_0)\big)$ is associated with $E(\mathcal{M},\tau)$ as well. Though the even p-summable case was settled in [Co1], the odd case may require introducing weak L_p-spaces (see [Co2, Section IV.2] and [CP, Section 2.A]). In other words, question *I* (with $F = \phi$) is prompted by and closely connected to the following problem.

II. Given the symmetric operator space $E(\mathcal{M},\tau)$, find a symmetric space $F(\mathcal{M},\tau)$ such that $\big(\mathcal{M}, \mathrm{sgn}(D_0)\big)$ is an odd bounded Breuer-Fredholm module associated with $E(\mathcal{M},\tau)$, provided $(\mathcal{M}, D_0)$ is an odd unbounded Breuer-Fredholm module associated with $F(\mathcal{M},\tau)$.

To answer Question *I*, we consider (see Sections 2–5) Lipschitz estimates of the type

$$\|\phi(D) - \phi(D_0)\|_{E(\mathcal{M},\tau)} \leq C \cdot \|D - D_0\| \tag{0.1}$$

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where $C > 0$ depends on D_0. In Section 6, we consider Hölder estimates of the type

$$\|\phi(D) - \phi(D_0)\|_{E(\mathcal{M},\tau)} \leq C \cdot \|D - D_0\|^{1/2}. \tag{0.2}$$

Our general approach to the estimates (0.1) is based on a systematic use of generalized s-numbers of measurable operators [FK] and symmetric operator space theory. After gathering in Section 1 below some basic facts concerning symmetric spaces of measurable operators, in Section 2 we reduce the problem of calculation of the norm $\phi(D) - \phi(D_0)$ in the norm of symmetric operator space $E(\mathcal{M}, \tau)$ to the calculation of some integral whose integrand depends only on the behaviour of generalised s-numbers of the operator $(1+D_0^2)^{-1/2}$ in an appropriate symmetric function space E (see Propositions 2.4, 2.6). This approach leads to the selection of a rearrangement-invariant ideal $J(E) \subseteq E$ consisting of all functions from E for which the integral converges. The study of $J(E)$, in the special cases of L_p-spaces, Lorentz and Marcinkiewicz spaces, is presented in Sections 3, 4 and 5 respectively. These sections contain an account of joint results of the author with A. Sedaev and E. Semenov (announced in [SSS]), and the author wishes to thank them for their kind permission to publish these results in this paper. We place earlier estimates of [CP] for L_p-spaces in the setting of non-commutative Orlicz and Lorentz spaces. This viewpoint allows improvement of several results given in [CP].

In Section 6 we present some results which indicate the intrinsic connection between our theme and the study of Hölder and Lipschitz continuity of the absolute value in the setting of operator spaces. Combined with the Birman-Koplienko-Solomyak inequality [BKS] this approach allows to obtain estimates of type (0.2).

All the results presented in this article may be easily reformulated to also contribute to the resolution of Question II. For example, Corollary 6.8 asserts, that for any odd unbounded p-summable Fredholm module $(\mathcal{H}, D_0)$, $1 \leq p \leq 2$, we have that $\big(\mathcal{H}, \operatorname{sgn}(D_0)\big)$ is a p-summable odd bounded Fredholm module. This complements results of A. Connes for the even case [Co1, Section 6]. Using the same method, it may be easily verified that the space $F(\mathcal{M}, \tau)$ arising in Question II may be taken to be $J(E)(\mathcal{M}, \tau)$.

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1 Preliminaries

We denote by $\mathcal{M}$ a semifinite von Neumann algebra on the Hilbert space $\mathcal{H}$, with a fixed faithful and normal semifinite trace τ. The identity in $\mathcal{M}$ is denoted by 1. A linear operator $x\colon \operatorname{dom}(x) \to \mathcal{H}$, with domain $\operatorname{dom}(x) \subseteq \mathcal{H}$, is called affiliated with $\mathcal{M}$ if $ux = xu$ for all unitary u in the commutant $\mathcal{M}'$ of $\mathcal{M}$. The closed and densely defined operator x, affiliated with $\mathcal{M}$, is called *τ-measurable* if for every $\epsilon > 0$ there exists an orthogonal projection $p \in \mathcal{M}$ such the $p(\mathcal{H}) \subseteq \operatorname{dom}(x)$ and $\tau(1 - p) < \epsilon$. The collection of all τ-measurable operators is denoted by $\widetilde{\mathcal{M}}$. With the sum and product defined as the respective closures of the algebraic sum and product, $\widetilde{\mathcal{M}}$ is a $*$-algebra.

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Given a self-adjoint operator x in $\mathcal{H}$ we denote by $e^x(\cdot)$ the spectral measure of x. Now assume that $x \in \tilde{\mathcal{M}}$. Then $e^{|x|}(B) \in \mathcal{M}$ for all Borel sets $B \subseteq \mathbb{R}$, and there exists $s > 0$ such that $\tau\big(e^{|x|}(s,\infty)\big) < \infty$. For $x \in \tilde{\mathcal{M}}$ and $t \geq 0$ we define

$$\lambda_s(x) := \tau\big(e^{|x|}(s,\infty)\big)$$

and

$$\mu_t(x) := \inf\{s \geq 0 : \tau\big(e^{|x|}(s,\infty)\big) \leq t\} = \inf\{s \geq 0 : \lambda_s(x) \leq t\}.$$

The function $\lambda_s(x)\colon [0,\infty) \to [0,\infty]$ is called the *distribution function* of x and the function $\mu(x)\colon [0,\infty) \to [0,\infty]$ is called the *generalized singular value function* (or decreasing rearrangement) of x; note that $\mu_t(x) < \infty$ for all $t > 0$. We note that a sequence $\{x_n\} \subseteq \tilde{\mathcal{M}}$ converges to 0 for the measure topology (see [FK]) if and only if $\mu_t(x_n) \to 0$ for all $t > 0$. Equipped with this measure topology, $\tilde{\mathcal{M}}$ is a complete topological $*$-algebra.

If we consider $\mathcal{M} = L_\infty(\mathbb{R}^+, m)$, where m denotes Lebesgue measure on $\mathbb{R}^+$, as an abelian von Neumann algebra acting via multiplication on the Hilbert space $\mathcal{H} = L_2(\mathbb{R}^+, m)$, with the trace given by integration with respect to m, it is easy to see that $\tilde{\mathcal{M}}$ consists of all measurable functions on $\mathbb{R}^+$ which are bounded except on a set of finite measure, and that for $x \in \tilde{\mathcal{M}}$, the generalized singular value function $\mu(x)$ (respectively, the distribution function $\lambda(x)$) is precisely the decreasing rearrangement of the function $|x|$ (respectively, the distribution function of $|x|$) and in this setting, $\mu(x)$ is frequently denoted by x^* (respectively, $n_{|x|}$). As usual, for $x \in \tilde{\mathcal{M}}$, we define

$$x^{**}(t) := \frac{1}{t}\int_0^t x^*(s)\,ds, \quad \text{for all } t > 0.$$

It is easy to see that $0 \leq x^*(t) \leq x^{**}(t)$, $t \in (0,\infty)$. We recall also that the *dilation operator* σ_s, $s > 0$ on $\tilde{\mathcal{M}}$ is defined by

$$\sigma_s x(t) = x(s^{-1}t), \quad x \in \tilde{\mathcal{M}}.$$

If $\mathcal{M} = \mathcal{L}(\mathcal{H})$ and τ is the standard trace, then $\tilde{\mathcal{M}} = \mathcal{M}$ and the measure topology coincides with the operator norm topology. If $x \in \mathcal{M}$, then x is compact if and only if $\lim_{t\to\infty} \mu_t(x) = 0$; in this case,

$$\mu_n(x) = \mu_t(x), \quad t \in [n, n+1), \quad n = 0, 1, 2, \ldots,$$

and the sequence $\{\mu_n(x)\}_{n=0}^\infty$ is just the sequence of eigenvalues of $|x|$ in non-increasing order and counted according to multiplicity.

Using the generalized singular value function, it is possible to construct certain Banach spaces of measurable operators. In particular, the non-commutative L_p-spaces ($1 \leq p \leq \infty$) associated with $(\mathcal{M}, \tau)$ can be defined by

$$L_p(\mathcal{M}, \tau) = \{x \in \tilde{\mathcal{M}} : \mu(x) \in L_p(\mathbb{R}^+, m)\},$$

equipped with the norm $\|x\|_p := \|\mu(x)\|_p$, $x \in L_p(\mathcal{M}, \tau)$. It is not difficult to see that this definition coincides with the definition of non-commutative L_p-spaces as in [Ne], [Te]. If

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$\mathcal{M} = \mathcal{B}(\mathcal{H})$ with standard trace, then these non-commutative L_p-spaces are precisely the *Schatten classes* $\mathcal{C}_p$, $1 \le p < \infty$.

We will consider non-commutative spaces of more general form and now briefly describe their construction.

By $L^0(\mathbb{R}^+, m)$, we denote the space of all $\mathbb{C}$-valued Lebesgue measurable functions on $\mathbb{R}^+$ (with identification m-a.e.). A vector subspace $E \subseteq L^0(\mathbb{R}^+, m)$ is said to be a *rearrangement-invariant ideal* if it follows from $x \in E$, $y \in L^0(\mathbb{R}^+, m)$ and $y^* \le x^*$ that $y \in E$. A Banach space $(E, \|\cdot\|_E)$, where $E \subseteq L^0(\mathbb{R}^+, m)$ is a rearrangement-invariant ideal and $y^* \le x^*$, $x \in E$ imply $\|y\|_E \le \|x\|_E$ is called a *rearrangement-invariant Banach function space*. Furthermore, $(E, \|\cdot\|_E)$ is called a *symmetric Banach function space* (respectively, *fully symmetric Banach function space*) if it has the additional property, that $x, y \in E$ and $y^{**} \le x^{**}$ (respectively, $x \in E$, $y \in L^0(\mathbb{R}^+, m)$ and $y^{**} \le x^{**}$) imply that $\|y\|_E \le \|x\|_E$ (respectively, $y \in E$ and $\|y\|_E \le \|x\|_E$). Any exact interpolation space for the pair $\big(L_1(\mathbb{R}^+, m), L_\infty(\mathbb{R}^+, m)\big)$ is fully symmetric (see [BS], [KPS]). Dilation operator σ_s, $s > 0$ acts boundedly in any rearrangement-invariant Banach function space $(E, \|\cdot\|_E)$ [KPS, Theorem II.4.4).

Recall (see [KPS]) that for an arbitrary rearrangement-invariant function space $E = E(0, \infty)$ the fundamental function of E, $\phi_E(\cdot)$, is given by

$$\phi_E(t) = \|\chi_{[0,t)}\|_E, \quad t > 0.$$

Given a semifinite von Neumann algebra $(\mathcal{M}, \tau)$ and a symmetric Banach function space $(E, \|\cdot\|_E)$ on $(\mathbb{R}^+, m)$ we define the corresponding non-commutative space $E(\mathcal{M}, \tau)$ by setting

$$E(\mathcal{M}, \tau) = \{x \in \tilde{\mathcal{M}} : \mu(x) \in E\}.$$

Equipped with the norm $\|x\|_{E(\mathcal{M},\tau)} := \|\mu(x)\|_E$, the space $\big(E(\mathcal{M}, \tau), \|\cdot\|_{E(\mathcal{M},\tau)}\big)$ is a Banach space and is called the (non-commutative) symmetric operator space associated with $(\mathcal{M}, \tau)$ corresponding to $(E, \|\cdot\|_E)$. If E is one of the familiar Lorentz (respectively, Marcinkiewicz) function spaces (see [KPS], [BS]), then the spaces $E(\mathcal{M}, \tau)$ given by the preceding construction coincide with Lorentz (respectively, Marcinkiewicz) operator spaces introduced in [O1], [O2].

The previous definition is still valid for rearrangement-invariant function spaces E provided that E is a subspace of an exact interpolation space for the pair $\big(L_1(\mathbb{R}^+, m), L_\infty(\mathbb{R}^+, m)\big)$ (see [CS], [DDP1], [DDP2], [SC]). Any rearrangement-invariant space $\big(E(\mathcal{M}, \tau), \|\cdot\|_{E(\mathcal{M},\tau)}\big)$ is continuously embedded into $\tilde{\mathcal{M}}$.

In the present article, we shall work mainly with the Banach space

$$\mathcal{E}(\mathcal{M}, \tau) := \mathcal{M} \cap \big(E(\mathcal{M}, \tau), \|\cdot\|_{E(\mathcal{M},\tau)}\big)$$

equipped with the norm

$$\|\cdot\|_{\mathcal{E}(\mathcal{M},\tau)} := \max\{\|\cdot\|, \|\cdot\|_{E(\mathcal{M},\tau)}\}.$$

It is easy to see that $\big(\mathcal{E}(\mathcal{M}, \tau), \|\cdot\|_{\mathcal{E}(\mathcal{M},\tau)}\big)$ is symmetric operator space corresponding to symmetric function space $\mathcal{E} := L_\infty(0, \infty) \cap E$.

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2 Lipschitz Estimates: Symmetric Operator Spaces

In what follows, we shall let $d_0 \in L_\infty(\mathbb{R}^+, m)$ be given by

$$d_0(t) := \mu_t(1 + D_0^2)^{-\frac{1}{2}}, \quad t > 0.$$

If $d_0 \in E$ then it is clear that $(1 + D_0^2)^{-1/2} \in E(\mathcal{M}, \tau)$. Moreover, if $\mathcal{M}$ is non-atomic, then it is easy to see that

$$(1 + D_0^2)^{-1/2} \in \mathcal{E}(\mathcal{M}, \tau) \Longleftrightarrow d_0 \in \mathcal{E}.$$

We denote by $\mathcal{M}_1 := \mathcal{M} \otimes M_2(\mathbb{C})$ the von Neumann algebra of all 2×2 matrices

$$[x_{ij}] = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

with $x_{ij} \in \mathcal{M}$, $i, j = 1, 2$, acting on the Hilbert space $\mathcal{H} \oplus \mathcal{H}$. If for $0 \le [x_{ij}] \in \mathcal{M}_1$, the trace τ_1 is defined by setting

$$\tau_1([x_{ij}]) = \tau(x_{11}) + \tau(x_{22}),$$

then $(\mathcal{M}_1, \tau_1)$ is a semifinite von Neumann algebra. Now suppose that $x \in \tilde{\mathcal{M}}$ and define

$$\pi(x)\colon \operatorname{dom}\bigl(\pi(x)\bigr) \to \mathcal{H} \oplus \mathcal{H}$$

by

$$\pi(x) = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix},$$

where $\operatorname{dom}\bigl(\pi(x)\bigr) = \operatorname{dom}(x) \oplus \mathcal{H}$. It is easy to see that $\pi(x)$ is affiliated with $\mathcal{M}_1$ and further $\pi(x) \in \tilde{\mathcal{M}}_1$.

We note that it follows immediately from [DDPS1, Lemma 2.1] that $\mu_t(x) = \mu_t\bigl(\pi(x)\bigr)$ for any $t > 0$ and any $x \in \tilde{\mathcal{M}}$, where $\mu_t\bigl(\pi(x)\bigr)$ is the generalised t-th singular number of the operator $\pi(x) \in \tilde{\mathcal{M}}_1$. If E is a symmetric Banach function space on $\mathbb{R}^+$, it follows that the operator $\pi(x)$ belongs to $E(\mathcal{M}_1, \tau_1)$ if and only if $x \in E(\mathcal{M}, \tau)$ and in this case

$$\|\pi(x)\|_{E(\mathcal{M}_1,\tau_1)} = \|x\|_{E(\mathcal{M},\tau)}.$$

Let $\mathcal{P}_{D_0}, \mathcal{P}_D \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ be orthogonal projections on the graphs of D_0 and D respectively.

For brevity we set

$$h(t) := 1 + \frac{1}{2}t^2 + \frac{1}{2}t(t^2 + 4)^{\frac{1}{2}}, \quad t > 0.$$

Our first result shows that distance between projections $\mathcal{P}_{D_0}$ and $\mathcal{P}_D$ calculated in the metric of the space $E(\mathcal{M}_1, \tau_1)$ is Lipschitz continuous with respect to small perturbations $D - D_0$.

Theorem 2.1 *Let E be a rearrangement-invariant Banach function space on $\mathbb{R}^+$, let $(1 + D_0^2)^{-\frac{1}{2}} \in E(\mathcal{M}, \tau)$ and let $(D - D_0) = (D - D_0)^* \in \mathcal{M}$. Then $\mathcal{P}_D - \mathcal{P}_{D_0} \in E(\mathcal{M}_1, \tau_1)$ and moreover*

$$\|\mathcal{P}_D - \mathcal{P}_{D_0}\|_{E(\mathcal{M}_1,\tau_1)} \le 2\|D - D_0\| \cdot (h^{\frac{1}{2}} + h)(\|D - D_0\|) \cdot \|(1 + D_0^2)^{-\frac{1}{2}}\|_{E(\mathcal{M},\tau)}.$$

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Since $h(1) < 3$ the preceding estimate implies immediately that

$$\|\mathcal{P}_D - \mathcal{P}_{D_0}\|_{E(\mathcal{M}_1,\tau_1)} \le 10\|d_0\|_E \cdot \|D - D_0\| \tag{2.1}$$

whenever $\|D - D_0\| \le 1$.

Proof It follows from the equality

$$\mathcal{P}_D - \mathcal{P}_{D_0} = \mathcal{P}_D(1 - \mathcal{P}_{D_0}) - (1 - \mathcal{P}_D)\mathcal{P}_{D_0}$$

that to prove the assertion of the theorem it suffices to show that

$$\|\mathcal{P}_D(1 - \mathcal{P}_{D_0})\|_{E(\mathcal{M}_1,\tau_1)} \le \|D - D_0\| \cdot (h^{\frac{1}{2}} + h)(\|D - D_0\|) \cdot \|(1 + D_0^2)^{-\frac{1}{2}}\|_{E(\mathcal{M},\tau)} \tag{2.2}$$

and

$$\|(1 - \mathcal{P}_D)\mathcal{P}_{D_0}\|_{E(\mathcal{M}_1,\tau_1)} \le \|D - D_0\| \cdot (h^{\frac{1}{2}} + h)(\|D - D_0\|) \cdot \|(1 + D_0^2)^{-\frac{1}{2}}\|_{E(\mathcal{M},\tau)}. \tag{2.3}$$

We shall check (2.3) first. Given $\omega \in \mathcal{H} \oplus \mathcal{H}$ there exists $\xi \in \operatorname{dom}(D_0)$ such that

$$\mathcal{P}_{D_0}\omega = \begin{pmatrix} \xi \\ D_0\xi \end{pmatrix}.$$

Taking into account that the matrix of projection $\mathcal{P}_D$ is given by (see [CP, Appendix A])

$$\mathcal{P}_D = \begin{pmatrix} (1 + D^2)^{-1} & D(1 + D^2)^{-1} \\ D(1 + D^2)^{-1} & D^2(1 + D^2)^{-1} \end{pmatrix}$$

and applying the spectral theorem we get

$$\begin{aligned} (1 - \mathcal{P}_D)\mathcal{P}_{D_0}\omega &= \begin{pmatrix} D^2(1 + D^2)^{-1} & -D(1 + D^2)^{-1} \\ -D(1 + D^2)^{-1} & (1 + D^2)^{-1} \end{pmatrix} \begin{pmatrix} \xi \\ D_0\xi \end{pmatrix} \\ &= \begin{pmatrix} D^2(1 + D^2)^{-1}\xi - D(1 + D^2)^{-1}D_0\xi \\ -D(1 + D^2)^{-1}\xi - (1 + D^2)^{-1}D_0\xi \end{pmatrix} \\ &= \begin{pmatrix} D(1 + D^2)^{-1}D\xi - D(1 + D^2)^{-1}D_0\xi \\ -(1 + D^2)^{-1}D\xi - (1 + D^2)^{-1}D_0\xi \end{pmatrix} \\ &= \begin{pmatrix} D(1 + D^2)^{-1}(D - D_0)\xi \\ -(1 + D^2)^{-1}(D - D_0)\xi \end{pmatrix}. \end{aligned} \tag{2.4}$$

Let

$$Q := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad U := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

be an orthogonal projection and a unitary operator from $\mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ respectively. Since $Q(\omega_1 \oplus \omega_2) = \omega_1 \oplus 0$, $U(\omega_1 \oplus \omega_2) = (\omega_2 \oplus \omega_1)$ for any $\omega_1, \omega_2 \in \mathcal{H}$, it follows from (2.4) that

$$Q(1 - \mathcal{P}_D)\mathcal{P}_{D_0}\omega = \pi\big(D(1 + D^2)^{-1}(D - D_0)\big)Q\mathcal{P}_{D_0}\omega \tag{2.5}$$

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and

$$(1-Q)(1-\mathcal{P}_D)\mathcal{P}_{D_0}\omega = U\pi\big(-(1+D^2)^{-1}(D-D_0)\big)Q\mathcal{P}_{D_0}\omega. \tag{2.6}$$

Using the fact that the uniform operator norms of the operators Q, U, $\mathcal{P}_{D_0}$ are all one and using the simplest properties of generalised singular numbers we get from (2.5) and (2.6)

$$\begin{aligned}\mu_t\big(Q(1-\mathcal{P}_D)\mathcal{P}_{D_0}\big) &\le \mu_t\Big(\pi\big(D(1+D^2)^{-1}(D-D_0)\big)\Big)\\ &= \mu_t\big(D(1+D^2)^{-1}(D-D_0)\big)\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\mu_t\big((1-Q)(1-\mathcal{P}_D)\mathcal{P}_{D_0}\big) &\le \mu_t\Big(\pi\big((1+D^2)^{-1}(D-D_0)\big)\Big)\\ &= \mu_t\big((1+D^2)^{-1}(D-D_0)\big)\end{aligned} \tag{2.8}$$

respectively. Since for any $a \in \mathcal{M}$ and $x \in E(\mathcal{M},\tau)$ we have

$$\|ax\|_{E(\mathcal{M},\tau)} \le \|a\|\cdot\|x\|_{E(\mathcal{M},\tau)}$$

we infer that (2.7) and (2.8) imply in turn

$$\|Q(1-\mathcal{P}_D)\mathcal{P}_{D_0}\|_{E(\mathcal{M}_1,\tau_1)} \le \|D-D_0\|\cdot\|D(1+D^2)^{-1}\|_{E(\mathcal{M},\tau)} \tag{2.9}$$

and

$$\|(1-Q)(1-\mathcal{P}_D)\mathcal{P}_{D_0}\|_{E(\mathcal{M}_1,\tau_1)} \le \|D-D_0\|\cdot\|(1+D^2)^{-1}\|_{E(\mathcal{M},\tau)}. \tag{2.10}$$

Via [CP, Appendix B, Lemma 6] and since $(1+D_0^2)^{-1} \le (1+D_0^2)^{-\frac{1}{2}}$, we have

$$\begin{aligned}\|(1+D^2)^{-1}\|_{E(\mathcal{M},\tau)} &\le h(\|D-D_0\|)\cdot\|(1+D_0^2)^{-1}\|_{E(\mathcal{M},\tau)}\\ &\le h(\|D-D_0\|)\cdot\|(1+D_0^2)^{-\frac{1}{2}}\|_{E(\mathcal{M},\tau)}.\end{aligned} \tag{2.11}$$

Since $D(1+D^2)^{-1} \le (1+D^2)^{-\frac{1}{2}}$ and again via [CP, Appendix B, Lemma 6] (combined with the fact that $0 \le x \le y$ implies $0 \le x^{\frac{1}{2}} \le y^{\frac{1}{2}}$) we have also

$$\begin{aligned}\|D(1+D^2)^{-1}\|_{E(\mathcal{M},\tau)} &\le \|(1+D^2)^{-\frac{1}{2}}\|_{E(\mathcal{M},\tau)}\\ &\le h^{\frac{1}{2}}(\|D-D_0\|)\cdot\|(1+D_0^2)^{-\frac{1}{2}}\|_{E(\mathcal{M},\tau)}.\end{aligned} \tag{2.12}$$

(2.3) follows now from (2.9), (2.10), (2.11) and (2.12).

The inequality (2.2) may be obtained via the same arguments (replacing D by D_0 and vice versa) used to establish (2.3) if it is noticed that

$$\|\mathcal{P}_D(1-\mathcal{P}_{D_0})\|_{E(\mathcal{M}_1,\tau_1)} = \|(1-\mathcal{P}_{D_0})\mathcal{P}_D\|_{E(\mathcal{M}_1,\tau_1)}.$$

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This completes the proof of Theorem 2.1. ∎

We further let for brevity

$$H(t) := 2t \cdot (h + h^{\frac{1}{2}})(t), \quad t > 0.$$

Corollary 2.2 *If* $(1 + D_0^2)^{-\frac{1}{2}} \in E(\mathcal{M}, \tau)$, *then*

$$\left\| \frac{D}{(1 + D^2)} - \frac{D_0}{(1 + D_0^2)} \right\|_{E(\mathcal{M},\tau)} \leq H(\|D - D_0\|) \cdot \|(1 + D_0^2)^{-\frac{1}{2}}\|_{E(\mathcal{M},\tau)}$$

and therefore

$$\left\| \frac{D}{(1 + D^2)} - \frac{D_0}{(1 + D_0^2)} \right\|_{E(\mathcal{M},\tau)} \leq 10\|D - D_0\| \cdot \|(1 + D_0^2)^{-\frac{1}{2}}\|_{E(\mathcal{M},\tau)}$$

whenever $\|D - D_0\| \leq 1$.

Proof The assertion of Corollary 2.2 follows from that of Theorem 2.1 combined with the following two easily seen facts:

$$Q(\mathcal{P}_D - \mathcal{P}_{D_0})(1 - Q) = \begin{pmatrix} 0 & \frac{D}{1+D^2} - \frac{D_0}{1+D_0^2} \\ 0 & 0 \end{pmatrix}$$

and

$$\mu_t \left(\begin{pmatrix} 0 & \frac{D}{1+D^2} - \frac{D_0}{1+D_0^2} \\ 0 & 0 \end{pmatrix} \right) = \mu_t \left(\frac{D}{1 + D^2} - \frac{D_0}{1 + D_0^2} \right), \quad t > 0.$$

This suffices to complete the proof of Corollary 2.2.

Corollary 2.3 *For any* $\lambda > 0$ *and any rearrangement-invariant operator space* $E(\mathcal{M}, \tau)$ *we have*

(2.13)
$$\left\| \frac{D}{1 + \lambda + D^2} - \frac{D_0}{1 + \lambda + D_0^2} \right\|_{E(\mathcal{M},\tau)} \leq \frac{H(\|D - D_0\|)}{1 + \lambda} \cdot \left\| \left(1 + \frac{D_0^2}{1 + \lambda}\right)^{-1/2} \right\|_{E(\mathcal{M},\tau)}.$$

Furthermore, if $\|D - D_0\| \leq 1$, *then*

$$\left\| \frac{D}{1 + \lambda + D^2} - \frac{D_0}{1 + \lambda + D_0^2} \right\|_{E(\mathcal{M},\tau)} \leq \frac{10\|D - D_0\|}{1 + \lambda} \cdot \left\| \left(1 + \frac{D_0^2}{1 + \lambda}\right)^{-1/2} \right\|_{E(\mathcal{M},\tau)}.$$

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Proof The assertion of Corollary 2.3 follows immediately from that of Corollary 2.2 and the definition of the function H:

$$\begin{aligned}
&\left\|\frac{D}{1+\lambda+D^2}-\frac{D_0}{1+\lambda+D_0^2}\right\|_{E(\mathcal{M},\tau)}\\
&\quad=\left\|(1+\lambda)^{-\frac{1}{2}}\cdot\left(\frac{(1+\lambda)^{-\frac{1}{2}}D}{1+\left((1+\lambda)^{-\frac{1}{2}}D\right)^2}-\frac{(1+\lambda)^{-\frac{1}{2}}D_0}{1+\left((1+\lambda)^{-\frac{1}{2}}D_0\right)^2}\right)\right\|_{E(\mathcal{M},\tau)}\\
&\quad\leq(1+\lambda)^{-\frac{1}{2}}\cdot H\left(\left\|\frac{D-D_0}{(1+\lambda)^{1/2}}\right\|\right)\cdot\left\|\left(1+\left(\frac{D_0}{(1+\lambda)^{1/2}}\right)^2\right)^{-1/2}\right\|_{E(\mathcal{M},\tau)}\\
&\quad\leq(1+\lambda)^{-1}H(\|D-D_0\|)\cdot\left\|\left(1+\frac{D_0^2}{1+\lambda}\right)^{-1/2}\right\|_{E(\mathcal{M},\tau)}.
\end{aligned}$$

We let

$$J_{D_0,E}:=\int_0^\infty\lambda^{-\frac{1}{2}}\cdot(1+\lambda)^{-1}\cdot\left\|\left(1+\frac{D_0^2}{1+\lambda}\right)^{-\frac{1}{2}}\right\|_{E(\mathcal{M},\tau)}d\lambda. \tag{2.14}$$

It should be pointed out that for an arbitrary D_0 the number $J_{D_0,\mathcal{M}}$ is finite and that $J_{D_0,\mathcal{E}}<\infty$ if and only if $J_{D_0,E}<\infty$. The next proposition gives the upper estimate for $\|\phi(D)-\phi(D_0)\|_{\mathcal{E}(\mathcal{M},\tau)}$ provided $J_{D_0,E}<\infty$ and thus shows that the study of (2.14) is crucial in our present approach.

Proposition 2.4 *Suppose that* $(1+D_0^2)^{-\frac{1}{2}}\in E(\mathcal{M},\tau)$ *and* $J_{D_0,E}<\infty$. *Then*

$$\frac{D}{(1+D^2)^{\frac{1}{2}}}-\frac{D_0}{(1+D_0^2)^{\frac{1}{2}}}\in\mathcal{E}(\mathcal{M},\tau),$$

and

$$\left\|\frac{D}{(1+D^2)^{\frac{1}{2}}}-\frac{D_0}{(1+D_0^2)^{\frac{1}{2}}}\right\|_{\mathcal{E}(\mathcal{M},\tau)}\leq\frac{1}{\pi}\cdot H(\|D-D_0\|)\cdot J_{D_0,\mathcal{E}}. \tag{2.15}$$

In particular,

$$\left\|\frac{D}{(1+D^2)^{\frac{1}{2}}}-\frac{D_0}{(1+D_0^2)^{\frac{1}{2}}}\right\|_{\mathcal{E}(\mathcal{M},\tau)}\leq\frac{10}{\pi}\|D-D_0\|\cdot J_{D_0,\mathcal{E}}$$

whenever $\|D-D_0\|\leq 1$.

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Proof For any $\lambda > 0$ we let

$$j(\lambda) := \lambda^{-\frac{1}{2}} \cdot \left(\frac{D}{1+\lambda+D^2} - \frac{D_0}{1+\lambda+D_0^2} \right).$$

It is clear that j is simultaneously $\mathcal{M}$-valued, $E(\mathcal{M},\tau)$-valued and $\mathcal{E}(\mathcal{M},\tau)$-valued function. In each case it may be checked directly (via the Spectral Theorem and simplest properties of symmetric operator spaces) that j is a continuous function. We show first that the (Riemann) integrals of the function $j(\cdot)$ over $(0,\infty)$ converge in all these three spaces.

By Corollary 2.3 applied to the symmetric operator space $(\mathcal{M}, \|\cdot\|)$ we have

$$\|j(\lambda)\| \le \frac{H(\|D-D_0\|)}{\lambda^{\frac{1}{2}}(1+\lambda)} \cdot \left\| \left(1 + \frac{D_0^2}{1+\lambda}\right)^{-\frac{1}{2}} \right\| \le \frac{H(\|D-D_0\|)}{\lambda^{\frac{1}{2}}(1+\lambda)},$$

whence

$$\int_0^\infty \|j(\lambda)\|\, d\lambda < \infty. \tag{2.16}$$

(2.16) means that the Riemann integral $(\mathcal{M}) - \int_0^\infty j(\lambda)\, d\lambda$ converges absolutely. Further, by Corollary 2.3 and the assumption $J_{D_0,E} < \infty$ we have

$$\int_0^\infty \|j(\lambda)\|_{E(\mathcal{M},\tau)}\, d\lambda \le H(\|D-D_0\|) \cdot J_{D_0,E} < \infty. \tag{2.17}$$

(2.17) means that the Riemann integral $\big(E(\mathcal{M},\tau)\big) - \int_0^\infty j(\lambda)\, d\lambda$ converges absolutely. Finally, it follows from (2.16) and (2.17) and remark before Corollary 2.4 that

$$\int_0^\infty \|j(\lambda)\|_{\mathcal{E}(\mathcal{M},\tau)} \le H(\|D-D_0\|) \cdot J_{D_0,\mathcal{E}} < \infty \tag{2.18}$$

and again, as before, (2.18) means that the Riemann integral $\big(\mathcal{E}(\mathcal{M},\tau)\big) - \int_0^\infty j(\lambda)\, d\lambda$ converges absolutely.

Since the spaces $\mathcal{M}$, $E(\mathcal{M},\tau)$ and $\mathcal{E}(\mathcal{M},\tau)$ are continuously embedded into $\tilde{\mathcal{M}}$ we infer that

$$(\mathcal{M}) - \int_0^\infty j(\lambda)\, d\lambda = \big(E(\mathcal{M},\tau)\big) - \int_0^\infty j(\lambda)\, d\lambda = \big(\mathcal{E}(\mathcal{M},\tau)\big) - \int_0^\infty j(\lambda)\, d\lambda.$$

Thus, there exists an operator $T \in \mathcal{E}(\mathcal{M},\tau)$ such that

$$T = \int_0^\infty \lambda^{-\frac{1}{2}} \cdot \left(\frac{D}{1+\lambda+D^2} - \frac{D_0}{1+\lambda+D_0^2} \right) d\lambda \tag{2.19}$$

where the integrand from the right side converges in any of the norms $\|\cdot\|$, $\|\cdot\|_{E(\mathcal{M},\tau)}$ and $\|\cdot\|_{\mathcal{E}(\mathcal{M},\tau)}$. From the convergence in the uniform operator norm $\|\cdot\|$ of the right hand side

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in (2.19) we see also that for every $\xi \in \mathcal{H}$ the Riemann integral $(\mathcal{H}) - \int_0^\infty \lambda^{-\frac{1}{2}} \cdot (\frac{D}{1+\lambda+D^2} - \frac{D_0}{1+\lambda+D_0^2})\xi \, d\lambda$ converges and furthermore

$$T\xi = \int_0^\infty \lambda^{-\frac{1}{2}} \cdot \left(\frac{D}{1+\lambda+D^2} - \frac{D_0}{1+\lambda+D_0^2} \right) \xi \, d\lambda. \tag{2.20}$$

From (2.18) and the simplest properties of vector-valued Riemann integrals we see that

$$\|T\|_{\mathcal{E}(\mathcal{M},\tau)} \leq \int_0^\infty \|j(\lambda)\|_{\mathcal{E}(\mathcal{M},\tau)} \, d\lambda \leq H(\|D - D_0\|) \cdot J_{D_0,\mathcal{E}} < \infty. \tag{2.21}$$

On the other hand from [CP, Appendix A, Lemma 4] we know that

$$\left(\frac{D}{(1+D^2)^{\frac{1}{2}}} - \frac{D_0}{(1+D_0^2)^{\frac{1}{2}}} \right) \xi = \frac{1}{\pi} \cdot \int_0^\infty \lambda^{-\frac{1}{2}} \cdot \left(\frac{D}{1+\lambda+D^2} - \frac{D_0}{1+\lambda+D_0^2} \right) \xi \, d\lambda \tag{2.22}$$

where $\xi \in \operatorname{dom}(D) = \operatorname{dom}(D_0)$ and the integrand on the right converges in $\mathcal{H}$. It follows from (2.20) and (2.22) that

$$\left(\frac{D}{(1+D^2)^{\frac{1}{2}}} - \frac{D_0}{(1+D_0^2)^{\frac{1}{2}}} \right) \xi = T\xi$$

for any $\xi \in \operatorname{dom}(D_0)$, whence

$$\left(\frac{D}{(1+D^2)^{\frac{1}{2}}} - \frac{D_0}{(1+D_0^2)^{\frac{1}{2}}} \right) \xi = T\xi \tag{2.23}$$

for any $\xi \in \mathcal{H}$. Thus we have just established the integral representation

$$\frac{D}{(1+D^2)^{\frac{1}{2}}} - \frac{D_0}{(1+D_0^2)^{\frac{1}{2}}} = \frac{1}{\pi} \cdot \big(\mathcal{E}(\mathcal{M},\tau)\big) - \int_0^\infty \lambda^{-\frac{1}{2}} \cdot \left(\frac{D}{1+\lambda+D^2} - \frac{D_0}{1+\lambda+D_0^2} \right) d\lambda. \tag{2.24}$$

The assertion follows immediately from (2.21), (2.23) and (2.24). ■

We shall present in Proposition 2.6 below different formulae which are intended to simplify the calculation of $J_{D_0,E}$. We need first the following well-known technical lemma.

Lemma 2.5 *If $0 \leq x \in \tilde{\mathcal{M}}$ then the equality*

$$\mu_t\big(\psi(x)\big) = \psi\big(\mu_t(x)\big)$$

holds for each $t > 0$ and any continuous increasing function ψ on $[0, \infty)$ with $\psi(0) \geq 0$.

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Proposition 2.6 *The following formulae hold for an arbitrary symmetric space* $(E, \|\cdot\|_E)$.

(i)

$$J_{D_0,E} = \int_0^\infty \frac{1}{\lambda^{\frac{1}{2}} \cdot (1+\lambda)^{\frac{1}{2}}} \cdot \left\| \frac{1}{\left(\frac{1}{d_0^2} + \lambda\right)^{\frac{1}{2}}} \right\|_E d\lambda. \tag{2.25}$$

(ii)

$$\frac{1}{4} J_{D_0,E} \le \int_0^\infty \frac{du}{u+1} \left\| \frac{1}{\frac{1}{d_0} + u} \right\|_E \le J_{D_0,E}. \tag{2.26}$$

(iii)

$$J_{D_0,E} < \infty \Longleftrightarrow \int_1^\infty \frac{du}{u} \left\| \min\left(\frac{1}{u}, d_0\right) \right\|_E < \infty. \tag{2.27}$$

(iv)

$$J_{D_0,E} < \infty \Longleftrightarrow \sum_{n=1}^\infty \frac{1}{n} \left\| \min\left(\frac{1}{n}, d_0\right) \right\|_E < \infty \Longleftrightarrow \sum_{n=1}^\infty \| \min(2^{-n}, d_0) \|_E < \infty. \tag{2.28}$$

Proof (i) Given $\lambda > 0$ we let

$$\psi_\lambda(x) := \left(\frac{1+\lambda}{\frac{1}{x^2} + \lambda} \right)^{\frac{1}{2}}, \quad x \in (0, \infty) \quad \text{and} \quad \psi_\lambda(0) = 0.$$

It is easily seen that $\psi_\lambda(\cdot)$ satisfies the assumptions of Lemma 2.5 for any $\lambda > 0$. If $x, y \in (0, \infty)$ are such that $x = (1+y)^{-\frac{1}{2}}$ then we have

$$\left(1 + \frac{y}{1+\lambda}\right)^{-\frac{1}{2}} = \left(\frac{1+\lambda+\frac{1}{x^2}-1}{1+\lambda} \right)^{-\frac{1}{2}} = \left(\frac{1+\lambda}{\frac{1}{x^2}+\lambda} \right)^{-\frac{1}{2}} = \psi_\lambda(x),$$

and it follows that

$$\left(1 + \frac{D_0^2}{1+\lambda}\right)^{-\frac{1}{2}} = \psi_\lambda\big((1+D_0^2)^{-\frac{1}{2}}\big).$$

By Lemma 2.5 and the definition of ψ_λ we have

$$\left\| \left(1 + \frac{D_0^2}{1+\lambda}\right)^{-\frac{1}{2}} \right\|_{E(\mathcal{M},\tau)} = \|\psi_\lambda\big(\mu_{(\cdot)}(1+D_0^2)^{-\frac{1}{2}}\big)\|_E = \left\| \frac{(1+\lambda)^{\frac{1}{2}}}{\left(\frac{1}{\mu_{(\cdot)}^2(1+D_0^2)^{-\frac{1}{2}}} + \lambda\right)^{\frac{1}{2}}} \right\|_E.$$

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The equality (2.25) now follows from (2.14)

$$J_{D_0,E} = \int_0^\infty \frac{1}{\lambda^{\frac12}\cdot(1+\lambda)^{\frac12}}\cdot\left\|\frac{1}{\left(\frac{1}{\mu^2_{(\cdot)}(1+D_0^2)^{-\frac12}}+\lambda\right)^{\frac12}}\right\|_E d\lambda$$

$$= \int_0^\infty \frac{1}{\lambda^{\frac12}\cdot(1+\lambda)^{\frac12}}\cdot\left\|\frac{1}{\left(\frac{1}{d_0^2}+\lambda\right)^{\frac12}}\right\|_E d\lambda.$$

(ii) Substituting in (2.25) $\lambda = u^2$, $\frac{d\lambda}{\sqrt{\lambda}} = 2du$ and taking into account that $a^{\frac12}+b^{\frac12} \le \sqrt{2}(a+b)^{\frac12} \le \sqrt{2}(a^{\frac12}+b^{\frac12})$, $0<a$, $0<b$ we get (2.26) as follows

$$\begin{aligned} J_{D_0,E} &= \int_0^\infty \frac{1}{\lambda^{\frac12}\cdot(1+\lambda)^{\frac12}}\cdot\left\|\frac{1}{\left(\frac{1}{d_0^2}+\lambda\right)^{\frac12}}\right\|_E d\lambda \\ &= \int_0^\infty \frac{2du}{(u^2+1)^{\frac12}}\cdot\left\|\frac{1}{\left(\frac{1}{d_0^2}+u^2\right)^{\frac12}}\right\|_E \\ &\le 4\int_0^\infty \frac{du}{u+1}\left\|\frac{1}{\frac{1}{d_0}+u}\right\|_E \\ &\le 4\int_0^\infty \frac{du}{(u^2+1)^{\frac12}}\cdot\left\|\frac{1}{\left(\frac{1}{d_0^2}+u^2\right)^{\frac12}}\right\|_E \\ &= 4J_{D_0,E}. \end{aligned}$$

(iii) Since it is obvious that

$$\int_0^1 \frac{du}{u+1}\left\|\frac{1}{\frac{1}{d_0}+u}\right\|_E \le \int_0^1 \frac{du}{u+1}\|d_0\|_E = \ln 2\cdot\|d_0\|_E < \infty$$

and that for $u \ge 1$

$$\left\|\frac{1}{\frac{1}{d_0}+u}\right\|_E \approx \left\|\frac{1}{\max\left(u,\frac{1}{d_0}\right)}\right\|_E = \left\|\min\left(\frac{1}{u},d_0\right)\right\|_E$$

(here and below $\approx$ signifies two-sided estimate) we get (2.27)

$$J_{D_0,E} < \infty \iff \int_1^\infty \frac{du}{u+1}\left\|\min\left(\frac1u,d_0\right)\right\|_E < \infty \iff \int_1^\infty \frac{du}{u}\left\|\min\left(\frac1u,d_0\right)\right\|_E < \infty.$$

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(iv) Since the function $\frac{1}{u}\|\min(\frac{1}{u}, d_0)\|_E$, $u \in [1, \infty)$ is decreasing we infer from (2.27) that

$$J_{D_0,E} < \infty \iff \int_1^\infty \frac{du}{u} \left\| \min\left(\frac{1}{u}, d_0\right) \right\|_E < \infty \iff \sum_{n=1}^\infty \frac{1}{n} \left\| \min\left(\frac{1}{n}, d_0\right) \right\|_E < \infty.$$

Finally we note that since

$$\begin{aligned} 2^n \cdot \frac{1}{2^{n+1}} \cdot \left\| \min\left(\frac{1}{2^{n+1}}, d_0\right) \right\|_E &\leq \sum_{i=2^n}^{2^{n+1}-1} \frac{1}{i} \left\| \min\left(\frac{1}{i}, d_0\right) \right\|_E \\ &\leq 2^n \cdot \frac{1}{2^n} \cdot \left\| \min\left(\frac{1}{2^n}, d_0\right) \right\|_E, \quad n \geq 1, \end{aligned}$$

we have that

$$\sum_{n=1}^\infty \frac{1}{n} \left\| \min\left(\frac{1}{n}, d_0\right) \right\|_E < \infty \iff \sum_{n=1}^\infty \| \min(2^{-n}, d_0)\|_E < \infty. \qquad \blacksquare$$

Throughout the rest of the article we shall be concerned with the description of the sets

$$\mathcal{J}(E) := \{d_0 \in E : J_{D_0,E} < \infty\}, \quad J(E) := \{f \in E : \mu(f) \in \mathcal{J}(E)\}$$

which are of interest in their own right. It turns out that $J(E)$ is a rearrangement-invariant ideal (see below Corollary 2.8). To see this we first present a slight generalization of [BM, Proposition 1.4].

Let $G \subseteq \{x = x^* : x \in L^1(\mathbb{R}^+, m) + L^\infty(\mathbb{R}^+, m)\}$ be such that $x, y \in G$ imply that $x + y \in G$.

Proposition 2.7 *The smallest rearrangement-invariant ideal N_G containing G coincides with the set $\mathcal{G}$ given by*

$$\begin{aligned} \mathcal{G} := \{x \in L^1(\mathbb{R}^+, m) + L^\infty(\mathbb{R}^+, m) :& \\ \exists\, q = q(x) > 0 \text{ such that } x^* \leq q\sigma_q(g^*) &\text{ for some } g \in G\}. \end{aligned}$$

Proof By [BM, Proposition 1.4], for an arbitrary $y \in L^1(\mathbb{R}^+, m) + L^\infty(\mathbb{R}^+, m)$ we have that the smallest rearrangement-invariant ideal N_y is given by the set $\{x \in L^1(\mathbb{R}^+, m) + L^\infty(\mathbb{R}^+, m) : \exists\, q = q(x) > 0$ such that $x^* \leq q\sigma_q(y^*)\}$, and it follows immediately that

$$G \subseteq \mathcal{G} \subseteq N_G.$$

It is easy to see that $y \in \mathcal{G}$, $|z| \leq |y|$ imply $z \in \mathcal{G}$ and therefore to complete the proof we need only to show that $\mathcal{G}$ is a vector space. To this end, let $y, z \in \mathcal{G}$. Then we have

$$y^* \leq q\sigma_q(v^*), \quad z^* \leq s\sigma_s(w^*)$$

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for some $v, w \in G$ and some positive q, s. Letting $n := \max\{q, s\}$ and $u = v + w \in G$ we see that

$$y^* \leq N\sigma_N(u^*), \quad z^* \leq N\sigma_N(u^*)$$

whence, via [KPS, I.2.23],

$$(y+z)^* \leq 2N\sigma_{2N}(u^*).$$ ■

Corollary 2.8 *For an arbitrary rearrangement-invariant Banach function space E the set $J(E)$ is the smallest rearrangement-invariant ideal containing $\mathfrak{J}(E)$.*

Proof Since

$$\min(a, b+c) \leq \min(a,b) + \min(a,c)$$

for an arbitrary $a, b, c \geq 0$ we immediately infer from (2.28) that $f_1, f_2 \in \mathfrak{J}(E)$ imply $f_1 + f_2 \in \mathfrak{J}(E)$. Again by (2.28) we have $g \in \mathfrak{J}(E)$, provided that $0 \leq g \leq f$, $f \in \mathfrak{J}(E)$. Therefore, by Proposition 2.7, it suffices to show that $\sigma_s(f) \in \mathfrak{J}(E)$ for an arbitrary $f \in \mathfrak{J}(E)$ and arbitrary $s > 0$. Since for any $s \in (0,1]$ and $x^* = x \in L_1(\mathbb{R}^+, m) + L_\infty(\mathbb{R}^+, m)$ we clearly have $\sigma_s x \leq x$, it follows that σ_s sends $\mathfrak{J}(E)$ into itself for any $s \in (0,1]$. Further, for $s \in (1, \infty)$ and $\alpha > 0$, we have that

$$\begin{aligned} n_{\sigma_s(f)}(\alpha) &= m\left\{t : \left|f\left(\frac{t}{s}\right)\right| > \alpha\right\} \\ &= m\{st : |f(t)| > \alpha\} \\ &= s n_f(\alpha), \end{aligned}$$

(see also [KPS, p. 98]) and therefore

$$\begin{aligned} \sigma_s\big(\min(\alpha, f)(t)\big) &= \sigma_s\big(\alpha\chi_{[0,n_f(\alpha))}(t) + f(t)\chi_{[n_f(\alpha),\infty)}(t)\big) \\ &= \alpha\chi_{[0,sn_f(\alpha))}(t) + f(t/s)\chi_{[sn_f(\alpha),\infty)}(t) \\ &= \alpha\chi_{[0,n_{\sigma_s(f)}(\alpha))}(t) + \sigma_s(f)(t)\chi_{[n_{\sigma_s(f)}(\alpha),\infty)}(t) \\ &= \Big(\min\big(\alpha, \sigma_s(f)\big)\Big)(t). \end{aligned}$$

Using the fact that σ_s boundedly acts in E [KPS, Theorem II.4.4] we deduce from the latter that

$$\Big(\sum_{n=1}^{\infty} \|\min(2^{-n}, f)\|_E < \infty\Big) \Longrightarrow \Big(\sum_{n=1}^{\infty} \|\min\big(2^{-n}, \sigma_s(f)\big)\|_E < \infty\Big).$$

The latter, again by (2.28), implies that $\sigma_s\big(\mathfrak{J}(E)\big) \subseteq \mathfrak{J}(E)$ for any $s > 1$ and this completes the proof of Corollary 2.8. ■

Since $J(E)$ is a rearrangement invariant ideal, it is natural to study it by means of the theory of rearrangement invariant spaces. In the next section we shall study $J(L_p)$ in terms of the asymptotic of d_0 (respectively, n_{d_0}) on infinity (respectively, in zero) and to this end

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we shall employ so called weak-L_p-spaces and a certain family of Orlicz spaces. In Section 4 a complete characterization of $J(E)$ is given in the setting of Lorentz spaces. Results of the fourth section will be subsequently used in Section 5 where the ideal $J(E)$ is studied in the setting of Marcinkiewicz spaces.

3 Lipschitz Estimates: L_p-Spaces

We begin with the simple and frequently encountered in the applications case when the function d_0 (equivalently, the operator $(1 + D_0^2)^{-\frac{1}{2}}$) satisfies a "Lorentz space"-type condition of the form

$$d_0(t) \le Ct^{-\frac{1}{r}} \quad C, r > 0, \tag{3.1}$$

for sufficiently large t; or, equivalently,

$$n_{d_0}(s) \le C's^{-r} \quad C', r > 0 \tag{3.2}$$

for sufficiently small s. It is convenient to employ the terminology and notation from the interpolation theory.

Definition 3.1 **[LT2]** For $1 \le p \le \infty$, $1 \le q < \infty$, $L_{p,q}(0, \infty)$ is the space of all locally integrable real valued functions g on $(0, \infty)$ for which

$$\|g\|_{p,q} = \left(q/p \int_0^\infty \left(t^{1/p} g^*(t)\right)^q dt/t \right)^{1/q} < \infty.$$

For $1 \le p \le \infty$, $L_{p,\infty}(0, \infty)$ is the space of all functions g as above so that

$$\|g\|_{p,\infty} = \sup_{t>0} t^{1/p} g^*(t) < \infty.$$

Recall (see [LT2, pp. 142–143]) that $L_{p,\infty} = L_{p,\infty}(0, \infty)$ is a linear space and though $\|\cdot\|_{p,\infty}$ does not satisfy the triangle inequality, nevertheless, the space $L_{p,\infty}$ can be made into a symmetric Banach space if $p > 1$ by introducing an actual norm $|||\cdot|||_{p,\infty}$ which satisfies $\|g\|_{p,\infty} \le |||g|||_{p,\infty} \le C(p)\|g\|_{p,\infty}$. In fact, the spaces $L_{p,1}$ and $L_{p,\infty}$ are special examples of Lorentz and Marcinkiewicz function spaces (see below, Sections 4 and 5 respectively).

The following inclusions are well-known (see [LT2, p. 143]) and may be verified directly via Hölder inequality:

$$\mathcal{L}_{r,\infty} = L_{r,\infty} \cap L_\infty \subseteq L_p, \quad r < p, \tag{3.3}$$

and

$$\mathcal{L}_p \subseteq \mathcal{L}_{p,\infty} \subseteq \mathcal{L}_{s,\infty}, \quad p < s. \tag{3.4}$$

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Theorem 3.2 *For every $p \in (1, \infty)$ we have*

$$\bigcup_{r<p} \mathcal{L}_{r,\infty} \subseteq J(L_p). \tag{3.5}$$

Proof Let $f(t) = t^{-\frac{1}{r}}$, $t \geq 1$ and let $f(t) \equiv 1$, $0 \leq t \leq 1$. Then, given $n \in \mathbb{N}$, we have $2^{-n} = f(2^{nr})$, whence

$$\|\min(2^{-n}, f)\|_p^p = 2^{-np} \cdot 2^{nr} + \int_{2^{nr}}^{\infty} t^{-\frac{p}{r}}\, dt = \frac{p}{p-r} 2^{n(r-p)}. \tag{3.6}$$

It follows immediately from (3.6) that

$$\sum_{n=1}^{\infty} \|\min(2^{-n}, f)\|_p < \infty. \tag{3.7}$$

By Proposition 2.6 we see from (3.7) that $f \in \mathcal{J}(L_p)$. The embedding (3.5) now follows from Corollary 2.8. ■

The next Corollary in an implicit form is contained in [CP, Proposition 2.4].

Corollary 3.3 *If $(1 + D_0^2)^{-\frac{1}{2}} \in L_p(\mathcal{M}, \tau)$ for some $p \in (1, \infty)$, then for given $\epsilon > 0$ there exists a constant C, depending on p, ϵ and D_0 only, such that*

$$\left\| \frac{D}{(1 + D^2)^{\frac{1}{2}}} - \frac{D_0}{(1 + D_0^2)^{\frac{1}{2}}} \right\|_{L_{p+\epsilon}} \leq C \cdot H(\|D - D_0\|)$$

and further, if $\|D - D_0\| \leq 1$

$$\left\| \frac{D}{(1 + D^2)^{\frac{1}{2}}} - \frac{D_0}{(1 + D_0^2)^{\frac{1}{2}}} \right\|_{L_{p+\epsilon}} \leq 10C \cdot \|D - D_0\|.$$

Proof By (3.4) the assumption $(1 + D_0^2)^{-\frac{1}{2}} \in \mathcal{L}_p(\mathcal{M}, \tau)$ implies that $(1 + D_0^2)^{-\frac{1}{2}} \in \mathcal{L}_{s,\infty}(\mathcal{M}, \tau)$ for any $p < s < p + \epsilon$, $\epsilon > 0$. The assertion follows now from Theorem 3.2 and Propositions 2.6, 2.4. ■

We shall further strengthen the result of Theorem 3.2. In this section this aim will be achieved via replacing of condition (3.1) (respectively, (3.2)) by

$$d_0(t) \leq Ct^{-\frac{1}{p}} \cdot \ln^{-\alpha} t \quad C, p, \alpha > 0, \tag{3.8}$$

for sufficiently large t (respectively,

$$n_{d_0}(s) \leq C' s^{-p} \cdot \ln^{-\alpha} \left(\frac{1}{s}\right) \quad C', p, \alpha > 0 \tag{3.9}$$

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for sufficiently small s.) In the next section another strengthening of Theorem 3.2 is achieved via the use of Lorentz spaces (see Corollary 4.5).

It should be pointed out that the conditions (3.8) and (3.9) are not equivalent to each other and therefore will be treated separately. It turns out that the most convenient way to consider a class of functions which satisfy (3.8) is to introduce a certain Orlicz space. We shall recall below some notation and definitions concerning Orlicz spaces (for more substantial information we refer to [KR], [LT2], [BS]).

Let Φ be an Orlicz function on $[0,\infty)$ (*i.e.*, Φ is a continuous convex increasing function on $[0,\infty)$ satisfying $\Phi(0) = 0$ and $\Phi(\infty) = \infty$). The Orlicz space $L_\Phi\big([0,\infty)\big)$ is the space of all measurable functions f on $[0,\infty)$ so that

$$\int_0^\infty \Phi\left(\frac{|f(t)|}{\rho}\right) dt < \infty$$

for some $\rho > 0$. The (Orlicz) norm in $L_\Phi\big([0,\infty)\big)$ is defined by

$$\|f\|_\Phi = \inf\left\{\rho > 0 : \int_0^\infty \Phi\left(\frac{|f(t)|}{\rho}\right) dt \le 1\right\}.$$

Definition 3.4 We let $L(p,q)$, $1 \le p, q < \infty$ be the Orlicz space $L_{\Phi_{(p,q)}}[0,\infty)$ with $\Phi_{(p,q)}$ given by

$$\Phi_{(p,q)}(u) := u^p \ln^q\left(\frac{1}{u} + e\right), \quad u > 0, \quad \Phi_{(p,q)}(0) := 0.$$

Clearly $\mathcal{L}(p,q) \subseteq \mathcal{L}_p$ for any $q \in [1,\infty)$ and $\|g\|_{L_p} \le \|g\|_{\Phi_{(p,q)}}$, for any $g \in \mathcal{L}(p,q)$. The next proposition shows that $\mathcal{L}(p,q)$ is bigger than $\bigcup_{r<p} \mathcal{L}_r$, provided $p < q$.

Proposition 3.5 *For $s \in [1,\infty)$ and for $q \in (p,\infty)$ we have*

$$\bigcup_{r<p} \mathcal{L}_r \subseteq \mathcal{L}(p,s) \quad \textit{and} \quad \mathcal{L}(p,q) \ne \bigcup_{r<p} \mathcal{L}_r. \tag{3.10}$$

Proof To establish the first assertion from (3.10) its suffices to show (see Definition 3.1) that the function ζ given by

$$\zeta(t) := \begin{cases} 1, & \text{if } 0 \le t < e; \\ t^{-\frac{1}{r}}, & \text{if } t \ge e \end{cases}$$

belongs to $\mathcal{L}(p,s)$ as soon as $r < p$. Indeed, since

$$\begin{aligned} \int_e^\infty \Phi_{(p,q)}\big(\zeta(t)\big)\, dt &= \int_e^\infty \left(\frac{1}{t^{\frac{1}{r}}}\right)^p \cdot \ln^q(t^{\frac{1}{r}} + e)\, dt \\ &\le \frac{e^q}{r} \int_e^\infty \frac{\ln^q(t)}{t^{\frac{p}{r}}}\, dt \\ &< \infty \end{aligned}$$

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we have, by Definition 3.4, that $\zeta \in \mathcal{L}(p, s)$.

To prove the second assertion from (3.10) we let α be a fixed positive real number such that $\alpha > \frac{1+q}{p}$ and

$$\xi(t) := \begin{cases} 1, & \text{if } 0 \le t < e; \\ t^{-\frac{1}{p}} \cdot \ln^{-\alpha} t, & \text{if } t \ge e. \end{cases}$$

It is obvious that for any $C > 0$ the inequality

$$\xi(t) \le C \cdot t^{-\frac{1}{r}}, \quad t > 0$$

does not hold (uniformly on t) if $r < p$. Therefore,

$$\xi \notin \bigcup_{r<p} \mathcal{L}_{r,\infty}.$$

On the other hand,

$$\begin{aligned} \int_e^\infty \Phi_{(p,q)}\big(\xi(t)\big)\,dt &= \int_e^\infty \left(\frac{1}{t^{\frac{1}{p}} \cdot \ln^\alpha t}\right)^p \cdot \ln^q(t^{\frac{1}{p}} \cdot \ln^\alpha t + e)\,dt \\ &\le \frac{e^{2q}}{p} \int_e^\infty \frac{\ln^q(t) + \alpha \ln^q\big(\ln(t)\big)}{t \cdot \ln^{\alpha p}(t)}\,dt \\ &\le \frac{e^{2q}}{p} \cdot (1+\alpha) \cdot \int_e^\infty \frac{1}{t \cdot \ln^{\alpha p - q}(t)}\,dt \\ &< \infty, \end{aligned}$$

and, again by the definition of $L(p, q)$, we have $\xi \in \mathcal{L}(p, q)$. ■

In view of Proposition 3.5, the following result is a strengthening of Theorem 3.2.

Theorem 3.6 *For any $p \in [1, \infty)$ and any $q \in (p, \infty)$ we have*

$$\mathcal{L}(p, q) \subseteq J(L_p).$$

Proof Let $f = f^* \in \mathcal{L}(p, q)$, $\|f\|_{\Phi_{(p,q)}} = 1$. Thanks to Proposition 2.6 we need only to show that

$$\sum_{k=1}^\infty \frac{1}{k} \left\| \min\left(\frac{1}{k}, f\right) \right\|_{L_p} < \infty.$$

Let $t_k = \inf\{t : f(t) < \frac{1}{k}\}$, $k = 1, 2, \dots$. Since

$$\left\| \min\left(\frac{1}{k}, f\right) \right\|_{L_p} \le \left\| \frac{1}{k} \chi_{[0,t_k)} \right\|_{L_p} + \left(\int_{t_k}^\infty f^p(t)\,dt \right)^{\frac{1}{p}}, \quad k = 1, 2, \dots$$

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it suffices to show that the following two series

$$\sum_{k=1}^{\infty}\frac{1}{k}\left\|\frac{1}{k}\chi_{[0,t_k)}\right\|_{L_p},\quad \sum_{k=1}^{\infty}\frac{1}{k}\left(\int_{t_k}^{\infty} f^p(t)\,dt\right)^{\frac{1}{p}} \tag{3.11}$$

converge. To show that the first series from (3.11) converges we note that

$$\left\|\frac{1}{k}\chi_{[0,t_k)}\right\|_{\Phi_{(p,q)}} \le \|f\chi_{[0,t_k)}\|_{\Phi_{(p,q)}} \le 1$$

whence

$$t_k\cdot\left(\frac{1}{k}\right)^p\cdot\ln^q(k+e)=\int_0^{t_k}\left(\frac{1}{k}\right)^p\cdot\ln^q(k+e)\,dt\le 1.$$

It follows immediately, that

$$\|\chi_{[0,t_k)}\|_{L_p}=t_k^{\frac{1}{p}}\le k\cdot\ln^{-\frac{q}{p}}(k+e).$$

Since, by the assumption, $p<q$ it follows that

$$\sum_{k=1}^{\infty}\frac{1}{k}\left\|\frac{1}{k}\chi_{[0,t_k)}\right\|_{L_p}\le\sum_{k=1}^{\infty}\frac{1}{k}\ln^{-\frac{q}{p}}(k+e)<\infty$$

and we are done with the first series.

To show that the second series from (3.11) converges we note that for $t\ge t_k$ one has $f(t)<\frac{1}{k}$ and it follows that

$$\begin{aligned}\ln^q(k+e)\int_{t_k}^{\infty} f^p(t)\,dt &\le \int_{t_k}^{\infty} f^p(t)\cdot\ln^q\left(\frac{1}{f(t)}+e\right)\,dt\\ &=\int_0^{\infty}\Phi_{(p,q)}\big(f(t)\big)\,dt\\ &\le 1.\end{aligned}$$

It follows, that

$$\left(\int_{t_k}^{\infty} f^p(t)\,dt\right)^{\frac{1}{p}}\le\ln^{-\frac{q}{p}}(k+e)$$

whence, again by the assumption $p<q$,

$$\sum_{k=1}^{\infty}\frac{1}{k}\left(\int_{t_k}^{\infty} f^p(t)\,dt\right)^{\frac{1}{p}}\le\sum_{k=1}^{\infty}\frac{1}{k}\ln^{-\frac{q}{p}}(k+e)<\infty.$$

It completes the proof of Theorem 3.6. ■

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Remark 3.7 (i) It follows from the proof of Proposition 3.5 and Theorem 3.6 that the function

$$\xi(t) := \begin{cases} 1, & \text{if } 0 \le t < e; \\ t^{-\frac{1}{p}} \cdot \ln^{-\alpha} t, & \text{if } t \ge e \end{cases}$$

belongs to $\mathfrak{J}(L_p)$ for any $\alpha > \frac{1+p}{p}$. It turns out that this estimate can not be improved in the terms of (3.8). Indeed, for the function

$$\eta(t) := \begin{cases} 1, & \text{if } 0 \le t < e; \\ t^{-\frac{1}{p}} \cdot \ln^{-\frac{1+p}{p}} t, & \text{if } t \ge e. \end{cases}$$

we have

$$\begin{aligned} \int_0^\infty \frac{du}{u+1} \left\| \frac{1}{\frac{1}{\eta} + u} \right\|_{L_p} &\ge \int_e^\infty \frac{du}{u+1} \left(\int_e^\infty \frac{dt}{(t^{\frac{1}{p}} \ln^{\frac{1+p}{p}} t + u)^p} \right)^{\frac{1}{p}} \\ &\ge \frac{1}{2} \int_e^\infty \frac{du}{u+1} \left(\int_e^\infty \frac{dt}{t \ln^{1+p} t + u^p} \right)^{\frac{1}{p}} \\ &\ge \frac{1}{2} \int_e^\infty \frac{du}{u+1} \left(\int_{u^p}^\infty \frac{dt}{t \ln^{1+p} t + u^p} \right)^{\frac{1}{p}} \\ &\ge \frac{1}{4} \int_e^\infty \frac{du}{u+1} \left(\int_{u^p}^\infty \frac{dt}{t \ln^{1+p} t} \right)^{\frac{1}{p}} \\ &\ge \frac{1}{8} \int_e^\infty \frac{1}{u+1} \cdot \frac{1}{\ln u^p} \, du \\ &= \infty, \end{aligned}$$

in other words (see (2.27)) $\eta \notin \mathfrak{J}(L_p)$.

(ii) It should be pointed out that the space $\mathcal{L}(1,1)$ coincides with $L_\infty \cap L \log L := \mathcal{L} \log \mathcal{L}$ where, the space $L \log L$ is the Zygmund space (see [BS, pp. 243, 266] and [BR]). It follows from Theorem 4.3 and Corollary 4.4 below that $J(L_1) = \mathcal{L} \log \mathcal{L}$.

Corollary 3.8 *If $p \in [1, \infty)$, $q \in (p, \infty)$ and $(1 + D_0^2)^{-\frac{1}{2}} \in \mathcal{L}(p,q)(\mathcal{M}, \tau)$, then there exists a constant C, depending on p, q and D_0 only, such that*

$$\left\| \frac{D}{(1+D^2)^{\frac{1}{2}}} - \frac{D_0}{(1+D_0^2)^{\frac{1}{2}}} \right\|_{\mathcal{L}_p(\mathcal{M},\tau)} \le C \cdot \|D - D_0\|$$

whenever $\|D - D_0\| \le 1$.

We shall now see what information concerning distribution function n_f of element $f \in \mathfrak{J}(L_p)$ may be obtained in terms of estimate (3.9). For brevity we let

$$B(p,q) = \{x = x^* \in \mathcal{L}_p : n_x(\lambda) \le C\lambda^{-p} \ln^{-q}(1/\lambda) \text{ for sufficiently small } \lambda > 0\}.$$

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Theorem 3.9 *If $q > p+1$, then*

$$B(p,q) \subseteq \mathfrak{J}(L_p) \subseteq B(p,p). \tag{3.12}$$

Proof For $x \in B(p,q)$ we have

$$\|\min(2^{-n},x)\|_{L_p} \leq \left(p\int_0^{2^{-n}} C\lambda^{-p}\ln^{-q}(1/\lambda)\lambda^{p-1}\,d\lambda\right)^{1/p} = \left(\frac{Cp}{q-1}(n\ln 2)^{-q+1}\right)^{1/p}.$$

By assumption $\frac{-q+1}{p} < -1$ and we can infer now that $\sum_{n=1}^{\infty}\|\min(2^{-n},x)\|_{L_p} < \infty$, whence, by (2.28), the first embedding in (3.12) is proven.

Conversely, given $x \in \mathfrak{J}(L_p)$ we have (again via (2.28)) that $\sum_{n=1}^{\infty}\|\min(2^{-n},x)\|_{L_p} < \infty$ and since the sequence $\{\|\min(2^{-n},x)\|_{L_p}\}_{n=1}^{\infty}$ decreases there exists a constant $C > 0$ such that $\|\min(2^{-n},x)\|_{L_p} \leq \frac{C}{n}$ for all $n = 1,2,\ldots$. In other words we have

$$\begin{aligned}\left(\frac{C}{n}\right)^p &\geq p\int_0^{2^{-n}} s^{p-1}n_x(s)\,ds\\ &\geq p\int_{2^{-n-1}}^{2^{-n}} s^{p-1}n_x(s)\,ds\\ &\geq n_x(2^{-n-1})(2^{-np}-2^{-(n+1)p})\\ &= n_x(2^{-n-1})2^{-np}(1-2^{-p}),\end{aligned}$$

or

$$n_x(2^{-k}) \leq C'2^{kp}k^{-p} = (C'\ln^p 2)(2^{-k})^{-p}\ln^{-p}(2^k)$$

for all $k = 1,2,\ldots$ and some $C' > 0$. It easily follows from that the latter inequality that $x \in B(p,p)$. This completes the proof of Theorem 3.9. ■

The following theorem shows that the description of n_x, $x \in \mathfrak{J}(L_p)$ given by Theorem 3.9 is exact in the terms of (3.9).

Theorem 3.10 *If $r > p$, then*

$$B(p,p+1) \not\subseteq \mathfrak{J}(L_p) \not\subseteq B(p,r).$$

Proof Let $y = y^* \in L_\infty(0,\infty)$ be such that

$$n_y(\lambda) = \begin{cases}\lambda^{-p}\ln^{-p-1}(1/\lambda), & \text{if } 0 < \lambda \leq 1/4;\\ 0, & \text{if } \lambda > 1/4.\end{cases}$$

Clearly, $y \in B(p,p)$. On the other hand,

$$\begin{aligned}\|\min(2^{-n},y)\|_{L_p}^p &= p\int_0^{2^{-n}} s^{p-1}n_y(s)\,ds\\ &\geq p\int_0^{2^{-n}} \frac{s^{-1}\,ds}{\ln^{p+1}(s^{-1})}\\ &= C\frac{1}{\ln^p(n)}\end{aligned}$$

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for some $C > 0$ and any $n = 2, 3, \ldots$. It follows immediately that $\sum_{n=1}^{\infty} \| \min(2^{-n}, y)\|_{L_p} = \infty$ and applying (2.28) we see that $y \notin \mathcal{J}(L_p)$.

To establish the second assertion of Theorem 3.10 we fix $p < s < r$ and set

$$z(t) = \sum_{j=0}^{\infty} 2^{-2^{j+1}} \chi_{(t_j, t_{j+1})}(t),$$

where $t_0 = 0$, $t_j = 2^{p2^j - js}$ for $j = 1, 2, \ldots$. Then $n_z(\lambda) = t_j$ if $2^{-2^{j+1}} < \lambda < 2^{-2^j}$, whence

$$\begin{aligned}
\lim_{j\to\infty} \left(\sup \left\{ \frac{n_z(\lambda)}{\lambda^{-p}(\ln \lambda)^{-r}}, 2^{-2^{j+1}} < \lambda < 2^{-2^j} \right\} \right) &= \lim_{j\to\infty} \frac{t_j}{(2^{-2^j})^{-p}(\ln 2^{2^j})^{-r}} \\
&= \lim_{j\to\infty} \frac{2^{p2^j - js}}{2^{p2^j - jr}(\ln 2)^{-r}} \\
&= \lim_{j\to\infty} (\ln 2)^r 2^{j(r-s)} \\
&= \infty,
\end{aligned}$$

i.e., $z \notin B(p, r)$. On the other hand for $2^j \le m < 2^{j+1}$ we have,

$$\begin{aligned}
\| \min(2^{-m}, z)\|_{L_p}^p &= 2^{-mp} t_j + \sum_{k=j}^{\infty} 2^{-p2^{k+1}} (t_{k+1} - t_k) \\
&\le \sum_{k=j}^{\infty} 2^{-p2^k} t_k = \sum_{k=j}^{\infty} 2^{-ks} \le 2^{-js+1}.
\end{aligned}$$

Hence,

$$\sum_{m=1}^{\infty} \| \min(2^{-m}, z)\|_{L_p} \le \sum_{j=1}^{\infty} 2^{(-js+1)/p} 2^j = 2^{1/p} \sum_{j=1}^{\infty} 2^{(1-s/p)j} < \infty,$$

i.e., $z \in \mathcal{J}(L_p)$. This completes the proof of Theorem 3.10. ∎

4 Lipschitz Estimates: Lorentz Spaces

In this section we shall study another special case of symmetric operator spaces $E(\mathcal{M}, \tau)$, namely Lorentz spaces. It is possible (see Theorem 4.3 below) to completely characterize the ideal $J(E)$ in this setting. It is interesting to note that this characterization may be applied further to general symmetric spaces, in particular another strengthening of Theorem 3.2, different in spirit from those given by Theorems 3.6, 3.9, 3.10, is presented in Corollary 4.6.

Definition 4.0 **[KPS], [LT2]** If $\psi \colon [0, \infty) \to [0, \infty)$ is a positive concave continuous function on $[0, \infty)$ with $\psi(0) = 0$, then Lorentz space $\Lambda_\psi = \Lambda_\psi[0, \infty)$ is the space of all measurable functions g on $[0, \infty)$ so that

$$\|g\|_\psi = \int_0^\infty g^*(t)\, d\psi(t) < \infty.$$

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Definition 4.1 Given symmetric space E we set

$$\mathcal{E}\log\mathcal{E} := \left\{x \in \mathcal{E} : \min(1, x^*) \cdot \ln\left(\frac{1}{\min(1, x^*)}\right) \in E\right\}.$$

In the case $E = \Lambda_\psi$ we denote $\mathcal{E}\log\mathcal{E}$ as $\Lambda_\psi \log \Lambda_\psi$.

Proposition 4.2 *$\mathcal{E}\log\mathcal{E}$ is a rearrangement-invariant ideal in $\mathcal{E}$.*

Proof The inclusion $x \in \mathcal{E}\log\mathcal{E}$ implies immediately that $\lambda x \in \mathcal{E}\log\mathcal{E}$ for any $\lambda \in \mathbb{C}$. The implication

$$\big(y^*(t) \le x^*(t), t > 0 \quad \text{and} \quad x \in \mathcal{E}\log\mathcal{E}\big) \Longrightarrow y \in \mathcal{E}\log\mathcal{E} \tag{4.1}$$

is also easy, if to take into account that the function $f(t) := t\ln(t)$, $t \in (0, 1)$ is decreasing. Further, it follows from [KPS, Theorem II.4.4] that $x \in \mathcal{E}$ implies $\sigma_s x \in \mathcal{E}$ for any $s > 0$, therefore, taking into account that $\big(x \cdot \ln(\frac{1}{x})\big)^* = x^* \cdot \ln(\frac{1}{x^*})$ (provided, $|x| < \frac{1}{e}$) and that $\sigma_s\big(x^* \cdot \ln(\frac{1}{x^*})\big) = \sigma_s(x^*) \cdot \ln(\frac{1}{\sigma_s(x^*)})$ we immediately infer that $x^* \in \mathcal{E}\log\mathcal{E}$ implies $\sigma_s(x^*) \in \mathcal{E}\log\mathcal{E}$. This fact and (4.1) combined with Proposition 2.7 give the assertion. ■

If E is a Lorentz space, then the space $\mathcal{E}\log\mathcal{E}$ is a special example of (so-called) Orlicz-Lorentz spaces and it may be equipped with a norm to become a symmetric function space (see *e.g.* [Ka]). In this setting the space $\mathcal{E}\log\mathcal{E}$ is a direct generalization of the Zygmund space $L\log L$ (see Remark 3.7).

We will assume, that $\psi(t) \to \infty$ as $t \to \infty$ so that Λ_ψ is separable, in particular, the space $L_\infty(0,\infty) \cap L_1(0,\infty)$ is dense in Λ_ψ (see [KPS, Corollary II.5.3, p. 110]) and therefore $\lim_{t\to\infty} g^*(t) = 0$ for any $g \in \Lambda_\psi$.

Without loss of generality we may (and shall) assume in this section that

$$d_0(t) \le \frac{1}{e}, \quad t > 0.$$

The following theorem is the main result of the present section.

Theorem 4.3 *If $E = \Lambda_\psi$, then*

$$J(E) = \mathcal{E}\log\mathcal{E},$$

and therefore for any $d_0 \in \mathcal{E}\log\mathcal{E}$, there exists a constant C, depending on ψ and D_0 only, such that

$$\left\| \frac{D}{(1+D^2)^{\frac{1}{2}}} - \frac{D_0}{(1+D_0^2)^{\frac{1}{2}}} \right\|_{\Lambda_\psi(\mathcal{M},\tau)} \le C \cdot \|D - D_0\|$$

provided $\|D - D_0\| \le 1$.

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Proof The second assertion follows immediately from the first assertion and Proposition 2.4. It follows from Proposition 2.6, Definition 4.2 and (2.26) that we need only to check that for $f = f^* \in E$

$$\int_0^\infty \frac{du}{u+1} \left\| \frac{1}{\frac{1}{f}+u} \right\|_\psi < \infty \Longleftrightarrow f \in \mathcal{E}\log\mathcal{E}.$$

To this end, noting that

$$\left(\frac{1}{u+\frac{1}{f}}\right)^*(t) = \frac{1}{u+\frac{1}{f(t)}}$$

and that

$$\left(f \cdot \ln\left(\frac{1}{f}\right)\right)^*(t) = f(t)\cdot \ln\left(\frac{1}{f(t)}\right)$$

we have

$$\begin{aligned}
\int_0^\infty \frac{du}{u+1} \left\| \frac{1}{\frac{1}{f}+u} \right\|_\psi &= \int_0^\infty \frac{du}{u+1} \int_0^\infty \frac{1}{u+\frac{1}{f(t)}}\, d\psi(t) \\
&= \int_0^\infty \psi'(t) \left(\int_0^\infty \frac{1}{u+1} \cdot \frac{1}{\frac{1}{f}+u}\, du \right) dt \\
&= \int_0^\infty \psi'(t) \left(\int_0^\infty \left(\frac{1}{u+1} - \frac{1}{\frac{1}{f}+u} \right) du \cdot \frac{1}{\frac{1}{f}-1} \right) dt \\
&= \int_0^\infty \frac{f(t)}{1-f(t)} \cdot \ln\left(\frac{1}{f(t)}\right) \psi'(t)\, dt \\
&\le 2 \left\| f \cdot \ln\left(\frac{1}{f}\right) \right\|_\psi \\
&\le 2 \int_0^\infty \frac{f(t)}{1-f(t)} \cdot \ln\left(\frac{1}{f(t)}\right) \psi'(t)\, dt \\
&= 2 \int_0^\infty \frac{du}{u+1} \left\| \frac{1}{\frac{1}{f}+u} \right\|_\psi .
\end{aligned}$$

∎

Corollary 4.4 *$\mathcal{J}(L_1) = \mathcal{L}\log\mathcal{L}$, and therefore for any $d_0 \in \mathcal{L}\log\mathcal{L}$, there exists a constant C, depending on D_0 only, such that*

$$\left\| \frac{D}{(1+D^2)^{\frac{1}{2}}} - \frac{D_0}{(1+D_0^2)^{\frac{1}{2}}} \right\|_{\mathcal{L}_1(\mathcal{M},\tau)} \le C \cdot \|D - D_0\|$$

provided $\|D - D_0\| \le 1$.

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Recall (see [KPS, p. 48]) that two positive functions ϕ and ψ on $(0,\infty)$ are said to be equivalent if there exist positive constants C_1 and C_2 such that

$$C_1\psi(t) \le \phi(t) \le C_2\psi(t), \quad \forall t > 0.$$

We further note, that the fundamental function ϕ_E is quasiconcave [KPS, Theorem II.4.7] and therefore equivalent to its smallest concave majorant $\widetilde{\phi_E}$ [KPS, Theorem II.1.1 and Corollary]. It is also well-known that $\Lambda_{\widetilde{\phi_E}}$ is the smallest among symmetric spaces with the fundamental function equivalent to ϕ_E (see *e.g.* [KPS, Theorem II.5.5], [BS, Corollary 2.5.14]).

Taking into account that the norm E is majorised by the norm of $\Lambda_{\widetilde{\phi_E}}$ [KPS, Theorem II.5.5] we arrive at the following corollary.

Corollary 4.5 *For an arbitrary symmetric function space E it follows from $d_0 \cdot \ln(\frac{1}{d_0}) \in \Lambda_{\widetilde{\phi_E}}$ that $d_0 \in \mathfrak{J}(E)$ and therefore there exists a constant C, depending on ϕ_E and D_0 only, such that*

$$\left\| \frac{D}{(1+D^2)^{\frac{1}{2}}} - \frac{D_0}{(1+D_0^2)^{\frac{1}{2}}} \right\|_{E(\mathcal{M},\tau)} \le C \cdot \|D - D_0\|$$

provided $\|D - D_0\| \le 1$.

In the special case $E = L_p$, $1 \le p < \infty$ we have $\phi_{L_p}(t) = \widetilde{\phi_{L_p}}(t) = t^{\frac{1}{p}}$, $0 < t < \infty$ and $\Lambda_{t^{\frac{1}{p}}} = L_{p,1}$ (see Definition 3.1). The following result yields new information concerning $J(L_p)$ comparatively with Theorem 3.2.

Corollary 4.6 *If $d_0 \cdot \ln(\frac{1}{d_0}) \in \mathcal{L}_{p,1}[0,\infty)$, $1 < p < \infty$, then $d_0 \in \mathfrak{J}(L_p)$ and therefore there exists a constant C, depending on p and D_0 only, such that*

$$\left\| \frac{D}{(1+D^2)^{\frac{1}{2}}} - \frac{D_0}{(1+D_0^2)^{\frac{1}{2}}} \right\|_{L_p(\mathcal{M},\tau)} \le 10C \cdot \|D - D_0\|$$

provided $\|D - D_0\| \le 1$.

To see that Corollary 4.6 indeed yields a strengthening of Theorem 3.2 we need to establish the following two assertions

$$\bigcup_{r<p} \mathcal{L}_r \subseteq \mathcal{L}_{p,1} \log \mathcal{L}_{p,1}, \quad \mathcal{L}_{p,1} \log \mathcal{L}_{p,1} \ne \bigcup_{r<p} \mathcal{L}_r. \tag{4.2}$$

We shall essentially follow to the arguments and notation following Proposition 3.5. The first assertion in (4.2) would follow from the embedding $\zeta \cdot \ln(\frac{1}{\zeta}) \in \mathcal{L}_{p,1}$ for any given $r < p$ (here we also used the fact that $\mathcal{L}_{p,1} \log \mathcal{L}_{p,1}$ is linear, rearrangement-invariant manifold). The following simple calculation validates this fact

$$\begin{aligned} \left\| \zeta \cdot \ln\left(\frac{1}{\zeta}\right) \right\|_{p,1} &= \int_0^1 dt^{\frac{1}{p}} + \int_1^\infty t^{-\frac{1}{r}} \ln(t^{-\frac{1}{r}})\, dt^{\frac{1}{p}} \\ &= 1 + \frac{1}{r} \int_1^\infty t^{-\frac{1}{r}+\frac{1}{p}-1} \ln(t)\, dt \\ &< \infty. \end{aligned}$$

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The second assertion will be proven if it is shown that $\xi \in \mathcal{L}_{p,1} \log \mathcal{L}_{p,1}$ for given $\alpha > 2$. It is confirmed by the following calculation

$$\begin{aligned}\int_e^\infty \frac{\ln(t^{\frac{1}{p}} \cdot \ln^\alpha t)}{t^{-\frac{1}{p}} \cdot \ln^{-\alpha} t} \cdot t^{\frac{1}{p}-1}\, dt &= \frac{1}{p} \int_e^\infty t^{-1} \cdot \ln^{-\alpha} t \cdot \ln t \, dt \\ &\quad + \frac{1}{\alpha} \int_e^\infty t^{-1} \cdot \ln^{-\alpha} t \cdot \ln(\ln t)\, dt \\ &\leq 2 \max\left\{\frac{1}{p}, \frac{1}{\alpha}\right\} \int_e^\infty \frac{dt}{t \cdot \ln^{\alpha-1} t} \\ &< \infty.\end{aligned}$$

The following proposition shows that in most cases $\mathcal{J}(E)$ is a proper subset of $\Lambda_{\widetilde{\phi_E}}$.

We let $\phi_E(\infty) := \lim_{t\to\infty} \phi_E(t)$ and note that if a natural embedding of E into its second associate space (see [KPS, II.4.6]) is an isometry, then $\phi_E(\infty) < \infty$ if and only if $L_\infty(0,\infty) \subseteq E$.

Proposition 4.7

(i) $J(E) \subseteq \Lambda_{\widetilde{\phi_E}}$ *for an arbitrary symmetric space* E.
(ii) If $\phi_E(\infty) = \infty$, *then* $J(E) \neq \Lambda_{\widetilde{\phi_E}}$.

Proof (i) If $x = x^* \in \mathcal{J}(E)$, $|x| \leq \frac{1}{2}$ then, via (2.28), we have that $\int_1^\infty \frac{du}{u} \|\min(\frac{1}{u}, x)\|_E < \infty$ and since

$$\begin{aligned}\left\|\min\left(\frac{1}{u}, x\right)\right\|_E &= \left\|\frac{1}{u} \cdot \chi_{[0,n_x(\frac{1}{u}))} + x \cdot \chi_{[n_x(\frac{1}{u}),\infty)}\right\|_E \\ &\leq \frac{1}{u} \|\chi_{[0,n_x(\frac{1}{u}))}\|_E + \|x \cdot \chi_{[n_x(\frac{1}{u}),\infty)}\|_E \\ &\leq 2 \left\|\min\left(\frac{1}{u}, x\right)\right\|_E\end{aligned}$$

we deduce that also $\int_1^\infty \frac{du}{u^2} \|\chi_{[0,n_x(\frac{1}{u}))}\|_E < \infty$, or, in other words,

$$\int_1^\infty \frac{du}{u^2} \phi_E\left(n_x\left(\frac{1}{u}\right)\right) < \infty. \tag{4.3}$$

Substituting $u = \frac{1}{t}$ in (4.3) we arrive at

$$\int_0^1 \phi_E\big(n_x(t)\big)\, dt < \infty. \tag{4.4}$$

Taking into account that $n_x(t) = 0$, $t > \frac{1}{2}$ we have from (4.4), from the fact that $\widetilde{\phi_E}$ and ϕ_E are equivalent and from [KPS, formula (II.5.4), p. 111] that

$$\|x\|_{\widetilde{\phi_E}} = \int_0^\infty \widetilde{\phi_E}\big(n_x(t)\big)\, dt < \infty.$$

(ii) We fix an arbitrary sequence of positive real numbers $\{a_i\}_{i=1}^{\infty}$ such that $\sum_{i=1}^{\infty} a_i < \infty$. Using the assumption $\phi_E(\infty) = \infty$ we may always select an increasing sequence of positive reals $\{t_i\}_{i=1}^{\infty}$ such that

$$s_i := \ln\big(\widetilde{\phi_E}(t_i)\big) - \ln(a_i), \quad i = 1, 2, \dots$$

is an increasing sequence of positive integers such that

$$\sum_{i=1}^{\infty}(s_{i+1} - s_i)a_{i+1} = \infty. \tag{4.5}$$

We further let

$$c_i := \frac{a_i}{\widetilde{\phi_E}(t_i)} (= e^{-s_i}), \quad x_i := c_i \chi_{[0,t_i)}, \quad i = 1, 2, \dots .$$

Clearly,

$$\|x_i\|_{\widetilde{\phi_E}} = c_i \widetilde{\phi_E}(t_i) = a_i, \quad i = 1, 2, \dots$$

and since $\sum_{i=1}^{\infty} \|x_i\|_{\widetilde{\phi_E}} = \sum_{i=1}^{\infty} a_i < \infty$ we may now set

$$x^* = x := \sum_{i=1}^{\infty} x_i \in \Lambda_{\widetilde{\phi_E}}.$$

Since the functions $\widetilde{\phi_E}$ and ϕ_E are equivalent we also have

$$\|x_i\|_E = c_i \phi_E(t_i) \geq C a_i \quad i = 1, 2, \dots$$

for some $C > 0$. It follows that for $s \in (0, s_i)$ we have $e^{-s} > c_i$ and

$$\|\min(e^{-s}, x)\|_E \geq \|\min(e^{-s}, x_i)\|_E = \|x_i\|_E \geq C a_i. \tag{4.6}$$

It now follows from (4.5) and (4.6) that

$$\sum_{s=1}^{\infty} \|\min(e^{-s}, x)\|_E = \sum_{i=1}^{\infty} \sum_{s_i \leq s < s_{i+1}} \|\min(e^{-s}, x)\|_E \geq C \sum_{i=1}^{\infty} (s_{i+1} - s_i) a_{i+1} = \infty.$$

The proof is completed since $\sum_{s=1}^{\infty} \|\min(2^{-s}, x)\|_E \geq \sum_{s=1}^{\infty} \|\min(e^{-s}, x)\|_E$ (see (2.28)). ■

Remark 4.8 Given symmetric space E with the fundamental function ϕ_E we always have

$$\Lambda_{\widetilde{\phi_E}} \subseteq E \subseteq M_{\frac{t}{\phi_E(t)}}$$

(see [BS, pp. 71–73], [KPS, Theorems II.5.5–5.7]), where $M_{\frac{t}{\phi_E(t)}}$ is a Marcinkiewicz space (see Section 5 below). If $E = L_p[0, \infty)$, $1 < p < \infty$, then $\phi_E(t) = t^{1/p}$ and $\Lambda_{\phi_E} = L_{p,1}$ and

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$M_{\frac{t}{\phi_E(t)}} = L_{p,\infty}$ (see Definition 3.1 and [KPS, Theorem II.5.3]). Proposition 4.7 (ii) shows that even in this situation the assumption $d_0 \in L_{p,1}$ does not guarantee that $d_0 \in \mathcal{J}(L_{p,\infty})$.

Remark 4.9 (i) Combining Proposition 3.5 with Corollary 4.6 we obtain the following strengthening of Theorem 3.2

$$\mathcal{L}(p,q) + \mathcal{L}_{p,1} \log \mathcal{L}_{p,1} \subseteq J(L_p), \quad \forall q \in (p,\infty).$$

Generally speaking, the latter result is stronger than Corollary 4.6. Indeed, if $\frac{1+q}{p} < \alpha \leq 2$, then a direct verification shows that the function ξ from the proof of Proposition 3.5 does not belong to $\mathcal{L}_{p,1} \log \mathcal{L}_{p,1}$ and therefore $\mathcal{L}(p,q)$ is not a proper subset of $\mathcal{L}_{p,1} \log \mathcal{L}_{p,1}$ for $q \in (p, 2p-1)$, $p > 1$. However, exact relationship between the spaces $\mathcal{L}(p,q)$, $p < q$ and $\mathcal{L}_{p,1} \log \mathcal{L}_{p,1}$ for $p \neq 1, q \neq 1$ is still unclear.

(ii) An analogue of Corollary 4.4 for L_p-spaces is not valid for $p > 2$. In other words, if $p > 2$ then $\mathcal{L}(p,1) \nsubseteq J(L_p)$ and, moreover, $\mathcal{L}(p,q) \nsubseteq J(L_p)$, $q \in [1, p-1]$. Indeed, let

$$\rho(t) := \begin{cases} 1, & \text{if } 0 \leq t < e; \\ \frac{1}{t^{1/p} \ln(t) \ln(\ln(10+t))}, & \text{if } t \geq e. \end{cases}$$

It is not difficult to verify that

$$\rho \in \mathcal{L}(p,q), \quad 1 \leq q \leq p-1 \quad \text{and, on the other hand,} \quad \rho \notin \mathcal{L}_{p,1}.$$

It now follows from Proposition 4.7 (i) that $\rho \notin J(L_p)$.

5 Lipschitz Estimates: Marcinkiewicz Spaces

We recall at first the definition and some simple properties of Marcinkiewicz spaces and associated concave functions, for more substantial information we refer to [KPS], [BS].

Let

$$\Psi := \{\psi : [0,\infty) \to [0,\infty) : \psi \text{ is concave and increasing with } \lim_{t\to\infty} \psi(t) = \infty\}$$

and let

$$\Psi_0 := \left\{ \psi \in \Psi : \liminf_{t\to\infty} \frac{\psi(2t)}{\psi(t)} > 1 \right\}.$$

For example, given $\alpha \in (0,1]$ we set

$$\phi_\alpha(t) := \ln^\alpha(t+e^2), \quad t > 0 \quad \text{and} \quad \psi_\alpha(t) := \frac{t}{\ln^\alpha(t+e^2)}, \quad t > 0.$$

As it is easily seen, we have $\phi_\alpha \in \Psi$ (however, $\phi_\alpha \notin \Psi_0$) and $\psi_\alpha \in \Psi_0$ for all $\alpha \in (0,1]$.

It follows from [KPS, Lemma II.1.4, p. 56] (see inequalities (II.1.30) and (II.1.23)) that any function $\psi \in \Psi_0$ is equivalent to the function $\int_0^t \psi(\tau)\tau^{-1}\,d\tau$, *i.e.*, there are two positive constants C_1, C_2 such that

$$C_1\psi(t) \leq \int_0^t \psi(\tau)\tau^{-1}\,d\tau \leq C_2\psi(t), \quad t > 0.$$

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For instance, it follows immediately, that given $\alpha \in (0,1]$ the function ψ_α is equivalent to the function Li_α (also belonging to Ψ_0) given by

$$\mathrm{Li}_\alpha(t) := \int_0^t \frac{du}{\ln^\alpha(u+e^2)}.$$

If $\alpha = 1$ we shall omit index in the notation Li_α and use just Li.

Definition 5.1 **[KPS]** For $\psi \in \Psi$ the Marcinkiewicz space $M_\psi = M_\psi[0,\infty)$ consists of all measurable functions x on $[0,\infty)$ for which

$$\|x\|_{M_\psi} := \sup_{t>0} \frac{1}{\psi(t)} \int_0^t x^*(s)\,ds < \infty.$$

It should be noted, that

$$\|x\|_{M_\psi} = \sup_{t>0} \psi_*(t) x^{**}(t) \tag{5.1}$$

where $\psi_*(t) := \frac{t}{\psi(t)}$ is the fundamental function of $(M_\psi, \|\cdot\|_{M_\psi})$ (see [KPS, Ch. II.6 and p. 101]). Recall that for an arbitrary $\psi \in \Psi$ the fundamental function ψ_* is quasiconcave [KPS, Theorem II.4.7] and therefore equivalent to its smallest concave majorant $\widetilde{\psi_*}$ [KPS, Theorem II.1.1 and Corollary].

For the sake of brevity in this section we shall employ the notation $(\mathrm{Li}^\alpha, \|\cdot\|_{\mathrm{Li}^\alpha})$ rather than $(M_{\mathrm{Li}_\alpha}, \|\cdot\|_{M_{\mathrm{Li}_\alpha}})$. It is clear, that since the functions ψ_α and Li_α are equivalent, we have $\mathrm{Li}^\alpha = M_{\psi_\alpha}$ and the norms $\|\cdot\|_{\mathrm{Li}^\alpha}$ and $\|\cdot\|_{M_{\psi_\alpha}}$ are equivalent for any $\alpha \in (0,1]$.

In the case $\psi \in \Psi_0$, a simpler formula to estimate norm $\|\cdot\|_{M_\psi}$ than (5.1) may be employed. Namely, the functional $F\colon M_\psi \to [0,\infty)$ given by

$$F(x) := \sup_{0<t<\infty} \psi_*(t) x^*(t), \tag{5.2}$$

is equivalent to $\|x\|_{M_\psi}$ (see [KPS, Theorem II.5.3]). Since the functions $\mathrm{Li}_\alpha(t)$ and $\psi_\alpha(t)$ are equivalent for any $\alpha \in (0,1]$, it follows immediately that $\frac{t}{\mathrm{Li}_\alpha(t)}$ is equivalent to $\frac{t}{\psi_\alpha(t)}$. Therefore, given $\alpha \in (0,1]$, there exist positive constants C_1, C_2 such that

$$C_1 \|x\|_{\mathrm{Li}^\alpha} \le \sup_{0<t<\infty} \ln^\alpha(t+e^2) x^*(t) \le C_2 \|x\|_{\mathrm{Li}^\alpha}, \quad x \in \mathrm{Li}^\alpha. \tag{5.3}$$

It is worth to note that if $(\mathcal{M},\tau)$ is $\mathcal{B}(\mathcal{H})$ equipped with the standard trace and $\alpha = 1$ (respectively, $\alpha = 1/2$), then $\mathrm{Li}^\alpha(\mathcal{M},\tau)$ coincides with the ideal $\mathrm{Li}(\mathcal{H})$ (respectively, $\mathrm{Li}^{1/2}(\mathcal{H})$) considered in [Co2, p. 391].

Recall also that the space $(M_\psi, \|\cdot\|_{M_\psi})$ is a fully symmetric Banach function space, that $\psi' \in M_\psi$ and that $\psi' \notin L_1[0,\infty)$, since $\psi(\infty) = \infty$. In other words, $M_\psi \not\subseteq L_1[0,\infty)$. If

$$\lim_{t\to\infty} \frac{t}{\psi(t)} = \infty, \tag{5.4}$$

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then $\chi_{[0,\infty)} \notin M_\psi$), in other words $L_\infty[0,\infty) \nsubseteq M_\psi$. If (5.4) holds, then the set of all $x \in M_\psi$ for which

$$\lim_{h\to 0,\infty} \frac{1}{\psi(t)} \int_0^t x^*(s)\,ds = 0$$

is denoted by M_ψ^0. The latter space is a fully symmetric subspace of M_ψ. An alternative description of M_ψ^0 is given by

$$x \in M_\psi^0 \quad \Longleftrightarrow \quad \lim_{s\to 0}\big(\|\min(s,x)\|_{M_\psi} + \|x^*\chi_{[0,s)}\|_{M_\psi}\big) = 0.$$

Indeed, the implication $\Longrightarrow$ follows from the fact that $(M_\psi^0, \|\cdot\|_{M_\psi})$ is a separable symmetric space [KPS, Theorem II.5.4]. The implication $\Longleftarrow$ follows from the fact that M_ψ^0 coincides with the closure of the set of all bounded functions of compact support in M_ψ [KPS, Lemma II.5.4]. It follows from the description just given and Proposition 2.6 that $J(M_\psi) \subseteq M_\psi^0$.

More detailed information on the set $J\big(M_\psi[0,\infty)\big)$ is contained in the following theorem, which is the main result of this section .

Theorem 5.2

(i) *For an arbitrary $\psi \in \Psi$ we have*

$$\Lambda_{\widetilde{\psi_*}} \log \Lambda_{\widetilde{\psi_*}} \subseteq J(M_\psi) \subseteq \Lambda_{\widetilde{\psi_*}} \subseteq M_\psi^0.$$

(ii) *For an arbitrary $\psi \in \Psi_0$ we have*

$$\mathcal{J}(M_\psi) \cap \mathcal{G}_{\psi_*} = \Lambda_{\widetilde{\psi_*}} \cap \mathcal{G}_{\psi_*}, \quad J(M_\psi) \cap G_{\psi_*} = \Lambda_{\widetilde{\psi_*}} \cap G_{\psi_*},$$

where

$$\mathcal{G}_{\psi_*} := \{x = x^* \in L_1(0,\infty) + L_\infty(0,\infty) : \exists\, t_0 \text{ such that } x(t)\psi_*(t) \text{ is decreasing for } t \geq t_0\}$$

and G_{ψ_*} is the smallest rearrangement-invariant ideal containing $\mathcal{G}_{\psi_*}$.

(iii) *For an arbitrary $\psi \in \Psi$ it follows from the embeddings*

$$x \in \mathcal{H}_{\psi_*} \quad \text{and} \quad x^{**} \in \Lambda_{\widetilde{\psi_*}}$$

where

$$\mathcal{H}_{\psi_*} := \{x = x^* \in L_1(0,\infty) + L_\infty(0,\infty) : \\ \exists\, t_0 \text{ such that } x^{**}(t)\psi_*(t) \text{ is decreasing for } t \geq t_0\},$$

that $x \in J(M_\psi)$.

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Proof (i) The second embedding follows immediately from Proposition 4.7 (i) and the first follows from Theorem 4.3 since $\Lambda_{\widetilde{\psi_*}}$ is continuously embedded into M_ψ (see also remark following Corollary 4.4 and Corollary 4.5). The third embedding follows immediately from [KPS, Theorem II.5.5].

(ii) The embedding

$$\mathcal{J}(M_\psi) \cap \mathcal{G}_{\psi_*} \subseteq \Lambda_{\widetilde{\psi_*}} \cap \mathcal{G}_{\psi_*}$$

follows immediately from (i). We shall now show that $x \in \Lambda_{\widetilde{\psi_*}} \cap \mathcal{G}_{\psi_*}$ implies $x \in \mathcal{J}(M_\psi)$. To ensure the latter embedding, it suffices to show that $\int_1^\infty \frac{du}{u} \| \min(\frac{1}{u}, x)\|_{M_\psi} < \infty$ (see (2.26) and (2.27)), or equivalently, that for some positive Δ we have $\int_\Delta^\infty \frac{du}{u} \| \min(\frac{1}{u}, x)\|_{M_\psi} < \infty$. By the assumption, there exists t_0 such that

$$x(t_1)\psi_*(t_1) \geq x(t_2)\psi_*(t_2), \quad t_0 \leq t_1 \leq t_2 \tag{5.5}$$

and without loss of generality we may take Δ so that $\frac{1}{\Delta} < x(t_0)$. It follows that $n_x(\frac{1}{u}) \geq t_0$, provided that $u \geq \Delta$. It should be noted that, since $x \in \mathcal{G}_{\psi_*}$ we have $m\big(\{t : x(t) = \frac{1}{u}\}\big) = 0$, $u \geq \Delta$, therefore

$$\min\left(\frac{1}{u}, x\right) = \frac{1}{u}\chi_{[0,n_x(\frac{1}{u}))} + x \cdot \chi_{[n_x(\frac{1}{u}),\infty)}.$$

Using (5.2) and (5.5) we now have

$$\begin{aligned}
&\int_\Delta^\infty \frac{du}{u} \left\| \min\left(\frac{1}{u}, x\right)\right\|_{M_\psi} \\
&\quad \approx \int_\Delta^\infty \frac{du}{u} F\left(\min\left(\frac{1}{u}, x\right)\right) \\
&\quad = \int_\Delta^\infty \frac{du}{u} \sup_{0<t<\infty} \left\{ \left(\frac{1}{u} \cdot \chi_{[0,n_x(\frac{1}{u}))}(t) + x \cdot \chi_{[n_x(\frac{1}{u}),\infty)}(t)\right) \psi_*(t)\right\} \\
&\quad = \int_\Delta^\infty \frac{du}{u} \max\left\{ \sup_{0<t<n_x(\frac{1}{u})} \left\{\frac{1}{u}\psi_*(t)\right\}, x\left(n_x\left(\frac{1}{u}\right)\right)\psi_*\left(n_x\left(\frac{1}{u}\right)\right)\right\} \\
&\quad = \int_\Delta^\infty \frac{du}{u^2} \psi_*\left(n_x\left(\frac{1}{u}\right)\right).
\end{aligned}$$

Since $x \in \Lambda_{\widetilde{\psi_*}}$ we may use the same arguments as in the proof of Proposition 4.7 (i) (see (4.3) and (4.4)) to show that $\int_\Delta^\infty \frac{du}{u^2}\psi_*\big(n_x(\frac{1}{u})\big) < \infty$. It completes the proof of the equality $\mathcal{J}(M_\psi) \cap \mathcal{G}_{\psi_*} = \Lambda_{\widetilde{\psi_*}} \cap \mathcal{G}_{\psi_*}$.

Let A (respectively, B) be the smallest rearrangement-invariant ideal containing $\mathcal{J}(M_\psi) \cap \mathcal{G}_{\psi_*}$ (respectively, $\Lambda_{\widetilde{\psi_*}} \cap \mathcal{G}_{\psi_*}$). It follows from the first part of the proof that $A = B$. Since $J(M_\psi) \cap G_{\psi_*}$ (respectively, $\Lambda_{\widetilde{\psi_*}} \cap G_{\psi_*}$) is a rearrangement-invariant ideal containing $\mathcal{J}(M_\psi) \cap \mathcal{G}_{\psi_*}$ (respectively, $\Lambda_{\widetilde{\psi_*}} \cap \mathcal{G}_{\psi_*}$) we see that $A \subseteq J(M_\psi) \cap G_{\psi_*}$ (respectively, $B \subseteq \Lambda_{\widetilde{\psi_*}} \cap G_{\psi_*}$). However, using Proposition 2.7, it is easy to see that in fact $A = J(M_\psi) \cap G_{\psi_*}$ and $B = \Lambda_{\widetilde{\psi_*}} \cap G_{\psi_*}$.

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(iii) Since

$$\min\left(\frac{1}{u}, x\right) \leq \min\left(\frac{1}{u}, x^{**}\right) = \frac{1}{u}\chi_{[0,n_{x^{**}}(\frac{1}{u}))} + x \cdot \chi_{[n_{x^{**}}(\frac{1}{u}),\infty)}$$

we, using the assumption $x \in \mathcal{H}_{\psi_*}$ and the same argument as in (ii), see that for sufficiently large Δ

$$\begin{aligned}
\int_\Delta^\infty &\frac{du}{u}\left\|\min\left(\frac{1}{u}, x\right)\right\|_{M_\psi} \\
&\leq \int_\Delta^\infty \frac{du}{u}\left\|\min\left(\frac{1}{u}, x^{**}\right)\right\|_{M_\psi} \\
&= \int_\Delta^\infty \frac{du}{u}\sup_{0<t<\infty}\left\{\left(\frac{1}{u}\cdot\chi_{[0,n_{x^{**}}(\frac{1}{u}))}(t) + x\cdot\chi_{[n_{x^{**}}(\frac{1}{u}),\infty)}(t)\right)\psi_*(t)\right\} \\
&= \int_\Delta^\infty \frac{du}{u^2}\psi_*\left(n_{x^{**}}\left(\frac{1}{u}\right)\right).
\end{aligned}$$

Since $x^{**} \in \Lambda_{\widetilde{\psi_*}}$ we have $\int_\Delta^\infty \frac{du}{u^2}\psi_*\left(n_{x^{**}}(\frac{1}{u})\right) < \infty$ and this completes the proof. ■

Remark 5.3 (i) It should be pointed out that the rearrangement-invariant ideal G_{ψ_*} introduced in Theorem 5.2 (ii) is a proper subset of M_ψ. Indeed, if it were that $G_{\psi_*} = M_\psi$ then, according to Theorem 5.2, we have $J(M_\psi) \cap \Lambda_{\widetilde{\psi_*}} = \Lambda_{\widetilde{\psi_*}}$ and this is a contradiction with Proposition 4.7 (ii).

(ii) Recall that symmetric space E has the *Hardy-Littlewood property* (notation: $E \in$ (HLP)) if and only if

$$x \in E \Longrightarrow x^{**} \in E$$

for every $x \in E$ (see, [KPS, II.6.7]). If $\Lambda_{\widetilde{\psi_*}} \in$ (HLP), then we may replace the assumption $x^{**} \in \Lambda_{\widetilde{\psi_*}}$ by the assumption $x \in \Lambda_{\widetilde{\psi_*}}$. By [KPS, Theorem II.6.6 and (II.4.20)], we have

$$\left(\Lambda_{\widetilde{\psi_*}} \in \text{(HLP)}\right) \Longleftrightarrow \left(\lim_{t\to\infty}\frac{1}{t}\|\sigma_t\|_{\Lambda_{\widetilde{\psi_*}}\to\Lambda_{\widetilde{\psi_*}}} = \lim_{t\to\infty}\frac{1}{t}\sup_{s>0}\frac{\psi_*(st)}{\psi_*(s)} = \lim_{t\to\infty}\sup_{s>0}\frac{\psi(s)}{\psi(st)} = 0\right).$$

We shall now specialize our considerations to the case, when $\psi = \psi_\alpha$, $\alpha \in (0,1]$ (*i.e.*, when $M_\psi = \mathrm{Li}^\alpha$), and show in particular that

$$\mathrm{Li}^\beta \subseteq J(\mathrm{Li}^\alpha), \quad 0 < \alpha < \beta \leq 1. \tag{5.6}$$

It is of interest to point out a certain resemblance with the situation in the L_p-setting; we refer here to the assertions of Theorem 3.2 and Corollary 4.6 (see also remarks given thereafter). For brevity we use below the notation $\Lambda_{\ln^\alpha}$ for the Lorentz space $\Lambda_{\ln^\alpha(\cdot+e^2)}[0,\infty)$.

Proposition 5.4 *For an arbitrary* $\alpha \in (0,1]$ *we have*

$$\bigcup_{\beta\in(\alpha,1]}\mathrm{Li}^\beta \subsetneq \Lambda_{\ln^\alpha}\log\Lambda_{\ln^\alpha} \subseteq J(\mathrm{Li}^\alpha).$$

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Proof We shall show first that

$$\mathrm{Li}^{\beta} \subseteq \Lambda_{\ln^{\alpha}} \log \Lambda_{\ln^{\alpha}}, \quad \beta \in (\alpha, 1]. \tag{5.7}$$

It is clear from (5.3) that $x \in \mathrm{Li}^{\beta}$ if and only if $x^*(t) \le \frac{C}{\ln^{\beta}(t+e^2)}$ for some $C > 0$. Therefore, to establish (5.7) we need only to show that

$$\frac{1}{\ln^{\beta}(t+2)} \in \Lambda_{\ln^{\alpha}} \log \Lambda_{\ln^{\alpha}}$$

or, equivalently (see Definition 4.2), that

$$\frac{1}{\ln^{\beta}(t+e^2)} \cdot \ln\bigl(\ln(t+e^2)\bigr) \in \Lambda_{\ln^{\alpha}}. \tag{5.8}$$

We shall check (5.8) via direct computations. Let $\epsilon = \frac{\beta-\alpha}{2} > 0$. Clearly, there exists such $t_0 > 0$ that

$$\ln\bigl(\ln(t+e^2)\bigr) \le \ln^{\epsilon}(t+e^2), \quad \forall t > t_0.$$

By Definition 4.1 and [KPS, (II.5.2), p. 108] we have

$$\begin{aligned}
&\left\| \frac{1}{\ln^{\beta}(t+e^2)} \cdot \ln\bigl(\ln(t+e^2)\bigr) \right\|_{\ln^{\alpha}} \\
&\quad \le 1 + \alpha \int_0^{\infty} \frac{1}{\ln^{\beta}(t+e^2)} \cdot \ln\bigl(\ln(t+e^2)\bigr) \cdot \frac{1}{(t+e^2)\ln^{1-\alpha}(t+e^2)}\,dt \\
&\quad = 1 + \alpha \int_0^{\infty} \frac{\ln\bigl(\ln(t+e^2)\bigr)}{(t+e^2)\ln^{1+\beta-\alpha}(t+e^2)}\,dt \\
&\quad = 1 + \alpha \left(\int_0^{t_0} \frac{\ln\bigl(\ln(t+e^2)\bigr)}{(t+e^2)\ln^{1+2\epsilon}(t+e^2)}\,dt + \int_{t_0}^{\infty} \frac{\ln\bigl(\ln(t+e^2)\bigr)}{(t+e^2)\ln^{1+2\epsilon}(t+e^2)}\,dt \right) \\
&\quad \le 1 + \alpha \left(\int_0^{t_0} \frac{\ln\bigl(\ln(t+e^2)\bigr)}{(t+e^2)\ln^{1+2\epsilon}(t+2)}\,dt + \int_{t_0}^{\infty} \frac{1}{(t+e^2)\ln^{1+\epsilon}(t+e^2)}\,dt \right) \\
&\quad < \infty.
\end{aligned}$$

To complete the proof of Proposition 5.4 we need to show that

$$\bigcup_{\beta \in (\alpha,1]} \mathrm{Li}^{\beta} \neq \Lambda_{\ln^{\alpha}} \log \Lambda_{\ln^{\alpha}}. \tag{5.9}$$

To prove (5.9), let us fix an arbitrary $\epsilon > 0$ and set

$$x^*(t) = x(t) := \frac{1}{\ln^{\alpha}(t+e^2)} \cdot \frac{1}{\ln^{2+\epsilon}\bigl(\ln(t+e^2)\bigr)}, \quad t > 0.$$

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Given $\beta > \alpha$ we have

$$\lim_{t\to\infty} x^*(t)\cdot \ln^{\beta}(t+e^2) = \frac{\ln^{\beta-\alpha}(t+e^2)}{\ln^{2+\epsilon}\big(\ln(t+e^2)\big)} = \infty,$$

whence $x \notin \bigcup_{\beta\in(\alpha,1]} \mathrm{Li}^{\beta}$. Thus, to establish (5.9) it suffices to show that

$$\left\| x\cdot \ln\left(\frac{1}{x}\right)\right\|_{\ln^{\alpha}} < \infty.$$

We have

$$\ln\left(\frac{1}{x(t)}\right) = \alpha\ln\big(\ln(t+e^2)\big) + (2+\epsilon)\ln\Big(\ln\big(\ln(t+e^2)\big)\Big)$$

and taking into account that

$$\ln\big(\ln(t+e^2)\big) > \ln\Big(\ln\big(\ln(t+e^2)\big)\Big)$$

for sufficiently large $t > 0$, we need to show only that

$$\|x(t)\ln\big(\ln(t+2)\big)\|_{\ln^{\alpha}} < \infty.$$

Again using [KPS, (II.5.2), p. 108] we have

$$\begin{aligned}
&\|x(t)\ln\big(\ln(t+e^2)\big)\|_{\ln^{\alpha}}\\
&\quad\le 1+\alpha\int_0^{\infty}\frac{\ln\big(\ln(t+e^2)\big)}{\ln^{\alpha}(t+e^2)}\cdot\frac{1}{\ln^{2+\epsilon}\big(\ln(t+e^2)\big)}\cdot\frac{1}{(t+e^2)\ln^{1-\alpha}(t+e^2)}\,dt\\
&\quad= 1+\alpha\int_0^{\infty}\frac{1}{(t+e^2)\ln(t+e^2)}\cdot\frac{1}{\ln^{1+\epsilon}\big(\ln(t+e^2)\big)}\,dt\\
&\quad= 1+\alpha\int_2^{\infty}\frac{ds}{s\cdot\ln^{1+\epsilon}s}\\
&\quad< \infty.
\end{aligned}$$

■

Remark 5.5 (i) The proof of (5.9) from the preceding proposition shows that if there exist $t_0, C_0, \epsilon > 0$ such that

$$x(t) \le C_0\cdot\frac{1}{\ln^{\alpha}(t+e^2)}\cdot\frac{1}{\ln^{2+\epsilon}\big(\ln(t+e^2)\big)},\quad t > t_0 \tag{5.10}$$

then $x(t) \in J(\mathrm{Li}^{\alpha})$. The asymptotics (5.10) may be further improved via using Theorem 5.2 (ii). Indeed, given $\epsilon > 0$, let

$$\theta_1(t) := \frac{1}{\ln^{\alpha}(t+e^2)}\cdot\frac{1}{\ln^{1+\epsilon}\big(\ln(t+e^2)\big)},\quad t > 0. \tag{5.11}$$

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Since

$$\theta_1(t)\ln^{\alpha}(t+e^2) \geq \theta_1(s)\ln^{\alpha}(s+e^2), \quad s \geq t > 0$$

it follows that $\theta_1 \in \mathcal{G}_{\ln^\alpha}$. Since $\theta_1 = \theta_1^*$ we have also

$$\begin{aligned}
\|\theta_1\|_{\ln^\alpha} &\leq 1 + \alpha \int_0^\infty \frac{1}{\ln^{\alpha}(t+e^2)} \cdot \frac{1}{\ln^{1+\epsilon}\big(\ln(t+e^2)\big)} \cdot \frac{1}{(t+e^2)\ln^{1-\alpha}(t+e^2)}\, dt \\
&= 1 + \alpha \int_0^\infty \frac{1}{\ln(t+e^2)} \cdot \frac{1}{\ln^{1+\epsilon}\big(\ln(t+e^2)\big)}\, d\big(\ln(t+e^2)\big) \\
&= 1 + \alpha \int_2^\infty \frac{ds}{s \cdot \ln^{1+\epsilon} s} \\
&< \infty,
\end{aligned}$$

i.e., $\theta_1 \in \Lambda_{\ln^\alpha}$. By Theorem 5.2 (ii), it follows that $\theta_1 \in \mathcal{J}(\mathrm{Li}^\alpha)$.

(ii) It should be noted that (via similar calculation to that given in the end of the proof of Proposition 5.4) $\theta_1 \notin \Lambda_{\ln^\alpha} \log \Lambda_{\ln^\alpha}$ whenever $\epsilon \in (0,1]$. It is also worth to mention, that similarly to L_p-setting (see Remark 3.7 (i), Theorem 3.9), we can not replace the power $1+\epsilon$ in (5.11) on 1. Indeed, the same calculations as in Remark 5.5 (i) show that the function

$$\theta'(t) := \frac{1}{\ln^{\alpha}(t+e^2)} \cdot \frac{1}{\ln\big(\ln(t+e^2)\big)}, \quad t > 0$$

does not belong to $\Lambda_{\ln^\alpha}$, whence, by Proposition 4.7 (i), does not belong to $\mathcal{J}(\mathrm{Li}^\alpha)$. However the asymptotics given by (5.11) may be improved as follows. For sufficiently large $t > 0$ and $1 \geq \epsilon > 0$ we let

$$\theta_2(t) := \frac{1}{\ln^{\alpha}(t+e^2)} \cdot \frac{1}{\ln\big(\ln(t+e^2)\big)} \cdot \frac{1}{\ln^{1+\epsilon}\Big(\ln\big(\ln(t+e^2)\big)\Big)}.$$

The same calculations as above show that $\theta_2 \in \mathcal{J}(\mathrm{Li}^\alpha)$. By obvious analogy we may define functions $\theta_k \in \mathcal{J}(\mathrm{Li}^\alpha)$ for any $k = 3, 4, \ldots$.

Corollary 5.6 *If there exist $t_0, C_0, \epsilon > 0$ and $k \in \mathbb{N}$ such that*

$$d_0\chi_{[t_0,\infty)}(t) \leq C_0 \cdot \frac{1}{\ln^{\alpha}(t+e^2)} \cdot \frac{1}{\ln^{1+\epsilon}\big(\ln(t+e^2)\big)} \chi_{[t_0,\infty)}(t)$$

then there exists a constant C, depending on ϵ and D_0 only, such that

$$\left\| \frac{D}{(1+D^2)^{\frac{1}{2}}} - \frac{D_0}{(1+D_0^2)^{\frac{1}{2}}} \right\|_{\mathrm{Li}^\alpha(\mathcal{M},\tau)} \leq C \cdot \|D - D_0\|$$

provided that $\|D - D_0\| \leq 1$.

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Proof It follows from Remark 5.5 (i) that that

$$\frac{1}{\ln^{\alpha}(\cdot + e^2)} \cdot \frac{1}{\ln^{1+\epsilon}\big(\ln(\cdot + e^2)\big)} \chi_{[t_0,\infty)}(\cdot) \in \mathcal{J}(\mathrm{Li}^{\alpha}).$$

By Corollary 2.8 we have $f \in J(\mathrm{Li}^{\alpha})$. The assertion follows now from Proposition 2.4. ■

Theorem 5.2 (ii) (respectively, 5.2 (iii)) yields a criterion as to whether a given rearrangement-invariant space E embeds into $J(M_{\psi})$, provided $\psi \in \Psi_0$ (respectively, $\psi \in \Psi$). We shall now adjust the arguments from that theorem to the special case that E is a Marcinkiewicz space. The next result is, in a certain sense, a generalization of the embedding (5.6).

Theorem 5.7 *If ψ, ϕ, $\frac{t}{\psi(t)} \in \Psi$ and $L(t) := \frac{\psi(t)}{\phi(t)}$ is an increasing function such that*

$$\sum_{k=1}^{\infty} \frac{1}{L\big(\mathbf{\Gamma}(e^k)\big)} < \infty, \tag{5.12}$$

where $\mathbf{\Gamma}$ is the inverse function of $\frac{t}{\phi(t)}$, then $M_{\phi} \subseteq J(M_{\psi})$.

Proof It suffices to show that for an arbitrary $x = x^* \in M_{\phi}$, $\|x\|_{M_{\phi}} = 1$ we have $x \in J(M_{\psi})$. We let for brevity $t_k := \mathbf{\Gamma}(e^k)$, *i.e.*, $e^k = \frac{t_k}{\phi(t_k)}$. Obviously, we have

$$\min\left(e^{-k}, \frac{\phi(t)}{t}\right) = e^{-k}\chi_{[0,t_k]}(t) + \frac{\phi(t)}{t}\chi_{[t_k,\infty)}(t). \tag{5.13}$$

By (5.1), without loss of generality, we may assume that $\sup_{t>0} x^{**}(t) \cdot \frac{t}{\phi(t)} \leq 1$. Noting that

$$\min(e^{-k}, x)^{**} \leq \min(e^{-k}, x^{**}), \quad k = 1, 2, \ldots$$

and using (5.1), (5.13) and (5.12) we have

$$\begin{aligned}
\sum_{k=1}^{\infty} \|\min(e^{-k}, x)\|_{M_{\psi}} &\leq \sum_{k=1}^{\infty} \left(\sup_{t>0} \min(e^{-k}, x)^{**} \cdot \frac{t}{\psi(t)}\right) \\
&\leq \sum_{k=1}^{\infty} \left(\sup_{t>0} \min(e^{-k}, x^{**}) \cdot \frac{t}{\psi(t)}\right) \\
&\leq \sum_{k=1}^{\infty} \left(\sup_{t>0} \min\left(e^{-k}, \frac{\phi(t)}{t}\right) \cdot \frac{t}{\psi(t)}\right) \\
&\leq \sum_{k=1}^{\infty} \left(\sup_{t>0} \left\{\left(e^{-k}\chi_{[0,t_k]} + \frac{\phi(t)}{t}\chi_{[t_k,\infty)}\right) \cdot \frac{t}{\psi(t)}\right\}\right) \\
&= \sum_{k=1}^{\infty} \left(\max\left\{\sup_{0\leq t\leq t_k} \left\{e^{-k} \cdot \frac{t}{\psi(t)}\right\}, \sup_{t_k\leq t<\infty} \left\{\frac{\phi(t)}{t} \cdot \frac{t}{\psi(t)}\right\}\right\}\right)
\end{aligned}$$

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$$= \sum_{k=1}^{\infty} \left(\max \left\{ e^{-k} \cdot \frac{t_k}{\psi(t_k)}, \frac{\phi(t_k)}{\psi(t_k)} \right\} \right)$$

$$= \sum_{k=1}^{\infty} \left(\max \left\{ \frac{\phi(t_k)}{t_k} \cdot \frac{t_k}{\psi(t_k)}, \frac{\phi(t_k)}{\psi(t_k)} \right\} \right)$$

$$= \sum_{k=1}^{\infty} \left(\frac{\phi(t_k)}{\psi(t_k)} \right)$$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{L(t_k)} \right)$$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{L\big(\mathbf{\Gamma}(e^k)\big)} \right)$$

$$< \infty.$$

By Proposition 2.6 we have $x \in \mathfrak{J}(M_\psi)$. ■

Corollary 5.8 *Given* $\epsilon, \alpha \in (0, 1]$, *let*

$$\phi = \frac{t + e^2}{\ln^{\alpha}(t) \cdot \ln^{1+\epsilon}\big(\ln(t + e^2)\big)}, \quad t > 0.$$

Then

$$\bigcup_{\alpha<\beta\leq 1} \mathrm{Li}^{\beta} \subsetneqq M_\phi \subseteq J(\mathrm{Li}^{\alpha}).$$

Proof Without loss of generality we may (and shall) assume that for sufficiently large t we have

$$L(t) = \frac{\psi_\alpha(t)}{\phi(t)} = \ln^{1+\epsilon}\big(\ln(t)\big) \quad \text{and} \quad \mathbf{\Gamma}(t) = e^{t^{1/\alpha}}.$$

Then

$$\sum_{k=1}^{\infty} \frac{1}{L\big(\mathbf{\Gamma}(e^k)\big)} = \sum_{k=1}^{\infty} \frac{1}{L(e^{e^{k/\alpha}})} = \sum_{k=1}^{\infty} \frac{1}{(k/\alpha)^{1+\epsilon}} < \infty.$$

The embedding $M_\phi \subseteq J(\mathrm{Li}^{\alpha})$ now follows immediately from Theorem 5.7. The embedding $\bigcup_{\alpha<\beta\leq 1} \mathrm{Li}^{\beta} \subseteq M_\phi$ follows from (5.2) if to take into account that

$$\phi_*(t) \leq \ln^{\beta}(t + e^2), \quad \alpha < \beta \leq 1$$

for sufficiently large $t > 0$. To see that $\bigcup_{\alpha<\beta\leq 1} \mathrm{Li}^{\beta} \neq M_\phi$, it suffices to notice that θ_1 (see Remark 5.5 (i)) belongs to M_ϕ and does not belong to Li^{β}, $\beta \in (\alpha, 1]$ (see the proof of Proposition 5.5). ■

Remark 5.9 Letting

$$\phi_k = (t + e^2) \cdot \theta_k(t), \quad t > 0, \quad k \in \mathbb{N}$$

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we see that essentially the same arguments as in Corollary 5.8 combined with the observation

$$\theta_k \notin M_{\phi_{k-1}}, \quad k \in \mathbb{N}$$

show that

$$\bigcup_{\alpha<\beta\leq 1} \mathrm{Li}^\beta \subsetneqq M_{\phi_1} \subsetneqq M_{\phi_2} \subsetneqq \cdots \subsetneqq M_{\phi_k} \cdots \subseteq J(\mathrm{Li}^\alpha).$$

Remark 5.10 This section presents a general approach to the study of (Breuer)-Fredholm modules motivated by special cases which appear in the existing literature. For the case when $\mathcal{M} = \mathcal{L}(\mathcal{H})$, τ is standard trace and $\psi = \psi_{1/2}$ we note that the notion of an odd Fredholm module associated with $M_\psi(\mathcal{M}, \tau)$ (see Definition 0.1) coincides with the notion of θ-summable Fredholm module (see *e.g.* [Co2, p. 391]). In the same setting, but with $\psi(t) = t^{1-1/p}$ the former notion coincides with the notion of (p, ∞)-summable Fredholm module (see [Co2, pp. 308–312]). It is natural to ask for meaningful examples of the same kind in the setting of general semifinite von Neumann algebra of type II. One such example is considered in [CP, Example II]. The operator D_0 constructed there is affiliated with II_∞ factor (N, τ) and $d_0(t) = (1 + t^2)^{-1/2}$, $t > 0$. Clearly, it is possible to treat (N, D_0) as an example of unbounded p-summable Breuer-Fredholm module for every $p \in (1, \infty)$ as it is done in [CP]. However, it is equally possible to consider (N, D_0) as an example of Breuer-Fredholm module associated with the Marcinkiewicz space $M_{\mathrm{Log}(t+2)}(N, \tau)$. The latter space was introduced in [DDPS2] as a (type II) analogue of the dual to Macaev ideal.

6 Hölder Estimates

We present some results which indicate the intrinsic connection between our theme and study of the Hölder and Lipschitz continuity of the absolute value in the setting of operator spaces. Among most recent publications concerning this setting we mention [Da], [Ko], [DD], [DDPS1], [DDPS2].

Definition 6.0 If $x, y \in \tilde{\mathcal{M}}$, then we say that x is submajorized by y and write $x \prec\prec y$ if and only if

$$\int_0^t \mu_s(x)\, ds \leq \int_0^t \mu_s(y)\, ds, \quad t \geq 0,$$

in other words, if and only if when $\big(\mu(x)\big)^{**} \leq \big(\mu(y)\big)^{**}$.

Recall that for an arbitrary fully symmetric operator space $E(\mathcal{M}, \tau)$ we have

$$\big(x \in \tilde{\mathcal{M}}, y \in E(\mathcal{M}, \tau), x \prec\prec y\big) \Longrightarrow \big(x \in E(\mathcal{M}, \tau), \|x\|_{E(\mathcal{M},\tau)} \leq \|y\|_{E(\mathcal{M},\tau)}\big).$$

We shall start from the auxiliary result which complements both [CS, Proposition 1.2] and [CP, Appendix B, Lemma 5].

Lemma 6.1 *Let $x = x^* \in \tilde{\mathcal{M}}$ and $0 \leq y \in \tilde{\mathcal{M}}$ and let $-y \leq x \leq y$. Then $|x|^{1/2} \prec\prec 2y^{1/2}$.*

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Proof Via the same arguments as in [CS] it suffices to consider the case when $\mathcal{M}$ is non-atomic and further that it suffices to establish that

$$\tau(p|x|^{1/2}) \leq 2\int_0^{\tau(p)} \mu_t(y^{1/2})\,dt$$

for an arbitrary projection $p \in \mathcal{M}$ commuting with x. Given such a projection p there are two projections $p_1, p_2 \in \mathcal{M}$ commuting with x such that $p = p_1 + p_2$ and $p|x|^{1/2} = (p_1x)^{1/2} + (-p_2x)^{1/2}$. Since $p_1xp_1 \leq p_1yp_1$ and since the square root is an operator monotone function, we have $p_1|x|^{1/2} = (p_1xp_1)^{1/2} \leq (p_1yp_1)^{1/2}$ and analogously $p_2|x|^{1/2} = (-p_2xp_2)^{1/2} \leq (p_2yp_2)^{1/2}$. It follows

$$\begin{aligned}
\tau(p|x|^{1/2}) &= \tau(p_1x^{1/2}) + \tau(p_2x^{1/2})\\
&= \tau\big((p_1xp_1)^{1/2}\big) + \tau\big((-p_2xp_2)^{1/2}\big)\\
&\leq \tau\big((p_1yp_1)^{1/2}\big) + \tau\big((p_2yp_2)^{1/2}\big)\\
&= \|(p_1yp_1)^{1/2}\|_1 + \|(p_2yp_2)^{1/2}\|_1\\
&\leq \|\mu^{1/2}(y)\chi_{[0,\tau(p_1))}\|_1 + \|\mu^{1/2}(y)\chi_{[0,\tau(p_2))}\|_1\\
&\leq 2\int_0^{\tau(p)} \mu_t(y^{1/2})\,dt
\end{aligned}$$

where in the penultimate step we used the inequality

$$\mu\big((qyq)^{1/2}\big) = \mu^{1/2}(qyq) \leq \mu^{1/2}(y)\chi_{[0,\tau(q))} = \mu(y^{1/2})\chi_{[0,\tau(q))}$$

which holds for an arbitrary projection $q \in \mathcal{M}$. ■

Theorem 6.2 *Let $E(0,\infty)$ be a fully symmetric function space, let $(\mathcal{M},\tau)$ be an arbitrary semifinite von Neumann algebra, let $D_0 = D_0^*$ be affiliated with $\mathcal{M}$ such that $d_0 \in E(0,\infty)$. Then there exists a constant $C > 0$ (depending on E and D_0) such that for all self-adjoint $D - D_0 \in \mathcal{M}$ we have*

$$\left\| \frac{|D|}{(1+D^2)^{1/2}} - \frac{|D_0|}{(1+D_0^2)^{1/2}} \right\|_{E(\mathcal{M},\tau)} \leq C\max\{\|D-D_0\|^{1/2}, \|D-D_0\|\}. \tag{6.1}$$

Proof It follows from the proof [CP, Appendix B, Proposition 10 (see also Lemma 6)] that

$$\begin{aligned}
-2\max\{\|D-D_0\|^2, \|D-D_0\|\}\cdot\frac{1}{(1+D_0^2)} &\leq \frac{1}{(1+D^2)} - \frac{1}{(1+D_0^2)}\\
&\leq 2\max\{\|D-D_0\|^2, \|D-D_0\|\}\cdot\frac{1}{(1+D_0^2)}
\end{aligned}$$

and, by Lemma 6.1, it follows that

$$\left|\frac{1}{(1+D^2)} - \frac{1}{(1+D_0^2)}\right|^{1/2} \prec\prec 2(2\max\{\|D-D_0\|^2, \|D-D_0\|\})^{1/2}\left(\frac{1}{(1+D_0^2)^{1/2}}\right).$$

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The latter inequality implies immediately that

$$\left\| \left| \frac{1}{(1+D^2)} - \frac{1}{(1+D_0^2)} \right|^{1/2} \right\|_{E(\mathcal{M},\tau)} \leq 2^{3/2} \max\{\|D-D_0\|^{1/2}, \|D-D_0\|\} \left\| \frac{1}{(1+D_0^2)^{1/2}} \right\|_{E(\mathcal{M},\tau)}$$

or, equivalently,

$$\left\| \left| \frac{D^2}{(1+D^2)} - \frac{D_0^2}{(1+D_0^2)} \right|^{1/2} \right\|_{E(\mathcal{M},\tau)} \leq C \max\{\|D-D_0\|^{1/2}, \|D-D_0\|\} \tag{6.2}$$

where $C := 2^{3/2} \| \frac{1}{(1+D_0^2)^{1/2}} \|_{E(\mathcal{M},\tau)}$.

We shall now apply the submajorization inequality due to M. S. Birman, L. S. Koplienko and M. Z. Solomyak presented in [BKS] for the case of symmetrically-normed ideals of compact operators and which was rediscovered by T. Ando in [An]. An extension of this inequality to measurable operators affiliated with an arbitrary semifinite von Neumann algebra $\mathcal{M}$ is due to H. Kosaki (it is given in the appendix to [HN]) with an alternative version of the proof given in [DD]. We need only a simplest case of this inequality for the (operator monotone) square root function. By Theorem 1.1 from [DD] (see also [BKS, Theorem 1]) we have

$$x^{1/2} - y^{1/2} \prec\prec |x-y|^{1/2} \tag{6.3}$$

for any $0 \leq x, y \in \tilde{\mathcal{M}}$. Combining (6.3) with (6.2) and taking into account that, by the assumption, the space $E(\mathcal{M},\tau)$ is fully symmetric we have

$$\begin{aligned} \left\| \frac{|D|}{(1+D^2)^{1/2}} - \frac{|D_0|}{(1+D_0^2)^{1/2}} \right\|_{E(\mathcal{M},\tau)} &= \left\| \left(\frac{D^2}{(1+D^2)} \right)^{1/2} - \left(\frac{D_0^2}{(1+D_0^2)} \right)^{1/2} \right\|_{E(\mathcal{M},\tau)} \\ &\leq \left\| \left| \frac{D^2}{(1+D^2)} - \frac{D_0^2}{(1+D_0^2)} \right|^{1/2} \right\|_{E(\mathcal{M},\tau)} \\ &\leq C \max\{\|D-D_0\|^{1/2}, \|D-D_0\|\}. \end{aligned}$$

Corollary 6.3 *Let the assumptions of Theorem 6.2 be satisfied. Then the functions*

$$\frac{\|D(1+D^2)^{-1/2} - D_0(1+D_0^2)^{-1/2}\|_{E(\mathcal{M},\tau)}}{\max\{\|D-D_0\|^{1/2}, \|D-D_0\|\}} \quad \textit{and} \quad \frac{\|(|D|-|D_0|)\cdot(1+D_0^2)^{-1/2}\|_{E(\mathcal{M},\tau)}}{\max\{\|D-D_0\|^{1/2}, \|D-D_0\|\}}$$

are bounded or unbounded simultaneously for all self-adjoint operators $D - D_0 \in \mathcal{M}$.

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Proof Since the operator D is affiliated with $\mathcal{M}$, the partial isometry v in the polar decomposition $D = v|D|$ belongs to $\mathcal{M}$. It easily follows that the functions

$$\frac{\||D|\big((1+D^2)^{-1/2}-(1+D_0^2)^{-1/2}\big)\|_{E(\mathcal{M},\tau)}}{\max\{\|D-D_0\|^{1/2},\|D-D_0\|\}} \quad \text{and}$$

$$\frac{\|D\big((1+D^2)^{-1/2}-(1+D_0^2)^{-1/2}\big)\|_{E(\mathcal{M},\tau)}}{\max\{\|D-D_0\|^{1/2},\|D-D_0\|\}}$$

are bounded or unbounded simultaneously for all self-adjoint operators $D - D_0 \in \mathcal{M}$. It follows from the equality

$$\begin{aligned} &|D|\left(\frac{1}{(1+D^2)^{1/2}}-\frac{1}{(1+D_0^2)^{1/2}}\right) \\ &\qquad = -(|D|-|D_0|)\left(\frac{1}{(1+D_0^2)^{1/2}}\right)+\frac{|D|}{(1+D^2)^{1/2}}-\frac{|D_0|}{(1+D_0^2)^{1/2}} \end{aligned} \tag{6.4}$$

and Theorem 6.2 that the functions

$$\frac{\||D|\big((1+D^2)^{-1/2}-(1+D_0^2)^{-1/2}\big)\|_{E(\mathcal{M},\tau)}}{\max\{\|D-D_0\|^{1/2},\|D-D_0\|\}} \quad \text{and} \quad \frac{\|(|D|-|D_0|)\cdot(1+D_0^2)^{-1/2}\|_{E(\mathcal{M},\tau)}}{\max\{\|D-D_0\|^{1/2},\|D-D_0\|\}}$$

are bounded or unbounded for all self-adjoint operators $D - D_0 \in \mathcal{M}$ simultaneously. Finally, it follows from the equality

$$D\left(\frac{1}{(1+D^2)^{1/2}}-\frac{1}{(1+D_0^2)^{1/2}}\right)=\frac{D}{(1+D^2)^{1/2}}-\frac{D_0}{(1+D_0^2)^{1/2}}+(D-D_0)\frac{1}{(1+D_0^2)^{1/2}} \tag{6.5}$$

that the functions

$$\frac{\|D\big((1+D^2)^{-1/2}-(1+D_0^2)^{-1/2}\big)\|_{E(\mathcal{M},\tau)}}{\max\{\|D-D_0\|^{1/2},\|D-D_0\|\}} \quad \text{and}$$

$$\frac{\|D(1+D^2)^{-1/2}-D_0(1+D_0^2)^{-1/2}\|_{E(\mathcal{M},\tau)}}{\max\{\|D-D_0\|^{1/2},\|D-D_0\|\}}$$

are bounded or unbounded for all self-adjoint operators $D - D_0 \in \mathcal{M}$ simultaneously. ■

If additional restrictions are imposed on the algebra $\mathcal{M}$ and/or the symmetric space E then some refinements are possible. In the next corollary we consider the class of symmetric function spaces E with non-trivial Boyd indices (see [LT2]). Such spaces are known to be interpolation spaces for some pair of non-trivial L_p-spaces. We assume, in addition, that $(D - D_0) \in E(\mathcal{M}, \tau)$. For the definition of the Fatou property in the setting of symmetric operator spaces, we refer to [DDP2] and [DDPS1].

Corollary 6.4 *Let $E(0,\infty)$ be an interpolation space for some couple $\big(L_p(0,\infty), L_q(0,\infty)\big)$, $1 < p, q < \infty$ with the Fatou property. If the self-adjoint operator $D - D_0$ belongs to $\mathcal{E}(\mathcal{M},\tau)$, then*

$$\|D(1+D^2)^{-1/2}-D_0(1+D_0^2)^{-1/2}\|_{\mathcal{E}(\mathcal{M},\tau)} \le C\max\{\|D-D_0\|_{\mathcal{E}(\mathcal{M},\tau)}^{1/2},\|D-D_0\|_{\mathcal{E}(\mathcal{M},\tau)}\}$$

for some $C > 0$.

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Proof It follows from [DDPS1, Theorem 3.4] that

$$\|(|D|-|D_0|)(1+D_0^2)^{-1/2}\|_{E(\mathcal{M},\tau)} \le C_E\|D-D_0\|_{E(\mathcal{M},\tau)}$$

for some $C_E > 0$. Letting $C_1 = \|(1+D_0^2)^{-1/2}\|_{E(\mathcal{M},\tau)}$ we clearly have

$$\|(D-D_0)(1+D_0^2)^{-1/2}\|_{E(\mathcal{M},\tau)} \le C_1\|D-D_0\|,$$

and by Theorem 6.2 we have that

$$\||D|(1+D^2)^{-1/2} - |D_0|(1+D_0^2)^{-1/2}\|_{E(\mathcal{M},\tau)} \le C_2\|D-D_0\|^{1/2}$$

for some $C_2 > 0$. Combining identities (6.5) and (6.4) with the preceding estimates and letting $M = 3\max\{C_1, C_2, C_E\}$, we obtain

$$\begin{aligned}
&\|D(1+D^2)^{-1/2} - D_0(1+D_0^2)^{-1/2}\|_{E(\mathcal{M},\tau)} \\
&\qquad \le C_1\|D-D_0\| + C_E\|D-D_0\|_{E(\mathcal{M},\tau)} + C_2\|D-D_0\|^{1/2} \\
&\qquad \le M\max\{\|D-D_0\|^{1/2}_{\mathcal{E}(\mathcal{M},\tau)}, \|D-D_0\|_{\mathcal{E}(\mathcal{M},\tau)}\}.
\end{aligned}$$

From [CP, Appendix A, Theorem 8] we have

$$\|D(1+D^2)^{-1/2} - D_0(1+D_0^2)^{-1/2}\| \le \|D-D_0\|.$$

The assertion (with $C = \max\{M, 1\}$) follows now from the two preceding estimates. ■

In the special case that $\mathcal{M} = \mathcal{B}(\mathcal{H})$, a stronger result may be achieved. We first present a result which is of interest in its own right. Its proof is a slight extension of original arguments of Yu. B. Farforovskaya.

Proposition 6.5 *Let f be a Lipschitz function with a constant* 1 *and let E be an interpolation space for the couple (l_1, l_2). If $T \in \mathcal{C}_E$ commutes with D_0, then $\big(f(D) - f(D_0)\big)T \in \mathcal{C}_E$ and, moreover*

$$\|\big(f(D)-f(D_0)\big)T\|_{\mathcal{C}_E} \le \|D-D_0\| \cdot \|T\|_{\mathcal{C}_E}.$$

Proof We shall present the proof assuming, in addition, that both D_0 and D have complete orthonormal systems of eigenvectors $\{e_i\}_{i=1}^{\infty}$ and $\{h_j\}_{j=1}^{\infty}$ respectively. It should be noted that in the special case $T = (1+D_0^2)^{-1/2}$ this assumption is satisfied automatically. Indeed, T has a complete orthonormal system of eigenvectors as a compact operator, whence the same holds for D_0 too. Using the first inequality from the proof of Theorem 6.2, the same arguments may be repeated for the operators $(1+D^2)^{-1/2}$ and D.

Using interpolation theorems from [Ar], it suffices to consider only the two cases $E = l_1$ and $E = l_2$. Further we may also identify T with the diagonal matrix (t_i) (with respect to the basis $\{e_i\}_{i=1}^{\infty}$). Let $\{k(i)\}_{i=1}^{\infty}$ and $\{l(j)\}_{j=1}^{\infty}$ be the systems of eigenvalues of D_0 and D corresponding to the systems $\{e_i\}_{i=1}^{\infty}$ and $\{h_j\}_{j=1}^{\infty}$ respectively. Now we consider the matrix

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representation $(f_{ij})_{i,j=1}^{\infty}$ of the difference $f(D)-f(D_0)$ with respect to the bases $\{e_i\}_{i=1}^{\infty}$ and $\{h_j\}_{j=1}^{\infty}$. It is given by the formulae

$$\begin{aligned} f_{ij} &= \Big(f\big(k(i)\big) - f\big(l(j)\big)\Big) \cdot \langle e_i, h_j\rangle \\ &= \frac{f\big(k(i)\big) - f\big(l(j)\big)}{k(i)-l(j)} \cdot \langle (D-D_0)e_i, h_j\rangle, \quad \text{if} \quad k(i) \neq l(j), \end{aligned}$$

and

$$f_{ij} = 0, \quad \text{if} \quad k(i) = l(j).$$

It follows from the preceding identities, from the fact that $(D-D_0) = (D-D_0)^* \in \mathcal{B}(\mathcal{H})$ and from the fact that f is a Lipschitz function with the constant 1 that

$$\sup_{i,j}\Big\{\sum_{j=1}^{\infty}|f_{ij}|^2, \sum_{i=1}^{\infty}|f_{ij}|^2\Big\} \leq \|D-D_0\|^2. \tag{6.6}$$

It follows from (6.6), that if P_j is the orthogonal projection on the linear span of the vector e_j, $j = 1, 2, \ldots$, we have

$$\big\|\big(f(D)-f(D_0)\big)P_j\big\|_2 = \big\|\big(f(D)-f(D_0)\big)P_j\big\|_1 = \Big(\sum_{i=1}^{\infty}|f_{ij}|^2\Big)^{1/2} \leq \|D-D_0\|. \tag{6.7}$$

Further, the matrix representation $(g_{ij})_{i,j=1}^{\infty}$ of the operator $\big(f(D)-f(D_0)\big)T$ (again with respect to the bases $\{e_i\}_{i=1}^{\infty}$ and $\{h_j\}_{j=1}^{\infty}$), is given by

$$g_{ij} = f_{ij}t_j, \quad i, j = 1, 2, \ldots .$$

We see now that if $T \in \mathcal{C}_1$ then $\|T\|_1 = \sum_{j=1}^{\infty}|t_j| < \infty$ and therefore it follows from (6.7) that

$$\begin{aligned} \sum_{j=1}^{\infty}\big\|\big(f(D)-f(D_0)\big)TP_j\big\|_1 &= \sum_{j=1}^{\infty}\Big(\sum_{i=1}^{\infty}|g_{ij}|^2\Big)^{1/2} \\ &= \sum_{j=1}^{\infty}|t_j|\sum_{i=1}^{\infty}(|f_{ij}|^2)^{1/2} \\ &\leq \|D-D_0\| \cdot \|T|_1. \end{aligned} \tag{6.8}$$

We infer from (6.8) that $\big(f(D)-f(D_0)\big)T \in \mathcal{C}_1$ and $\big\|\big(f(D)-f(D_0)\big)T\big\|_1 \leq \|D-D_0\| \cdot \|T\|_1$.

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If $T \in \mathcal{C}_2$ then $\|T\|_2 = (\sum_{j=1}^{\infty} |t_j|^2)^{1/2} < \infty$ and therefore it follows from (6.6) and (6.7) that

$$\begin{aligned}\|\big(f(D) - f(D_0)\big)T\|_2^2 &= \sum_{j=1}^{\infty} \|\big(f(D) - f(D_0)\big)TP_j\|_2^2 \\ &= \sum_{j=1}^{\infty}\Big(\sum_{i=1}^{\infty} |g_{ij}|^2\Big) \\ &= \sum_{j=1}^{\infty} |t_j|^2 \sum_{i=1}^{\infty} |f_{ij}|^2 \\ &\leq \|D - D_0\|^2 \cdot \|T\|_2^2.\end{aligned}$$

It suffices to complete the proof of Proposition 6.5. ∎

Corollary 6.6 *Given an interpolation space E for the couple (l_1, l_2) and the operator $D_0 = D_0^*$ with $d_0 \in E$ we have $(|D| - |D_0|)(1 + D_0^2)^{-1/2} \in \mathcal{C}_E$ for all self-adjoint $D - D_0 \in \mathcal{B}(\mathcal{H})$. Moreover*

$$\|(|D| - |D_0|)(1 + D_0^2)^{-1/2}\|_{\mathcal{C}_E} \leq \|D - D_0\| \cdot \|d_0\|_E.$$

Combining now Corollary 6.3 with Corollary 6.6 we arrive at the following result.

Corollary 6.7 *Let E be an interpolation space for the couple (l_1, l_2) and let $d_0 \in E$. There exists a positive constant C such that for all self-adjoint $(D - D_0) \in \mathcal{B}(\mathcal{H})$ we have*

$$\|D(1 + D^2)^{-1/2} - D_0(1 + D_0^2)^{-1/2}\|_{\mathcal{C}_E} \leq C \max\{\|D - D_0\|^{1/2}, \|D - D_0\|\}.$$

Let A be a unital Banach $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. We shall assume below that the span $u(\mathcal{A})$ of all unitary elements from $\mathcal{A}$ generate $\mathcal{A}$ (see Definition 0.1).

Corollary 6.8 *If E is an interpolation space for the couple (l_1, l_2) and $(\mathcal{H}, D_0)$ is an odd unbounded Fredholm module associated with $\mathcal{C}_E$ and A, then $\big(\mathcal{H}, \operatorname{sgn}(D_0)\big)$ is an odd bounded Fredholm module associated with $\mathcal{C}_E$ and A.*

Proof We first show that $\big(\mathcal{H}, \phi(D_0)\big)$ is an odd bounded Fredholm module associated with $\mathcal{C}_E$ and A. For an arbitrary $u \in u(\mathcal{A})$ we have

$$[\phi(D_0), u] = \phi(D_0)u - u\phi(D_0) = u\big(u^*\phi(D_0)u - \phi(D_0)\big) = u\big(\phi(u^*D_0u) - \phi(D_0)\big).$$

By the assumption $u^*D_0u - D_0 = u^*[D_0, u]$ is a bounded self-adjoint operator from $\mathcal{B}(\mathcal{H})$, therefore, letting $D = u^*D_0u$ we have by Corollary 6.7, that $\phi(D) - \phi(D_0) \in \mathcal{C}_E$. It follows, immediately that $[\phi(D_0), u] \in \mathcal{C}_E$ and, since $\big(1 - \phi(D_0)^2\big)^{1/2}$ obviously belongs to $\mathcal{C}_E$, it follows that $\big(\mathcal{H}, \phi(D_0)\big)$ is an odd bounded Fredholm module associated with $\mathcal{C}_E$ and A. It is now easy to verify that $\big(\mathcal{H}, \operatorname{sgn}(D_0)\big)$ is an odd bounded Fredholm module associated

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with $\mathcal{C}_E$ and A. Indeed, condition (1) from Definition 0.1 obviously holds. To verify that condition (2) holds, we note that

$$\begin{aligned}\big(\operatorname{sgn}(D_0)-\phi(D_0)\big)\big(\operatorname{sgn}(D_0)+\phi(D_0)\big) &= \operatorname{sgn}(D_0)^2-\phi(D_0)^2\\ &= \operatorname{sgn}(D_0)^2-D_0^2(1+D_0^2)^{-1}\\ &= (1+D_0^2)^{-1}|\operatorname{sgn}(D_0)|\\ &\le (1+D_0^2)^{-1/2}\in\mathcal{C}_E,\end{aligned}$$

and since

$$\Big(\big(\operatorname{sgn}(D_0)+\phi(D_0)\big)|\operatorname{sgn}(D_0)|\Big)^{-1}\in\mathcal{B}(\mathcal{H})$$

it follows

$$\operatorname{sgn}(D_0)-\phi(D_0)=(1+D_0^2)^{-1}|\operatorname{sgn}(D_0)|\big(\big(\operatorname{sgn}(D_0)+\phi(D_0)|\operatorname{sgn}(D_0)|\big)^{-1}\in\mathcal{C}_E,$$

whence

$$[\operatorname{sgn}(D_0),u]=[\operatorname{sgn}(D_0)-\phi(D_0),u]+[\phi(D_0),u]\in\mathcal{C}_E$$

for any $u\in u(\mathcal{A})$. ■

References

[An] T. Ando, *Comparison of norms $\|f(A)-f(B)\|$ and $\|f(|A|)-f(|B|)\|$.* Math. Z. **197**(1988), 403–409.

[Ar] J. Arazy, *Some remarks on interpolation theorems and the boundedness of the triangular projection in unitary matrix spaces.* Integral Equations Operator Theory **1**(1978), 453–495.

[BKS] M. S. Birman, L. S. Koplienko and M. Z. Solomyak, *Estimates for the spectrum of the difference between fractional powers of two selfadjoint operators.* (Russian) Izvestia VUZ (3) **154**(1975), 3–10.

[BM] M. Braverman and A. Mekler, *The Hardy-Littlewood property for symmetric spaces.* (Russian) Sib. Mat. Zhur. **18**(1977), 522–540.

[BR] C. Bennett and K. Rudnick, *On Lorentz-Zygmund spaces.* Dissertationes Math. **175**(1980), 5–67.

[BS] C. Bennett and R. Sharpley, *Interpolation of operators.* Pure Appl. Math. **129**, 1988.

[BSo1–3] M. Sh. Birman and M. Z. Solomyak, *Double Stilties operator integrals I; II; III.* Problemy Mat. Fiz. **1**(1966), 33–67; **2**(1967), 26–60; **6**(1973), 27–53.

[CP] A. Carey and J. Phillips, *Unbounded Fredholm modules and spectral flow.* Canad. J. Math. **50**(1998), 673–718.

[Co1] A. Connes, *Noncommutative differential geometry.* Publ. Math. Inst. Hautes Études Sci. **62**(1985), 41–144.

[Co2] ———, *Noncommutative geometry.* Academic Press, 1994.

[CS] V. I. Chilin and F. A. Sukochev, *Weak convergence in non-commutative symmetric spaces.* J. Operator Theory **31**(1994), 35–65.

[Da] E. B. Davies, *Lipschitz continuity of functions of operators in the Schatten classes.* J. London Math. Soc. **37**(1988), 148–157.

[DD] P. G. Dodds and T. K. Dodds, *On submajorization inequality of T. Ando.* Oper. Theory Adv. Appl. **75**(1995), 113–133.

[DDP1] P. G. Dodds, T. K. Dodds and B. de Pagter, *Non-commutative Banach function spaces.* Math. Z. **201**(1989), 583–597.

[DDP2] ———, *Non-commutative Köthe duality.* Trans. Amer. Math. Soc. **339**(1993), 717–750.

[DDPS1] P. G. Dodds, T. K. Dodds, B. de Pagter and F. A. Sukochev, *Lipschitz continuity of the absolute value and Riesz projection in symmetric operator spaces.* J. Funct. Anal. **148**(1997), 28–69.

[DDPS2] ———, *Lipschitz continuity of the absolute value in preduals of semifinite factors.* Integral Equations Operator Theory **34**(1999), 28–44.

[FK] T. Fack and H. Kosaki, *Generalized s-numbers of τ-measurable operators.* Pacific J. Math. **123**(1986), 269–300.

https://doi.org/10.4153/CJM-2000-037-8 Published online by Cambridge University Press

[HN] F. Hiai and Y. Nakamura, *Distance between unitary orbits in von Neumann algebras.* Pacific J. Math. **138**(1989), 259–294.

[Ka] A. Kaminska, *Some remarks on Orlicz-Lorentz spaces.* Math. Nachr. **147**(1989), 29–38.

[Ko] H. Kosaki, *Unitarily invariant norms under which the map $A \to |A|$ is continuous.* Publ. Res. Inst. Math. Sci. **28**(1992), 299–313.

[KPS] S. G. Krein, Ju. I. Petunin and E. M. Semenov, *Interpolation of linear operators.* Transl. Math. Monographs **54**, Amer. Math. Soc., 1982.

[KR] M. A. Krasnoselskii and Ya. B. Rutickii, *Convex functions and Orlicz spaces.* P. Noordhoff, Groningen, 1961.

[LT2] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II.* Springer-Verlag, 1979.

[Ne] E. Nelson, *Notes on non-commutative integration.* J. Funct. Anal. **15**(1974), 103–116.

[O1] V. I. Ovchinnikov, *Symmetric spaces of measurable operators.* Soviet Math. Dokl. **11**(1970), 448–451.

[O2] ———, *Symmetric spaces of measurable operators.* (Russian) Trudy Inst. Mat. VGU **3**(1971), 88–107.

[P1] J. Phillips, *Self-adjoint Fredholm operators and spectral flow.* Canad. Math. Bull. **39**(1996), 460–467.

[P2] ———, *Spectral flow in type I and II factors—A new approach.* Fields Inst. Commun. **17**(1997), 137–153.

[SC] F. A. Sukochev and V. I. Chilin, *Symmetric spaces on semifinite von Neumann algebras.* Soviet Math. Dokl. **42**(1991), 97–101.

[SSS] A. A. Sedaev, E. M. Semenov and F. A. Sukochev, *Perturbation estimate of the operator function $D(1+D^2)^{-1/2}$ in symmetric spaces via a certain nonlinear functional.* (Russian) Dokl. Akad. Nauk **369**(1999), 316–319.

[Te] M. Terp, *L^p-spaces associated with von Neumann algebras.* Notes, Copenhagen University, 1981.

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