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The Quaternion and its Depreciators.

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Of late years there has arisen a clique of vector analysts who refuse to admit the quaternion to the glorious company of vectors. There are others again who take exception to some of Hamilton's most fundamental principles, and make corrections as they deem them, which logically revolutionise the whole basis of the calculus.

These rebellious ones do not agree at all amongst themselves; but their disloyal sentiments may be conveniently discussed under three headings.

First, there is the broad question as to the value of the quaternion as a fundamental geometrical conception.

Second, there is the question of notation.

Third, there is the question of the sign of the square of a vector when quaternion expressions are to be transformed into ordinary algebraic expressions.

In discussing these points, I shall give what seems to me to be the most natural geometrical approach to the calculus of quaternions. The position of the innovators will thus be better understood.

I. THE QUATERNION AS A GEOMETRICAL CONCEPTION.

(FIGURES 21, 22, 23).

In the preface to the third edition of his *Quaternions*, Professor Tait speaks of Professor Willard Gibbs as one of the retarders of quaternion progress, and of his system of notation as "a sort of hermaphrodite monster compounded of the notations of Hamilton and Grassmann" Professor Gibbs, in a letter published in *Nature*, April 2, 1891, virtually admits both impeachments. For he proceeds to give reasons for his antagonistic attitude, first, to Quaternions as an algebra of vectors, and, second, to Hamilton's notation. His objection to Hamilton's selective system of notation is based upon the dogma that the quaternion product cannot claim a funda-

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mental place in a system of vector analysis. In support of this contention, Professor Gibbs presents a broad argument from geometry, which he thinks he strengthens by a reference to trigonometrical usage. He says :—

"*It will hardly be denied that sines and cosines play the leading* part in trigonometry. Now the notations $V\alpha\beta$ and $S\alpha\beta$ represent the sine and cosine of the angle included between α and β, combined in each case with certain other simple notions. But the sine and cosine combined with these auxiliary notions are incomparably more amenable to analytical transformation than the simple sine and cosine of trigonometry, exactly as, etc., etc."

What does this argument amount to? Certainly no quaternionist ever denied the importance of the sine and the cosine in trigonometry; and Hamilton was unquestionably the first to show forth the analytical power of the functions $S\alpha\beta$ and $V\alpha\beta$. But because these functions are so incomparably more amenable to analytical transformation than their trigonometrical ghosts, are we to infer that they are necessarily superior to or more fundamental than *anything else?* Yet that is the remarkable logic we are treated to.

Mr Heaviside, in his series of articles on "Electromagnetic Theory," published in the *Electrician*, seems to be referring to this argument when he says:—"The justification for the treatment of scalar and vector products as fundamental ideas in vector algebra is to be found in the distributive property they possess." *A fortiori*, the justification for the treatment of the quaternion product as a fundamental idea in vector algebra is to be found in the distributive and associative property it possesses.

Moreover, as Professor Macfarlane points out, the angle itself is of greater fundamental importance than its sine or cosine. So, on the principle of answering a wise man according to his wisdom, I say :—

It will hardly be denied that angles and their functions play the leading part in trigonometry. Now the notation $\alpha\beta^{-1}$ represents the angle included between α and β combined with certain other simple notions. But the angle combined with these auxiliary notions is incomparably more amenable to analytical transformation than the simple angle of trigonometry, and so on—which statement proves just as much and just as little as the great original itself.

But the real argument advanced by Professor Gibbs is as follows :—

"$V\alpha\beta$ represents in magnitude the area of the parallelogram determined by the sides α and β, and in direction the normal to the plane of the parallelogram. $S\gamma V\alpha\beta$ represents the volume of the parallelepiped determined by the edges $\alpha\beta\gamma$. These conceptions are the very foundations of geometry.* . . . I do not know of anything which can be urged in favour of the quaternion product as a *fundamental* notion in vector analysis, which does not appear trivial or artificial in comparison with the above considerations. The same is true of the quaternionic quotient and of the quaternion in general."

"These conceptions"—what conceptions? It can hardly be the conceptions of vector and scalar products of vectors, for these are altogether of the nineteenth century, whereas geometry is of all centuries. It must then be simply the conceptions of the parallelogram as the typical area, and of the parallelepiped as the typical volume. But to speak of these conceptions, and these conceptions *only*—as must be understood if the argument means anything—to speak of these as the very foundations of geometry is surely a misuse of terms, to put it most mildly. Is not the inclination of two lines as fundamental a conception as either of these? Indeed, underlying all the recognised theorems of parallelograms and parallelepipeds there is the great axiom of parallel lines. That lies at the foundation of geometry, if anything so lies.

To appreciate the real character of this argument, let us consider the meaning and purpose of a vector analysis. Having formed the conception of a vector, we have next to find what relations exist between any two vectors. We have to compare one with another; and this we may do by taking either their difference or their ratio. The geometry of displacements and velocities suggests the well-known addition theorem

$$\alpha + \delta = \beta,$$

in which, by adding the vector δ, we pass from the vector α to the vector β. But this method, which is always given first as the simplest, does not seem to me to be more fundamental geometrically than the other method which gives us the quaternion. When we wish to compare fully two lengths, a and b, we divide the one by the other. We form the quotient a/b, and this quotient is defined as the factor which changes b into a. Now a vector is a directed

* The part omitted here is the part already given about the sines and cosines.

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length. By an obvious generalisation, therefore, we compare two vectors by taking their quotient α/β, and by defining this quotient as the factor which changes the vector β into the vector α. This is the germ out of which the whole of vector analysis naturally grows. A more fundamental conception it is impossible to make. Yet Gibbs calls it trivial and artificial! Far more fundamental—we are told—are the conceptions of a vector-bounded area and of a vector-bounded volume, whose bounding vectors may have an infinity of values. Again, a vector is an embodiment of direction; and to know how to change a direction is surely demanded of a vector analyst from the very beginning. But a change of direction is an angular displacement—that is, a versor. Or take the case of a body strained homogeneously. The vector joining any pair of points changes by a process which is a combination of stretching and turning. A simpler description cannot be imagined. It is completely symbolised by the quaternion with its tensor and versor factors. And *this*, we are taught, is trivial and artificial! On the contrary, so fundamental and natural is the conception of the quotient of two vectors that it can be made intelligible to any one. We all unconsciously perform the operation when estimating the time that must be allowed to catch a train.

There is a certain superficial plausibility in the argument that the quaternion product of two vectors is *in that form* less suggestive of geometric significance than the scalar and vector parts taken severally. But when Professor Gibbs says that "the same is true of the quaternionic quotient," he invites the severest criticism. For not only is the quaternion quotient, as a geometrical conception, more fundamental and direct than its own scalar and vector parts; but, if *simplicity* of conception be a guide, it is infinitely more fundamental than even the much-lauded vector and scalar products. To a quaternionist, however, the product $\alpha\beta$ is fundamentally as intelligible as the quotient; for it is simply α/β^{-1}.

Professor Gibbs would have us base the whole of vector analysis on the two geometrical ideas embodied in the formulae $V\alpha\beta$ and $-S\gamma V\alpha\beta$. These are defined, and from the definitions, combined with recognised geometrical truths, the calculus is developed. Clifford, in his *Dynamic*, starts in this very way; and such a method may have an apparent advantage in introducing an otherwise ignorant student rapidly to the merits of a concise and expressive notation. It is "spoon meat," as Mr MacAulay puts it in his recent letter to *Nature* (December 15, 1892). But the average student will probably make little real progress along these lines. He will probably fail to grasp the unity of the calculus as developed from its broad quaternion basis. His faith—his credulity indeed—is severely tested from the very outset. Certain geometric conceptions are put forward and represented by a symbolism of a distinctly arbitrary character. For example, $V\alpha\beta$ does not really mean the area of the parallelogram determined by the vectors α and β, but is a mode of representing that area by a vector line perpendicular to its plane.* And, again, the transition from the parallelepiped $S\gamma V\alpha\beta$ to the uniplanar projection $S\gamma\delta$ cannot but seem to be a piece of legerdemain, involving the transformation of an area into a line. The method requires indeed a succession of definitions, and a careful geometrical discussion of the properties of the quantities so defined. In quaternions, however, the whole is a beautiful and compact development from the fundamental conception of the factor (α/β) which changes β into α. Corresponding to every such quaternion, there is another quaternion known as the conjugate, which will turn β into a particular vector α', equal in length to α, but lying equally inclined to β on the opposite side of it. In short, $\alpha\beta\alpha'$ lie in one plane, and α' is, so to speak, the reflection of α in β (regarded as a mirror).

The geometry is of the very simplest. Suppose, for example, that the quaternion α/β does not change the length of β, but simply its direction—in other words, that it is a versor merely.

Call it q, and its conjugate Kq. Then if $\overline{OB}$, $\overline{OA}$, $\overline{OA'}$ (Fig. 21) are β, α, α' respectively, we get at once

$$q.\overline{OB} = \overline{OA} \quad \text{or} \quad q\beta = \alpha$$
$$Kq.\overline{OB} = \overline{OA'} \quad \text{or} \quad Kq\beta = \alpha'.$$

Hence
$$(q + Kq)\beta = \alpha + \alpha' = \overline{OO} = \beta \times 2\cos\theta,$$

and
$$(q - Kq)\beta = \alpha - \alpha' = \overline{OD} = \overline{OB'}.2\sin\theta.$$

But OB′ is simply OB turned through a right angle. Hence if

* This is very clearly brought out in O'Brien's system of vector analysis, briefly described further on.

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we take i to represent the quadrantal versor, having the same axis as q, $\overline{\mathrm{OB}'}=i\beta$.

Thus we get

$$q+\mathrm{K}q=2\cos\theta$$
$$q-\mathrm{K}q=2i\sin\theta.$$

That is, the sum of a quaternion and its conjugate is a scalar quantity; while the difference is a quadrantal quaternion, changing the length of the vector on which it acts in the ratio of 1 to $2\sin\theta$.

The quadrantal quaternion is evidently of great importance, and it has a property of peculiar value.

Thus let ii' be any two given quadrantal versors, and let them be represented by double arrow-headed unit lines in the directions of their axis, as shown in Figure 22.

Take the unit vector at right angles to both. Then assuming the distributive law, we have

$$(i+i')\beta=i\beta+i'\beta$$
$$=\alpha\ \ +\alpha'.$$

But if we construct on i and i' a parallelogram like that which gives us the resultant vector $\alpha+\alpha'$, we get for its diagonal a directed line parallel to the axis of the quaternion which will turn β into $\alpha+\alpha'$. Not only so, but the length of this diagonal has the same ratio to the length of $\alpha+\alpha'$, which the length of i (or i') has to the length of β. We may therefore regard this diagonal as representing the quadrantal quaternion $(i+i')$. The conclusion is that quadrantal versors and (by an easy extension) quadrantal quaternions are compounded just like vectors. Since, so far, no definition of a vector acting on another vector has been given, we may (if no inconsistency arises) identify quadrantal quaternions and vectors. It is this identification which so wonderfully simplifies the calculus, and yet in no way destroys its generality. We shall refer to this later on.

Meanwhile the point to be noted is that, with this identification of quadrantal quaternion and vector, we conclude that

$$q+\mathrm{K}q=2\mathrm{S}q,\ \text{ a scalar.}$$
$$q-\mathrm{K}q=2\mathrm{V}q,\ \text{ a vector,}$$

where the meanings of $\mathrm{S}q$ and $\mathrm{V}q$ are easily detected. If q is a versor, $\mathrm{S}q$ is the *cosine* of the angle through which q turns a vector perpendicular to its axis; and $\mathrm{V}q$ is the vector (or quadrantal quaternion) measured along this axis, and of length equal to the *sine* of the same angle. By a simple extension, if a and b are the lengths (or *tensors*) of α and β, we find

$$\frac{\alpha}{\beta}=\mathrm{S}\frac{\alpha}{\beta}+\mathrm{V}\frac{\alpha}{\beta},$$

where $$\mathrm{S}\frac{\alpha}{\beta}=\frac{a}{b}\cos\theta \text{ and } \mathrm{V}\frac{\alpha}{\beta}=i\frac{a}{b}\sin\theta.$$

And now consider the result of operating by two quadrantal versors in succession. Let $i'i''$ be these versors (Fig. 23). Draw the planes perpendicular to them, and let γ be a vector along the line of intersection. Take β perpendicular to γ and i', so that $i'\beta=\gamma$.

Then $$i''(i'\beta)=i''\gamma=\alpha,$$

or, assuming the associative law, we get

$$i''i'.\beta=\alpha.$$

Hence $i''i'$ is the quaternion α/β, which (as is obvious from the figure) has its axis perpendicular to i' and i'', and turns β through an angle equal to the complement of the angle between i' and i''. Consequently we find, θ being the angle between i' and i'',

$$\mathrm{S}i''i=\mathrm{S}\frac{\alpha}{\beta}=-\cos\theta$$

$$\mathrm{V}i''i=\mathrm{V}\frac{\alpha}{\beta}=\ i\sin\theta.$$

From this we readily see that, with the identification of vectors and quadrantal quaternions,

$$\alpha\beta=\mathrm{S}\alpha\beta+\mathrm{V}\alpha\beta,$$

and $$\mathrm{S}\alpha\beta=-ab\cos\theta$$

$$\mathrm{V}\alpha\beta=i.ab\sin\theta,$$

where i is a unit vector perpendicular to the plane $\alpha\beta$. Thus the geometric meanings of $\mathrm{S}\alpha\beta$ and $\mathrm{V}\alpha\beta$ grow naturally out of the original conception of the quaternion quotient of two vectors, taken in conjunction with the identification of vectors and quadrantal quaternions, and with the assumption that the distributive and associative laws hold.

In any vector analysis which begins by separately defining the parts of the complete quaternion product, there is a want of cohesion from the very beginning, and there is nothing that can be compared with the beauty and solidarity of the quaternion calculus.

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Take, by way of comparison, the symbolic algebra of the Rev. M. O'Brien, to which the systems affected by Gibbs, Heaviside, and Macfarlane have a strong family likeness. O'Brien, at that time Professor of Astronomy and Natural Philosophy in King's College, London, published his most important paper in the *Philosophical Transactions* (1852). He begins by defining what he calls the longitudinal and lateral translations of the vector β with reference to the vector α. These are symbolised as products in the form $\alpha \times \beta$ and $\alpha.\beta$—the reason being because they are distributive. It is readily seen that $\alpha \times \beta$ is the product of the lengths of α and β into the cosine of the angle between them; in fact, Hamilton's $-S\alpha\beta$, Grassmann's "inner" product, and Gibbs's "direct" product $(\alpha.\beta)$. In $\alpha.\beta$ O'Brien recognises the area of the parallelogram, of which α and β are the sides. In developing his system, he finds that the line perpendicular to the plane containing these two vectors is of fundamental importance. He calls it the directrix, and uses for it the symbol D. Thus $D\alpha.\beta$ corresponds geometrically to Hamilton's $V\alpha\beta$. It will be noticed that O'Brien keeps quite distinct the conception of the product $\alpha.\beta$, and that of its directrix $D\alpha.\beta$. From the definitions it follows that $\alpha.\alpha$ is zero, so that $\alpha \times \alpha$ may be written α^2 without any fear of ambiguity. Then α^2 is assumed to be the square of the length of α. It is abundantly evident that O'Brien's vector in multiplication is not intended to have any versor characteristic. He sees that the square of every unit vector must be the same, and confessedly *assumes* it to be $+1$, pointing out, however, that if he could see any reason for making it -1, his system would be the same as Hamilton's. We shall return to this further on.

Meanwhile, take another of the arguments accumulated by Professor Gibbs in favour of the non-quaternionic basis of vector analysis. He writes:—

"How much more deeply rooted in the nature of things are the functions $S\alpha\beta$ and $V\alpha\beta$ than any which depend on the definition of a quaternion will appear in a strong light if we try to extend our formulae to space of four or more dimensions. It will not be claimed that the notions of quaternions will apply to such a space, . . . But vectors exist in such a space, and there must be a vector analysis for such a space. The notions of geometrical addition and the scalar product are evidently applicable to such a space. As we cannot define the direction of a vector in space of four or more dimensions by the condition of perpendicularity to two given vectors,

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the definition of $V\alpha\beta$, as given above, will not apply *totidem verbis* to space of four or more dimensions. But a little change in the definition, which would make no essential difference in three dimensions, would enable us to apply the idea at once to space of any number of dimensions."

To elucidate the "nature of *things*" by an appeal to the fourth dimension—to solve the Irish Question by a discussion of social life in Mars—it is a grand conception, worthy of the scorner of the trivial and artificial quaternion of three dimensions. But is it not the glory of quaternions that it is so pre-eminently a tri-dimensional calculus? Geometers who look forward to a four dimensional existence may think their time in three dimensions best employed by confining their attention only to such mathematical methods as *seem* to be applicable to the higher space. But he lives best who works best in the particular environment of the moment. The man who fasts a whole week in prospect of a feast of unique magnificence is hardly rational. And note that Professor Gibbs has to make "a little change in the definition" of $V\alpha\beta$, ere he can make it serviceable in his evanescing vision of four dimensional space. Even his own vector analysis does not apply at once; and, with the admission of the necessity of change, the argument loses all point.

"There must be a vector analysis in such a space"—true, and there must be in space of n dimensions an M-in-one corresponding to the 4-in-one in 3-dimensional space. Moreover, the geometrical significance of a quaternion, as the factor that changes one vector into another, must have its analogue in space of four or higher dimensions. For if there be vectors, there must be modes of changing one into another.

In further pursuit of his end, Professor Gibbs draws a comparison between the quaternion and the linear and vector function, which latter he regards as quite enough for all purposes. He asserts that "nothing is more simple than the definition of a linear vector function, while the definition of a quaternion is far from simple." Observe, it is the simplicity of the *definition* that is here spoken of; but a definition will appear simple or the reverse, according to the degree of previous knowledge possessed. I question very much that a vector function of a vector is an easy conception to make on the part of one who is just entering upon the study of a vector analysis. It is only by a study of its properties, geometrical and dynamical, that the linear vector function becomes intelligible. Not until the

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thing symbolised is got a hold of by the mind can the definition of the symbol convey any adequate meaning. But, on the other hand, if the conception of a vector be realised at all, the further conception of the geometric meaning of the quotient and product of two vectors is a very simple step indeed. A simpler can hardly be imagined.

When Professor Gibbs speaks of the definition of a quaternion being far from simple, he probably has in mind the truth that a quaternion is expressible as the sum of a scalar and a vector. Mr Heaviside says: "The quaternion is regarded as a complex of scalar and vector." The pure analyst may think of it so; but the physicist should think of it in its purely geometrical significance as made up of tensor and versor. Its property of being decomposable into scalar and vector parts with geometric meanings, at first sight so distinct from its own fundamental characteristic, is an absolutely invaluable one. The quaternion includes within itself the conception of a rotation, a stretching, a vector area, and a projection. You may choose whichever part or parts may serve your purpose for the moment—they are all there uniquely determined when the quaternion is given. There truly is a king of quantities. "Upon earth there is not his like."

Still another argument, advanced in all seriousness by both Gibbs and Heaviside, is that even the avowed quaternionist comparatively rarely uses the quaternion, but is constantly manipulating his scalar and vector products. Now, it is true that the symbols S and V throng the pages of Hamilton and Tait; but the expression $V\alpha\beta$ does not hide the truth that $\alpha\beta$ is a quaternion. It rather displays it. By way of illustration, let us apply the Gibbs-Heaviside argument to trigonometry. In any treatment of this subject, the quantities $\sin\theta$ and $\cos\theta$ occur a hundred times at least for once that θ occurs singly. Is the angle, then, of no fundamental importance in trigonometry? There is more than an apparent analogy here. For just as *sin* and *cos* are selective symbols operating on θ, so are V and S selective symbols operating on q.

II. Comparison of Notations.

Professor Gibbs, having to his own satisfaction got rid of the "trivial and artificial" quaternion, is, for consistency's sake, obliged to object to the selective system of notation. This is not, however, the ostensible ground on which he recommends the adoption of a notation in which vector and scalar products of two vectors are

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indicated by symbols inserted between the quantities. This he regards as the natural mode of representation. Consequently he suggests $\alpha \times \beta$ to represent what he calls the "skew" product, and $\alpha.\beta$ to represent what he calls the "direct" product.* The skew product is Hamilton's vector product, which is certainly an infinitely more suitable name, even from Gibb's own limited point of view. The "direct" product—a most inappropriate name, it seems to me—is the product of the lengths of the vectors into the cosine of the angle between them, and corresponds to Hamilton's $-S\alpha\beta$. It is obvious that, though there may be a saving of labour in writing $\alpha.\beta$ instead of $S\alpha\beta$, no such advantage attaches to $\alpha \times \beta$ as compared with $V\alpha\beta$. But it is when more than two vectors have to be joined together that the inferiority of the suggested notation becomes painfully evident. Thus the expression $S\alpha\beta\gamma$ must be written $-\alpha.\beta \times \gamma$, which is less compact and less symmetrical than Hamilton's form. Again, the expression $VV\alpha\beta V\gamma\delta$ must be written $(\alpha \times \beta) \times (\gamma \times \delta)$, where the brackets are all-essential. The quantity $V\alpha\beta\gamma$ cannot be expressed by Gibbs at all in simple form, but has to be given in the expanded form

$$-\alpha.\beta \times \gamma + \alpha \times (\beta \times \gamma).$$

Such an expression as $V\alpha.\beta V\gamma\delta$ can only be displayed in the extraordinary form

$$-\alpha\beta.\gamma \times \delta + \alpha \times (\beta \times (\gamma \times \delta)).$$

It is occasionally necessary to use brackets in somewhat complex quaternion formulae, although in general a separating "dot" suffices to prevent ambiguity. But, in Gibbs's system, brackets have to be introduced just as soon as we begin to pass to the simplest formulae involving three vectors. The *cross* and *dot* are, in short, quite unequal to the task of distinguishing vector and scalar quantities.

Heaviside, in his notation, retains Hamilton's V, but drops the S, so that where no initial V exists, the product is taken to be the scalar product. Thus he would write $S\alpha\beta\gamma$ in the form $-\alpha V\beta\gamma$, in which, it appears to me, the symmetry of the expression is, to a large extent, lost, and in which there is no gain in compactness. The possibility of cyclically permuting $\alpha\beta\gamma$ without altering the value of $S\alpha\beta\gamma$ is by no means so evident in Heaviside's form.

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* These are O'Brien's very symbols, but used with the meanings interchanged.

One of the peculiar merits of Hamilton's notation is the way in which vector quantities stand out in relief among quantities of a different character. Small Greek letters are in general used for vectors; small Roman letters for scalars. The selective symbols V, S, T, U, K are evident at a glance, and we know what a quantity is before we have to inquire narrowly into its constitution. Not so with Gibbs's notation, in which any really complex expression becomes bewildering in its dots, crosses, and brackets. Heaviside has to a large extent destroyed the perspicuity of Hamilton's notation by employing capitals for the frequently occurring single quantities, so that the very important symbol V is not conspicuous. He distinguishes vectors from scalars by using heavy type. This distinguishes them sufficiently, no doubt, in print; but vector analysis is a thing *to be used*, and it is hopeless to write, easily and rapidly, capital letters and thick-lined capital letters with pencil, pen, or chalk. His own suggestion of a suffix notation to be used in manuscript is an unconscious condemnation of his whole system. A good notation in vector analysis requires these three things: (1) rapidity and ease in *writing* the frequently recurring quantities; (2) a distinction, evident at a glance, between vectors and scalars; and (3), as important as any, the vector and scalar parts of products thrown out in clear relief. It is abundantly evident that, in these respects, Hamilton's notation easily holds its own.

Apart altogether from the comparison that has just been made, there is, I think, a fundamental objection to a notation like O'Brien's and Gibbs's. It is that, corresponding to either product, there is no process by which a generalised quotient can be formed by taking one of the members over to the other side of the equation. Thus the equation

$$\alpha \times \beta = \gamma$$

suggests by its very form that there ought to be a transformation like

$$\alpha = \gamma \div \beta.$$

But there is no such, for obvious geometrical reasons. In other words, given γ and β, α is not determined. This simply shows, of course, that $\alpha \times \beta$ has no claim whatever to being regarded as a complete or generalised product. Exactly the same is true of $\alpha . \beta$. Now in quaternions we have $\alpha\beta = q$, where any one is determined uniquely when the other two are given. We are able at once to write $\alpha = q\beta^{-1}$ or $\beta = \alpha^{-1}q$. But in the equations

$$V\alpha\beta = \gamma \text{ and } S\alpha\beta = a$$

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there is no suggestion of the possibility of taking β or a to the other side as a kind of divisor. By the very law of their being, S and V are selective symbols, and (like *sin*, *cos*, *log*, etc.) operate on the whole quantity $a\beta$. But in Gibbs's notation we have two quantities having all the appearance of ordinary products, to which, however, the familiar transformations which are suggested by their form are inapplicable. Such a restriction is surely inexpedient, especially when the desired end can be attained by a less objectionable and infinitely more perspicuous notation such as Hamilton has provided.

III. The Versorial Character of Vectors.

The identification of quadrantal quaternions and vectors has already been described as constituting one of the most important simplifications effected in the calculus. If a quadrantal quaternion operate *twice* on the same vector perpendicular to its axis, it will turn that vector through *two* right angles, and change the length of the vector in the ratio of a^2 to 1, where a is the tensor or stretching part of the quaternion. In symbols, if a is the quadrantal quaternion *or vector*, and β the perpendicular vector acted on, we get

$$\alpha\alpha\beta = -a^2\beta,$$

because the direction of β is simply inverted; or $\alpha^2 = -a^2$. In words, the square of a vector is equal to *minus* the square of its length. If i, j, k are unit vectors, then $i^2 = j^2 = k^2 = -1$. This negative sign, which O'Brien puzzled over long ago, is a stumbling-block and rock of offence to both Mr Heaviside and Dr Macfarlane. It reappears whenever the quantity $\mathrm{S}\alpha\beta$ is transformed into its value in ordinary algebraic quantities. Heaviside apparently was the first to kick against this peculiarity of quaternions. In his earlier papers he used the symbolism of quaternions because of its expressive compactness; and having found it irksome to be continually changing signs of scalar products, when he had occasion to transform these into ordinary algebraic symbols, he determined to take the scalar product as *plus* the product of the tensors into the cosine of the angle between the vectors. This O'Brien touch seems *so far* to have led to no confusion. Heaviside's formulae are quasi-quaternionic, and are a considerable simplification on the corresponding Cartesian expressions. But as the change involves the very fundamental one of making i^2, j^2, k^2 each *plus* unity, it is certain that the system is not quaternions. What, then, is it? To what,

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if fully developed, would it lead us? Macfarlane completely answers this question. In his pamphlet, *The Algebra of Physics*, he works out very fully O'Brien's and Heaviside's vector analysis, and obtains a system very similar up to a certain point to Hamilton's quaternions, but departing widely therefrom in certain of its higher developments. It is much more complicated, YET NO MORE GENERAL. When Dr Heaviside has realised the complication which is the logical outcome of his imagined simplification, we trust he will return into the paths of quaternionic rectitude. In his recent paper on the *Forces, Stresses, and Fluxes in the Electromagnetic Field* (Phil. Trans. 1892), he writes that his system "is simply the elements of quaternions without the quaternions, with the notation simplified to the uttermost, and with the very inconvenient *minus* sign before scalar products done away with." As we shall see presently, the first nine words of this sentence are fundamentally inconsistent with the last twelve.

Let us consider, first, what is common to quaternions, and to the system advocated by Heaviside and Macfarlane. It is well known that quaternions may be built up analytically upon the properties of i, j, k, three unit vectors (or right versors), at right angles to one another. Now Heaviside and Macfarlane admit the relations

$$ij = k = -ji,\ jk = i = -kj,\ ki = j = -ik.$$

which also hold in quaternions. Gibbs, it may be noted, does not use the complete product at all, but writes his relations thus:

$$i \times j = k = -j \times i, \text{ etc, ; } i.i = 1, \text{ etc.}$$

O'Brien and his unconscious followers, however, boldly put $i^2 = j^2 = k^2 = +1$, thereby clashing at once with quaternions.

Taking, then, what is common to the two, namely, the set of equations represented by $ij = k = -ji$, let us consider the product of the three vectors, i, $i+j$, j, the values of i^2, j^2, k^2 being meanwhile left undetermined.

Then by one mode of association,

$$i(i+j)j = i(ij + j^2) = ik + ij^2 = -j + ij^2,$$

and by another mode of association,

$$i(i+j)j = (i^2 + ij)j = i^2j + kj = i^2j - i.$$

Here the distributive law is assumed. Now if these quantities are to be the same, that is, if the associative law is also to hold, we must have $i^2 = j^2 = -1$.

If we use $+1$, we get opposite vectors, and the associative law does not hold in vector products. The above, of course, is a very simple case. In the completely general case in this rival system, the products $(\alpha\beta)\gamma$ and $\alpha(\beta\gamma)$ are different quantities, giving the same scalar part, but quite different vector parts. It is surprising that this aspect of the question should have escaped O'Brien.

Let us represent vectors in Heaviside's and Macfarlane's system by Roman letters *abcd* ..., and corresponding Hamiltonian vectors by $\alpha\beta\gamma\delta$ .... Then it is easy to see that, since the scalar part of the product *ab* is equal to $-\mathrm{S}\alpha\beta$,

$$ab = -\mathrm{S}\alpha\beta + \mathrm{V}\alpha\beta = -\mathrm{K}\alpha\beta = -\beta\alpha,$$

and it may be shown that

$$\begin{aligned}(ab)c &= -\gamma\alpha\beta\\ a(bc) &= -\beta\gamma\alpha.\end{aligned}$$

Now in quaternions we get in general *six* different quantities by permutations of α, β, γ; and at first sight it might seem that this new vector algebra gives *twelve* different products, since each arrangement such as *abc* gives two products by different associations. But inquiry soon shows that this is not so; for although there are two quantities got by different associations of any given arrangement, each quantity so obtained is reproduced in a particular association of some other particular arrangement. We easily see, in fact that

$$\begin{aligned}\alpha\beta\gamma &= -(bc)a = -c(ab)\\ \gamma\alpha\beta &= -(ab)c = -b(ca),\end{aligned}$$

and so on. It is evident that the O'Brien system gives us absolutely nothing more than is given by quaternions, but simply adds complexity. In quaternions we get all possible products by permutation *only*; in this other system we get the same number of quantities, partly by association, partly by permutation. The complexities of the system are still more pronounced when we pass to products of four or more vectors. Macfarlane glories in his five products obtained by different associations, namely,

$$((ab)c)d,\ (a(bc))d,\ (ab)(cd),\ a((bc)d),\ a(b(cd)).$$

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But then we find that each one of these is reproduced in four other associations of particular arrangements. For example,

$$(ab)(cd) = ((ad)b)c = (d(ba))c = b((dc)a) = b(c(ad)).$$

All this hopeless confusion is the result of putting i^2, j^2, k^2 each equal to unity. Well may we be grateful to Hamilton for having given us an associative vector algebra of the utmost generality. A most interesting discussion of this very point is given in §§ 50–56 of the Preface to Hamilton's *Lectures on Quaternions.* It is there shown, from general considerations of the symmetry of space, that, when the rules for the multiplication of vectors are made to differ as little as possible from the usual rules for the multiplication of numbers in algebra, the result is the quaternion system of vector analysis, the *commutative law only* being departed from. These sections should be carefully considered by all would-be innovators.

The question naturally arises—What meaning are we to attach to the equations $ij = k$, $jk = i$, etc? Heaviside and Macfarlane seem to regard i and j as mutually perpendicular vectors, which, by their product, give a third vector perpendicular to both.* In quaternions the meaning is obvious, for i is the versor which, acting on j, turns it into k. Moreover, Professor Gibbs, on page 6 of his pamphlet, explicitly enunciates the same principle when he says that "the effect of the skew [*i.e.*, vector] multiplication by α [any unit vector] upon vectors in a plane perpendicular to α is simply to rotate them all 90° in that plane." To which, by way of commentary, we may quote the following from Heaviside :

"In a given equation" [in quaternions, that is], "one vector may be a vector, and another a quaternion. Or the same vector, in one and the same equation, may be a vector in one place and a quaternion (versor or turner) in another. This amalgamation of the vectorial and quaternionic functions is very puzzling. You never know how things may turn out."

Puzzling?—then should Mr Heaviside find his own system as puzzling as any. For when he writes the vector product

$$ij = k,$$

he is simply acting on j by i, or on i by j, and turning it through a

* O'Brien seems to be much more consistent here, for his product $\alpha.\beta$ is the area, and he uses $D\alpha.\beta$ as the symbolism for the quantity $V\alpha\beta$. Where Heaviside and Macfarlane cease to be O'Brienites, they become inconsistent.

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right angle. It is impossible to get rid of this versorial effect of a vector. It stares you in the face from the very beginning. It is the only rational way of impressing the meaning of the equations.

Leaving Gibbs and Heaviside to harmonise, if possible, their differences, I shall here call attention briefly to one distinction between Hamilton's quaternions and Grassmann's *Ausdehnungslehre*. In the *Ausdehnungslehre* of 1862, Grassmann explains the meaning of his units e_1, e_2, e_3. . . . The *essential* feature of these is, that $e_1e_2 = -e_2e_1$, $e_1e_3 = -e_3e_1$, and so on for any pairs. Since the units are supposed to be of the same kind, it follows that $e_1e_1 = -e_1e_1$ also, an equation which cannot be true unless e_1^2 vanishes. Similarly the squares of all the units vanish. Grassmann also suggests that another algebra is given if we assume e_1^2, e_2^2, e_3^2 . . . to be each equal to $+1$, and all the products to be zero.

It is evident that the whole mode of looking at the question is fundamentally different in the two cases; and that it is impossible to identify Grassmann's units with Hamilton's i, j, k. Grassmann's "outer" and "inner" products in the *Ausdehnungslehre* of 1844 correspond to Hamilton's $V\alpha\beta$ and $-S\alpha\beta$; but there is no doubt that Grassmann failed to see that these quantities could be combined by subtraction, so as to give a new quantity having a very simple geometrical meaning, namely the quaternion of Hamilton.

IV. General Conclusions.

The general conclusions at which we have arrived may be summarised briefly as follows:

(1) The quaternion quotient is as fundamental a geometrical conception as the vector sum, the vector product, and the scalar product of two vectors, so that Professor Gibbs's argument, which is based upon the assertion that it is certainly not so, is void and meaningless.

(2) Whatever demerits may exist in Hamilton's own notation, there has not as yet been suggested anything that can be regarded as an improvement. The changes introduced by Gibbs and Heaviside destroy some of the most perspicuous and symmetrical features of the quaternion notation. Leaving out of account a few very exceptional cases, these suggested notations cannot for a moment compare with Hamilton's in clearness, compactness, and facility for manipulation.

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(3) In the original conception of a vector (as involved in the addition theorem, for example), there is nothing inconsistent with its versorial character in multiplication. The truth is, that many physical quantities, which are symbolised by vectors, are essentially rotational. It is not merely displacement, or velocity, or acceleration that is so symbolised. Moments of velocities and forces, rotations themselves, vortex axes, and a whole host of similar quantities in electricity and magnetism, are either simple vectors or localised vectors. Or again, it is universally admitted that a displacement may be regarded as a rotation about an infinitely distant axis. Every vector in space may be regarded as a vector arc upon a spherical surface of infinite radius. But a vector arc on a spherical surface is a versor. On what physical ground, then, can any one object to a vector having a versorial quality? Indeed, notwithstanding all assertions to the contrary, Heaviside and Macfarlane really use the vector as a versorial operator; for what other meaning can be attached to the equation $ij = k$? Gibbs, as we said, explicitly uses the vector as a versor. The versorial character of a vector being thus admitted, there is no sufficient reason for regarding the square of a vector as other than *minus* the square of its length.

(4) The vector algebra, which is built upon the *assumption* that $i^2 = j^2 = k^2 = +1$ is non-associative in its products. And yet, notwithstanding this *appearance* of greater generality, it gives us absolutely no new thing. Its non-associative character is partly balanced by the fact that its non-commutativeness is incomplete. It simply muddles what is beautifully clear in quaternions.

In this paper I have limited myself to the consideration of the fundamental differences that exist between quaternions and the systems advocated by Gibbs, Heaviside, and Macfarlane. To complete the discussion, however, it would be necessary to review the systems in themselves as they have been developed. Of the three, Professor Gibbs has given us the most consistent system in his pamphlet, *The Elements of Vector Analysis*. In a paper communicated to the Royal Society of Edinburgh, I have entered at some length into a criticism of the contents of this pamphlet.

I show that Professor Gibbs, although ostensibly excluding the quaternion, introduces it in a covert way in his treatment of the

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linear and vector function. Not only so, but in certain volume surface and line integrals he uses *the quaternion product itself*, thereby perjuring his whole position, as described in his letter to *Nature*. Then there is his treatment of the quantities and operations that cluster round the quaternion operator ∇. By their tinkering processes, Gibbs, Heaviside, and Macfarlane all reduce this beautiful operator to a mere make-believe, which, in the simpler applications, appears to have all the essential attributes of the true ∇, but utterly fails when higher things are demanded of it.

It is a fair question—What has induced these scientific writers to take up their antagonistic attitude to the quaternion calculus? Heaviside and Macfarlane confess that their grievance is the *minus* sign. It is marvellous—indeed, almost ludicrous—to have mathematicians take fright at such a very simple matter. To the beginner, perhaps, who is constantly translating the quaternion quantities into ordinary analytical form, the necessity of changing the sign before scalar quantities is at first a little irksome. But, with a very little experience, the irksomeness quite vanishes away. It is no more formidable than re-arranging the terms of an equation by shifting them to different sides. Possibly, however, this preliminary peculiarity may have deterred many from continuing their study of quaternions. Heaviside, with inimitable assurance, thinks his system is what the physicist wants. An algebra non-associative in its products! When once the physicist has realised the full meaning of this, he will surely take courage, and tackle the quaternion analysis in earnest.

Gibbs, however, although he uses a symbolism for $-S\alpha\beta$, and thereby appears to side with Heaviside, nowhere confesses to have been repulsed by the "unnatural" and "inconvenient" *minus* sign. Why, then, does he object to Hamilton's system? His ostensible reasons, as given in the first letter to *Nature*, have been shown to be based on a complete misapprehension. Evidently he has not taken the trouble to get into the spirit of quaternions—and this, I believe, to be the true explanation of the apathy amongst physicists towards quaternion analysis—or (if we may judge from his second letter to *Nature*) he has so convinced himself as to the all-efficiency of Grassmann's methods, that he is determined to bar out the great thing in Hamilton's system which is lacking in Grassmann's. With what success, or non-success rather, he manages this, is shown in my paper communicated to the *Royal Society of Edinburgh*.

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